Measuring Tax Efficiency: A Tax Optimality Index

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ABSTRACT: This paper introduces an index of tax optimality that measures the distance of some current tax structure from the optimal tax structure in the presence of public goods. This index is defined on the \([0, 1]\) interval and measures the proportion of the optimal tax rates that will achieve the same welfare outcome as some arbitrarily given initial tax structure. We call this number the Tax Optimality Index. We also show how the basic methodology can be altered to derive a revenue equivalent uniform tax, which measures the tax burden implied by the public sector. A numerical example is used to illustrate the method developed, and extensions of the analysis to handle models with multiple households and nonlinear taxation structures are undertaken.

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1. Introduction

The present paper contributes to the theory of measuring the efficiency costs of taxes. To address this issue the standard approach in the public finance literature is to use the excess burden of taxation, which measures the efficiency cost of taxes in money terms. In this paper, we propose a non-money metric index that contains normative information about the current level of taxes in a quite intuitive way.

Despite its wide appeal, the excess burden measure has some well-known weaknesses in its application. Quoting Auerbach (1985), the excess burden of taxation is defined "as the amount that is lost in excess of what the government collects. Unfortunately, while this definition makes intuitive sense, it is too vague to permit a single interpretation" (p.67). Thus, while in Mohring (1971) the excess burden is the amount in excess of taxes being collected that the consumer would give up in exchange for the removal of all taxes (an Equivalent Variation calculation), in Diamond and McFadden (1974) the excess burden is the amount that the government must supply to the consumer to allow her to maintain the initial level of utility (a Compensating Variation calculation). The results, as we know, differ, but, of course, either one chosen provides a correct indicator of welfare change. More importantly, and independently of which method we use, the problem with excess burden is that it represents a money metric aggregate. As such it is influenced by the level of prices in the economy, i.e., it is not homogenous of degree zero in prices. This clearly precludes any reasonable comparison of the efficiency costs of taxes over time or between countries.

A simple approach to this problem, and one that is widely used in economic policy, is to normalize the excess burden of taxation measure by a measure of economic activity, e.g., gross domestic product (GDP). However, there will still be problems with such a procedure. On the one hand, there will be an issue of what to use for the normalization. In comparing the distortionary effects of taxes over time or between countries, should we normalize by using GDP, or should we use the size of the public sector measured by tax revenues or public expenditures? Different choices will lead to different numbers and thus to different orderings in our comparison. On the other hand, assuming that we can agree on a particular normalization, the normalized excess burden measure is not necessarily homogenous of degree zero in prices. In particular, when taxes are not proportional to prices, i.e., when they are specific and not ad valorem, use of readily available measures of economic activity will not remove the homogeneity problem (see
Chau et al., 2003, pp. 1089).¹

In the present paper we suggest a measure of tax efficiency that is not expressed in money terms and is free of all the above problems.² Building on distance function techniques recently utilized by Anderson and Neary (1996, 2003, 2005) in the international trade context, we propose an index that measures the welfare burden of a given tax configuration as its distance from optimal taxes (where the welfare burden is taken to be zero). We call it the Tax Optimality Index (TOI). Our measure has an immediate, and intuitive, interpretation: for example, a TOI equal to 0.6 implies that current taxes are 60% efficient, or, in other words, that a 40% reduction of optimal tax rates will reduce welfare to the level that exists at the current tax rates. Being a distance measure of tax efficiency, it is not expressed in money terms and can be directly compared between different countries and/or time periods without at all addressing the normalization issue (with all the above mentioned problems).

We should, however, mention that the gain from using our index comes at a price. Our measure says nothing about how costly it is to be away from the optimal level of taxes. For example, while a comparison between a \(\text{TOI}_i = 0.8\) and a \(\text{TOI}_j = 0.6\) would indicate that taxes are more efficient in case \(i\) than in case \(j\), the costs of reforming taxes towards their optimal level in case \(i\) can be much higher than in case \(j\) (assuming that the two cases do not have the same preferences and technology). To that purpose, a normalized measure of the excess burden of taxation would be more appropriate. However, this issue notwithstanding, our measure does provide a theoretically more sound way to rank countries or time periods in terms of tax efficiency.³

As a final introductory comment, our tax optimality index and the traditional equivalent and compensating variation concepts provide alternative ways of measuring welfare differences between different taxation equilibria. The variation concepts express welfare differences in dollar terms by measuring the consumer costs of attaining different utility levels. Our TOI measures

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¹Basically, while the numerator (say an equivalent variation measure of the excess burden of taxation) is in terms of the undistorted prices, the denominator (some economic activity measure) is in terms of the distorted prices, and thus any proportional change of prices will not cancel out when price distortions are specific. A way to surpass this problem will be to normalize by a measure of economic activity that is based on undistorted prices. This type of measure, however, is not readily available from official statistical sources.

²Kay and Keen (1988) use the Debreu (1951) coefficient of resource utilization in order to derive a measure of tax inefficiency. In that sense, our measure is an alternative and, in our opinion, more intuitive measure than their’s.

³It is worth emphasizing here that the main assumption needed for any method of measuring the efficiency of taxes is the existence of some social utility function. This permits the derivation of a first best (with lump-sum taxes) or a second best (with distortionary taxation) taxation policy, which is then used for calculating either the excess burden of taxation or our tax optimality index.
the welfare differences in terms of tax rates relative to the optimal tax rates and expresses the results in terms of an index. The two sets of concepts have different interpretations as a consequence. The equivalent and compensating variation measures have a natural interpretation in terms of expenditures needed to achieve utility outcomes. Our TOI has a more appealing interpretation from the point of view of the taxation authority, since it measures how large tax rates can be set, relative to optimal taxes, to achieve different welfare outcomes.

In addition to the TOI, we also propose a Revenue Equivalent Uniform Tax (REUT) measure to address an issue frequently considered in public economics, viz. the uniform tax rate that keeps the provision of public goods unchanged. This corresponds to a reform of taxes that changes the non-uniform initial tax structure to a uniform one that yields the same amount of tax revenues and thus the same provision of public goods. We argue that the resulting uniform tax rate is a convenient and easily interpreted measure of the tax burden implied by the public sector containing none of the index number problems of existing measures.

The rest of the paper proceeds as follows. In the following section, a model of a small open economy with public good provision and linear distortionary taxation is presented. This model comprises production, household, government and foreign sectors and equilibrium is expressed in terms of the private sector and public sector budget constraints. Section 3 uses this model of the economy to construct the definition of the tax optimality index, whose properties and interpretation are discussed. The derivation of the revenue-equivalent average tax is presented in section 4. To illustrate the techniques used and the nature of our tax optimality index and our revenue-equivalent average tax, a numerical example is presented in section 5. In section 6, we extend the theoretical model to allow for a household sector comprising many households and to allow for nonlinear taxation structures. It is demonstrated that our previous analysis can be readily extended to allow the construction of a tax optimality index and a revenue-equivalent average tax in these more complex situations. Finally, section 7 concludes the paper with further discussion of the implications of our analysis.

2. A Small Open Economy with Public Good Production

Consider a small open economy that faces fixed world prices on goods that it trades with the rest of the world. Assume that the number of traded goods is $1 + M$, with $M$ being the number of non-numeraire goods. The world prices of these non-numeraire goods are denoted by the
The government raises revenues by taxing consumption. Denote \( t \) as the \((M \times 1)\) vector of ad-valorem taxes, which create a wedge between world and producer prices \( p \) and consumer prices \( q = (1 + t) \cdot p \).

The tax revenues finance the production of a public good \( g \) that is returned to consumers free of charge.

We assume the existence of a representative agent that achieves utility level \( u \) by consuming private and public goods and raising income through its (fixed) factor supply. The consumer decisions are characterized by the expenditure function \( e(q, g, u) \), which denotes the minimum expenditure needed to achieve utility level \( u \), given consumer prices \( q \) and a level \( g \) of the public good. Standard properties of this function (see Dixit and Norman, 1980, Woodland, 1982, and Cornes, 1992) imply that \( e_q \equiv \partial e/\partial q \) is the vector of compensated demand functions for the private goods, \( -e_g \) is the marginal willingness to pay for the public good and \( e_u \) is the inverse of the marginal utility of income.

Let the restricted revenue function \( r(p, g) \) denote the value of total income generated in the private sector given producer prices \( p \) and the level of the provision of the public good \( g \). The gradient of this function with respect to producer prices \( (r_p \equiv \partial r/\partial p) \) gives the vector of domestic supplies of private goods and \( r_g \) is the supply shadow price of the public good \((-r_g > 0 \text{ is the unit cost of producing the public good}) \). It is assumed that the production technology exhibits constant returns to scale, implying that \( r_{gg} \equiv \partial^2 r/\partial g^2 = 0 \).

As is well known, the difference \( e_q - r_p \) is the vector of utility-compensated excess demands. Moreover, the difference \((-e_g) - (-r_g)\) denotes the wedge between the marginal willingness to pay for the public good \((-e_g > 0\) and the marginal cost of producing it \((-r_g > 0\). Clearly,

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4 We use the first of the traded goods as a numeraire, its price being normalised to unity. In addition, it is assumed, without loss of generality, that this good is not taxed. Unless otherwise mentioned, all vectors are taken to be column vectors and the transpose of a vector \( x \) is denoted by \( x' \).

5 Extending the model to include both income taxes and consumption taxes is straightforward. Similarly, taxes and subsidies on international trade can easily be incorporated into the model. In principle, we could extend the model to include all feasible tax bases.

6 Specific taxes could easily be incorporated without any influence on the methodology that we use.

7 The notation “\( \cdot \)” denotes the horizontal product of two vectors; if \( z = x \cdot y \) then \( z_i = x_i y_i \). In the expression for \( q \), 1 is a vector of ones.

8 The unit price of the numeraire good is also an argument of the expenditure function, but it is suppressed for simplicity.

9 The term \( e_q \) represents the reduction in expenditure on the private goods as a result of an extra unit consumption of the public good, holding utility level constant. In that sense, \( e_q \) is the shadow demand price of the public good and \(-e_q(>0)\) is then the marginal willingness to pay for the public good.

10 As with the expenditure function, the unit price of the numeraire good is suppressed for simplicity. In addition, we suppress the (fixed) factor endowments vector as an argument of the revenue function.
if this wedge is positive (negative) an extra unit of the public good will increase (decrease) welfare.\textsuperscript{11}

We can now describe the equilibrium of our economy by the following two equations:

\[ e(q, g, u) = r(p, g) - r_g(p, g)g \]  
\[ (t \cdot p)' e_q(q, g, u) = -r_g(p, g)g. \]  

The first equation is the private sector’s budget constraint expressed in domestic prices. It demands that the money consumers spent on private goods comes from the income generated by working both in the private sector and in the public sector (\(-r_gg\) is the total cost of producing \(g\) units of the public good, which, under the assumption of constant returns to scale, is equal to total income generated in that sector). The second equation is the public sector budget constraint, equating tax revenues and total public sector costs. Given world prices \(p\) and the tax vector \(t\), these two equations simultaneously determine the level of utility \(u\) and the level of provision of public goods \(g\).

The public sector budget constraint (2) may be solved for the quantity of the public good \(g\) as a function of \(t\) and \(u\), i.e., \(g = g(t, u)\). Substituting this solution into equation (1) and re-writing the private sector budget constraint as the balance of trade function \(B(t, u)\), we obtain:

\[ B(t, u) \equiv e((1 + t) \cdot p, g(t, u), u) - r(p, g(t, u)) + r_g(p, g(t, u))g(t, u) = 0. \]  

Equation (3) represents the general equilibrium budget constraint for the economy, making sure that the public good market is in equilibrium and that consumers and the government cannot spend more money than they earn. It is expressed in terms of the tax rates and the level of utility. The utility level that satisfies equation (3) is given by the indirect utility function expressed as \(u = U(t)\).

3. The Tax Optimality Index

Having expressed the economy’s equilibrium with a single compact equation (3), we now construct a measure of the efficiency of the tax structure in the presence of public goods.

Suppose that we observe a country with tax rates and utility given by \((t^1, u^1)\) and a level of

\textsuperscript{11}The condition \(-e_g = -r_g\), or \(MRS = MRT\), is the so-called Samuelson rule for optimal public good provision in a closed economy without distortionary taxation. This rule does not apply here as we consider a small open economy with distortionary taxation.
public good provision given by \( g^1 = g(t^1, u^1) \). Let \((t^0, u^0)\) be the welfare maximizing choice of taxes and corresponding utility level and let the corresponding optimal public good provision be \( g^0 = g(t^0, u^0) \).\(^{12}\) This welfare optimal solution is obtained by maximizing the indirect utility function \( U(t) \) with respect to the tax vector \( t \), yielding solution \( t^0 \). The objective is to obtain a measure of how well the observed tax-public good situation compares with the optimal tax-public good solution.

With these preliminaries in hand, we can now define the Tax Optimality Index (TOI) as the distance function

\[
T(t^0, u^1) \equiv \min \{ \delta : B(\delta t^0, u^1) = 0, \delta > 0 \}.
\]

The solution \( \delta \) to this problem determines a tax vector \( \delta t^0 \equiv \delta t^0 \) that yields the reference utility level \( u^1 \). This new tax vector has the property that it is a contraction of the optimal tax vector \( t^0 \) towards the origin and thus lies on the ray from the origin to \( t^0 \) in tax space. The solution \( \delta \) to the minimization problem in (4) is the scaling factor by which the optimal tax vector is contracted. Thus, the Tax Optimality Index \( T(t^0, u^1) \) is the proportion of the optimal tax vector that achieves the same level of welfare \( u^1 \) as achieved by the observed tax vector \( t^1 \).\(^{13,14}\)

If the economy is already at the welfare optimum, then \( t^1 = t^0 \) and \( u^1 = u^0 \) and so the index takes a value of unity. If the country is not at the welfare optimum, its level of utility is \( u^1 < u^0 \) and so a proportionally smaller vector of tax rates than \( t^0 \) will allow the country to maintain its level of utility \( u^1 \). Our index will therefore be less than unity. In an extreme case, the observed tax vector is \( t^1 = 0 \) and there is a zero provision of public goods \( g^1 = 0 \). In this case, the solution to the above problem is \( \delta = 0 \) and so the tax optimality index takes the value of zero. Thus, in short, the tax optimality index \( T(t^0, u^1) \) ranges from zero to unity. Zero indicates that there is, effectively, no public sector in the observed situation. Unity indicates that the government’s choice of taxation rates (and hence the provision of public goods) is optimal. In between, a higher index indicates greater proximity to the optimum. We can refer to this index, therefore,

\(^{12}\) We should clarify here that by optimality we mean a constrained optimality where lump sum taxes do not exist.

\(^{13}\) This definition of the TOI is expressed as a distance function in tax space. Anderson and Neary (1996) use a distance function in commodity price space to derive measures of the trade restrictiveness of tariffs, but their measure can also be expressed in tax space. Chau et.al (2003) use distance functions in quantity space to derive measures of economic inefficiency of tariffs. These authors provide also a comparison between their efficiency measure and the coefficient of resource utilisation (Debreu, 1951), the open economy index of deadweight loss (Diewert, 1985), and the well-known equivalent variation measure.

\(^{14}\) There will also exist another set of taxes that is higher than \( t^0 \) that can maintain the same level of utility. The index then will be greater than unity, as optimal taxes will have to inflated. The distance function definition considers only the lower than \( t_0 \) taxes (hence the min in equation (4)).
Figure 1: The Tax Optimality Index

as a *Tax Optimality Index (TOI)*.

The index is illustrated in Figure 1. The figure depicts iso-utility contours in a three (two taxable) good small open economy with public good provision. Since we assume without loss of generality that one good (the numeraire) is not taxed, the index may be illustrated in the two-dimensional tax space \( (t_2, t_3) \). Point \( W \) represents the optimal tax situation where non-zero taxes finance the production of public goods, while point \( A \) is the assumed current tax situation for the economy. The contours (indifference curves) show the sets of tax rates that yield various levels of utility, point \( W \) being on the highest feasible indifference curve.\(^{15}\)

A proportional contraction of the optimal taxes given by \( W \) yield tax vectors on the ray passing through the origin and \( W \). In particular, one such deflation of the optimal taxes takes us to point \( B \) in Figure 1, which produces the same level of utility as at the current tax equilibrium given by point \( A \). The tax optimality index for the current tax equilibrium is therefore given by the ratio \( TOI = OB/OW \). This tax vector \( B \), as defined above, is (a) a uniform contraction of the optimal tax vector and (b) yields the same level of utility as the initial tax situation. It should be noted that the level of provision of the public good will generally be different at points \( W, A \) and \( B \), but these differences have no bearing on the tax optimality index \( TOI \), which

\(^{15}\)Any well-behaved quasi-concave utility function will produce such indifference curves, indicating that the costs of taxes are balanced by the gains of having a public good. Thus, there will be case where for a level of, say, \( t_2 \) there will exist two levels of \( t_3 \) (a high and a low) that will give the same utility level.
focuses on welfare alone.\textsuperscript{16} The TOI index in based on welfare comparisons; points $B$ and $A$ are welfare equivalent and yield lower welfare than point $W$.

If the initial tax situation is given by point $C$, the same point $B$ is obtained and so the index takes the same value as for situation $A$. This is as it should be; even though they may produce different levels of the public good, both have the same utility and so they are equal in a welfare sense. At initial situation $D$, on the other hand, the level of utility is lower than at $A$ and $C$ and so the tax optimality index will also be lower. Tax configuration $D$ is further from the welfare optimum $W$ than are $A$ and $C$ in the sense that it is on a lower indifference curve. Its tax optimality index is given by $OF/OW$, which is lower than $A$’s index.

In summary, our TOI measures the distance of any initial tax vector from the optimal tax vector, distance being measured along the ray $OW$. This distance, relative to the distance $OW$, accurately ranks initial tax situations according to their levels of utility relative to the optimal utility point $W$. Thus, we measure the true welfare cost of taxes as the distance from the optimal tax structure.\textsuperscript{17}

Note that there can be two scalars that will do the job: if the current tariff structure is described by point $A$, the optimal tariff rates can be both proportionally increased and decreased to get the same utility level (points $E$ and $B$, respectively). Our convention, enshrined in the definition of the TOI as a min, is to consider only points on the $OW$ line (that is, only point $B$ is considered) and then measure the TOI by the ratio $OB/OW$.

The distance $BW$ measures (in tax space) the loss of welfare associated with initial point $A$ compared to the best attainable welfare at $W$. The ratio $BW/OW$ can therefore be interpreted as a Tax Inefficiency Index (TII). Of course, the two indices are related by the equation $TII = 1 - TOI$.

Some of the properties of the TOI are apparent from the above discussion. The main properties are brought together as follows:

1. The TOI has the range $[0, 1]$. If an economy has optimal choices of both public goods

\textsuperscript{16}It is possible, of course, for the level of provision of public goods to be the same at points $A$ and $W$. The level of utility at $A$ will be lower than at $W$ by choosing a sub-optimal tax vector to finance the public sector.

\textsuperscript{17}As we have mentioned, the techniques we use are based on the work of Anderson and Neary (1996), whose method measures the welfare-preserving uniform tax when optimal taxes are zero. However, when optimal taxes are not zero, such a measure may not exist. To see this, consider Figure 1: clearly the 45 line (which depicts uniform taxes) may not intersect with the iso-utility contour corresponding to a current tax configuration. The only welfare-preserving tax that we can always find, is the tax that lies on the line that connects the origin with the optimal tax configuration. The position of this welfare-preserving tax on this line is exactly what our Tax Optimality Index measures.
Tax Optimality Index

provision and commodity tax rates, then $TOI = 1$. If an economy has sub-optimal choices of either public goods provision or commodity tax rates, then $TOI < 1$.

2. The $TOI$ is monotonically increasing in utility. Thus, higher values for $TOI$ indicate higher welfare; same values for $TOI$ indicate the same level of welfare.

3. The $TOI$ is homogeneous of degree zero in world prices, and hence independent of the choice of numeraire.

The first two properties are easily proved and follow from the definition of the TOI. The final property follows from the homogeneity properties of the revenue and expenditure functions.\(^\text{18}\)

These properties make the TOI particularly appropriate for measuring the optimality or, conversely, inefficiency of tax structures and for undertaking international or inter-temporal comparisons. Property 2 states that the TOI and the level of welfare are positively and uniquely related. Thus, in comparing various alternative tax/public good situations for a given economy the TOI is a perfectly accurate measure of welfare.\(^\text{19}\) For example, index calculations of $TOI^1 = 0.9$ and $TOI^2 = 0.8$ indicate that situation 1 is more efficient than situation 2 and that $u^1 > u^2$.

Because the tax optimality index is homogeneous of degree zero in world prices (Property 3) and so is a "pure number", it can be used for comparisons between different countries and/or different time periods. Different countries may have different preferences and technologies and, thus, different optimal taxes and different optimal levels of public good provision. Nevertheless, comparisons on the basis of the TOI are valid, as the TOI measures the distance of the current tax structure from its optimal tax configuration. Thus, the TOI can be used to rank countries in terms of distance from optimality. Such rankings of the TOI generally cannot be translated into welfare rankings, of course (note footnote 2 above and its corresponding text).\(^\text{20}\) Rather, the international rankings of TOI values are rankings of how well each country’s taxation policy performs relative to its own optimal tax-public good policy equilibrium.

\(^\text{18}\) The homogeneity of the $TOI$ in prices follows from the homogeneity properties of the expenditure and revenue functions. If all world prices (including the numeraire’s price) are doubled, then expenditure and revenue double (via functions $e$ and $r$). In addition, the shadow supply price $r_s(p, g)$ doubles as does tax revenues. Thus, the solution for the real variables $u$ and $g$ in equations (1) and (2) remain unchanged as a result of the world price inflation. Hence the balance of trade function $B(t, u)$ is homogeneous of degree zero in prices and this implies that the $TOI$ is also homogeneous of degree zero in prices.

\(^\text{19}\) Of course, we are dealing with the special case of a single household economy here. Further below (in section 6) we show how to handle many households.

\(^\text{20}\) On the other hand, if two countries have the same technologies, preferences and endowments then a comparison of their TOIs will enable an accurate welfare comparison. A similar remark applies to a comparison of the same economy in different time periods.
When only small tax changes have occurred between time periods, we can use calculus to uncover some properties of the TOI. Totally differentiating (4) around the initial equilibrium, holding $t^0$ fixed, we get that

$$B'_t(\delta t^0, u^1) t^0 d\delta + B_u(\delta t^0, u^1) du = 0.$$  \tag{5}

At the same time, total differentiation of $B(t^1, u^1) = 0$ gives

$$du = -\frac{B'_t(t^1, u^1)}{B_u(t^1, u^1)} dt.$$  \tag{6}

Recalling that $\overline{t^1} = \delta t^0$ and substituting (6) into (5) yields the relationship

$$\frac{d\delta}{\delta} = \frac{B_u(\overline{t^1}, u^1)}{B_u(t^1, u^1)B'_t(t^1, u^1)\overline{t^1}} B'_t(t^1, u^1) dt$$ \tag{7}

where $B'_t(t^1, u^1) dt \equiv \sum_i \partial B(t^1, u^1)/\partial t_i dt_i$.

We can easily see from the above formula that the proportional change of the $\delta$ deflator is related (via a scaling factor) to the weighted sum of the tax changes ($dt$), with the weights being the marginal welfare effects of taxes ($B'_t$) evaluated at the initial equilibrium. The structure of these weights is obtained by totally differentiating (3) to get\(^{21}\)

$$B_u du + B'_t dt = 0,$$ \tag{8}

where

$$B_u = e_u - \frac{e_g}{r_g + (t \cdot p)'e_{qq}}(t \cdot p)'e_{qu},$$ \tag{9a}

$$B'_t = \left[ e'_q - \frac{e_g}{r_g + (t \cdot p)'e_{qq}}(t \cdot p)'e_{qq} \right] p.$$ \tag{9b}

The derivative $B_u$ denotes the welfare gain from a unit increase of the economy’s purchasing power, while $B'_t$ denotes the marginal welfare effects of tax changes. Since the balance of trade

\(^{21}\)Differentiating (1) and (2) gives:

$$e_u du = -e'_q dq - e_g dg$$

$$-(r_g + (t \cdot p)'e_{qq}) dg = (t \cdot p)'e_{qq} dq + (t \cdot p)'e_{qu} du.$$  

Substituting the second equation into the first, and noting that $dq = pdt$, immediately gives (8), (9a) and (9b).
equation can be interpreted as determining the foreign exchange needed (zero in our case) to sustain utility $u$ given world prices $p$, taxes $t$, and public good provision $g$, the change in $B$ following a change in taxes gives the money metric measure of the resulting welfare effect. Clearly, and due to the existence of public goods, $B'_t$ is not always negative and, thus, an optimal level of taxes can be derived by setting $B'_t = 0$.

The first term on the right hand side of the $B'_t$ term in equation (9b) gives the "snapshot" effect of a tax change, ignoring the general equilibrium effects incorporated in the remaining term. If only this "snapshot" effect were to be taken into account, the expression for (7) would be the approximation $d\delta/\delta = -c'dt/c't$, where $c = e_q$ is the consumption vector at the initial equilibrium. This expression weights the marginal changes in the tax rates by the consumption vector and measures the (snapshot) percentage change in consumption tax revenue. However, while this is easy to compute and "intuitive", it ignores important general equilibrium effects. The appropriate marginal index is given by (7) and (9a-9b). This is the marginal version of our TOI.

The TOI, as defined above, compares the existing tax situation with the optimal tax situation. The optimal tax situation is one in which the economy chooses its consumption tax vector and the level of public good provision to maximize utility. A variation on this tax optimality index may be defined for the situation where the level of public goods provision is not endogenously determined. Consider a tax structure that maximizes utility subject to the requirement that the level of public good provision is equal to an exogenously determined public revenue requirement. In this case, the constrained optimal tax structure maximizes utility $u$ subject to the private budget constraint (1) in which $g = g^1$ is given, that is, subject to

$$B(t, u, g^1) \equiv E((1 + t) \cdot p, g^1, u) + r_g(p, g^1)g^1 = 0. \tag{10}$$

Call the solution for the tax vector $t^0$. The constrained tax optimality index (CTOI) can then be defined, analogously to the TOI as

$$T(t^0, u^1, g^1) \equiv \min \left\{ \delta : B(\delta t^0, u^1, g^1) = 0, \; \delta > 0 \right\}. \tag{11}$$

This index measures the efficiency or optimality of the existing tax structure relative to the optimal tax structure that achieves the same public good provision outcome as the initial equilibrium. If the initial taxes are optimal for this purpose then the index is unity; if sub-
optimal for this purpose, the index is less than one. This index, therefore, does not assume optimality of the public good provision choice.\footnote{Another type of constrained tax optimality index may be readily constructed. Specifically, we could extend the model to require that a subset of tax rates are fixed at their initial levels and the optimization required to construct the tax optimality index is then undertaken over the non-fixed tax rates. By using such a restricted tax optimality index, we could separately determine the optimality of different classes of taxes, e.g., consumption taxes, trade taxes, production taxes and income taxes.}

4. The Revenue-Equivalent Uniform Tax

We now make a different application of the techniques used in the previous section. We will derive a uniform tax that holds the provision of public goods, and thus the tax revenue, constant.\footnote{Anderson and Neary (2003) make a similar application in which they keep fixed the initial trade vector.}

For this, we use the private sector’s budget constraint (1) to solve for utility $u$ as a function of $t$ and $g$, i.e., $u = u(t, g)$, and substitute the solution into the public sector’s budget constraint (2). Rewriting the public sector budget constraint, we have:

$$\Pi(t, g) \equiv (t \cdot p)' e_p((1 + t)p, g, u(t, g)) + r_g(p, g)g = 0. \quad (12)$$

Equation (12) represents the general equilibrium public sector budget constraint for the economy. Denoting by $t^1$ and $g^1$ the current level of taxes and public good provision, we define the revenue equivalent uniform tax $T_g$ as

$$T_g(t^1, g^1) \equiv \{T_g : \Pi(1 \cdot T_g, g^1) = \Pi(t^1, g^1) = 0, \quad (13)$$

where $T_g$ is a scalar and 1 is the unit vector.

According to this definition, $T_g$ is the uniform tax rate that will yield the reference level of public good provision ($g^1$) for the economy. Since this uniform tax rate satisfies the general equilibrium public sector budget constraint (12), both the public and private sector budget constraints are satisfied.

Figure 2 depicts this revenue-equivalent uniform tax (REUT) rate. In the figure, point $A$ is the initial tax rate configuration ($t^1$), while $W$ is the optimal vector of taxes. The contours concentric to point $W$ represent different levels of utility. The locus of points given by the solid curve through point $A$ defines the set of taxes that can support the public good provision ($g^1$) given at the initial situation. The shape of this locus is determined by the general equilibrium public sector budget constraint, i.e., equation (12), but it will be downward sloping. This
constant-$g$ contour cuts the 45 degree line at point $A'$. Hence, the uniform tax ($T_g$) that reproduces the same level of the public good ($g^1$) is then given by point $A'$. Clearly, imposing this uniform tax rate may increase or decrease welfare (as Figure 2 is drawn, welfare falls).

We can now use this uniform tax as a measure of the tax burden implied by the public sector in different time periods within a country or between different countries. Even if the shape of the constant-public-good locus is not the same for all countries, all loci will cut the 45 degree line thus allowing for a comparison. As an example, consider a second country’s tax configuration at point $B$, where both taxes are higher than the tax configuration at point $A$, but the tax revenues is exactly the same as that of country $A$. However, given that the second country may be different in many dimensions from the first country, its constant-public-good locus is depicted by the stippled curve passing through points $B$ and $B'$.

The second country’s revenue equivalent uniform tax vector is then $B'$, which is smaller than $A'$, indicating that the second country $B$ has a lower tax burden than country $A$. What makes the revenue equivalent uniform tariff the correct tool for comparing the two countries is, of course, the fact that it takes into account the general equilibrium effects of changing taxes and is independent of the choice of numeraire.

To put our revenue equivalent uniform tax measure into perspective, we briefly discuss the existing measure of tax burden used in the literature, viz. the average effective tax. It is defined as the ratio of tax revenues over tax base. For the case of, say, consumption taxes, the
average effective consumption tax \( (t_c) \) is defined as 
\[
t_c = \sum_i \frac{t_i p_i c_i}{C},
\]
where \( c_i \) is the consumption of good \( i \), \( t_i \) is the consumption tax rate and 
\[
C = \sum_i (1 + t_i) p_i c_i
\]
is total value of consumption at consumer prices. Thus, the resulting scalar is constructed by using current tax rates \( (t_i) \) weighted by current activity information \( (c_i) \). Clearly, its popularity is its simplicity: reference to the national statistics of a given country provides all the information needed for its calculation.\(^{24}\)

However, the above method is theoretically problematic. The use of current activity information is the culprit of a classic index-number problem: the activity that is highest taxed weighs less in the index! As the tax on a particular good rises, the weight put on that good (here, its consumption level) falls. At prohibitively high levels of taxation the weight is zero and thus the constructed index underestimates the true tax burden in the economy. These are typical index number problems that could be addressed by applying appropriate index number techniques.\(^{25}\)

Clearly, these types of problems do not exist in the calculation of our revenue-equivalent uniform tax as it uses correct general equilibrium weights.

5. Calculating the TOI: A Numerical Example

In order to illustrate the use of the TOI, we present a numerical example. The model we use has four commodities - three private and one public good. As in our theoretical model, the economy is a small open economy with the prices of the private goods being given by world market conditions. Commodity 1 is taken to be the numeraire.

5.1. Model Specification. Rather than specify a functional form for the revenue function \( r(p, g) \) directly, it is convenient to use an indirect approach that makes it simpler to obtain a functional form consistent with economic theory. Accordingly, the production side of the economy is described by the revenue function \( R(p, \rho) \), where \( \rho \) is the ‘shadow price’ for the public good. Setting \( g = \partial R(p, \rho)/\partial \rho \), the supply function for the public good, we can solve for the shadow supply price for the public good \( \rho^S \), which can then be eliminated from the

\(^{24}\) Consequently, the literature on measuring the average effective tax has concentrated on the details of what should be included in the numerator and denominator of the above expression. A recent overview can be found in Sørensen, 2004.

\(^{25}\) The problem in its basic form is the same as the one encountered in the price index literature concerning the relation between Paasche, Laspeyres, and Konüs Indexes, viz. \( P_L > P_K > P_P \) (see Dievert, 1981). While the former two have the advantage of using easily available information, it is only the latter that truly expresses the true cost of living. This latter index, however, presupposes knowledge of the utility function and as such it is difficult to use. The solution is to calculate the Fisher ideal price index, which is a combination of the Paasche and Laspeyres indexes \( P_P = [P_L P_P]^{0.5} \), and which can closely approximate the Konüs index. Such a development could also be done in the average effective tax literature, where the current measure underestimates the true tax burden. A similar procedure, to a different issue, is also argued by Cornes (1996).
revenue function to get the restricted revenue function that we define in section 2, i.e., \( r(p, g) = R(p, \rho^S(p_1, p_2, p_3, g)) \). We do not have to perform this solution and substitution analytically as it can be done numerically in the example.

Similarly, we use an indirect approach to specify the functional form for the expenditure function \( e(q_1, q_2, q_3, g, u) \). We specify a functional form for the expenditure function \( E(q, \rho, u) \). Setting \( g = \partial E(p, \rho, v) / \partial \rho \), the demand function for the public good, we can solve for the shadow demand price for the public good \( \rho^D \), which can then be eliminated from the expenditure function to get the expenditure that we define in section 2, i.e., \( e(q, g, u) = E(q, \rho^D(p, g, u), u) \). Again, we do not have to perform this solution and substitution analytically as a numerical solution suffices.

The functional forms chosen for \( R \) and \( E \), along with the implied supply and demand functions and parameter values, are provided in the Appendix. Given the parameter choices, we set the world prices of the private goods \( (p) \) so that the country exports the numeraire good (good 1) and imports the other two goods (goods 2 and 3) (although the trade pattern does not matter here). Specifically, \( p = (1 \ 0.7 \ 0.5)' \) and the endowment is normalized to unity, i.e., \( \nu = 1 \).

5.2. **Numerical Results.** Table 1 below summarizes the results of the equilibrium for the open economy at the initial or reference tax vector \( t_1 = [0, 0.16, 0.21]' \) and at the optimal tax vector \( t_0 \). In the table, \( t_i, i = 2, 3 \) are the ad valorem consumption taxes on the non-numeraire goods, \( c \) is the consumption vector, \( g \) is the quantity of public production and \( TB^i \) denotes the traditional tax burden indexes.

<table>
<thead>
<tr>
<th></th>
<th>Reference equilibrium</th>
<th>Optimal equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_2 )</td>
<td>0.16</td>
<td>0.2290</td>
</tr>
<tr>
<td>( t_3 )</td>
<td>0.21</td>
<td>0.2080</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>0.4161</td>
<td>0.4176</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>0.5124</td>
<td>0.4854</td>
</tr>
<tr>
<td>( c_3 )</td>
<td>0.5377</td>
<td>0.5414</td>
</tr>
<tr>
<td>( g )</td>
<td>0.1116</td>
<td>0.1265</td>
</tr>
<tr>
<td>( u )</td>
<td>1.0601</td>
<td>1.0609</td>
</tr>
<tr>
<td>( TB )</td>
<td>0.1814</td>
<td>0.2197</td>
</tr>
</tbody>
</table>

\[ TB^1/TB^0 = 0.8258 \]

\[ REUT = 0.1812 \]

\[ TOI = 0.7972 \]
The initial equilibrium has the production sector producing a vector \( y^1 = (0.5998, 0.4198, 0.2999)' \) of private goods giving a revenue of \( r = 1.0436 \). The consumer faces the price vector \( (1.0, 0.812, 0.605)' \) and consumes the vector \( c^1 = (0.4161, 0.5124, 0.5377) \), spending income of 1.1574 and attaining a utility level of 1.0601. The economy exports good 1 and imports the other two private goods, the balance of trade being zero. The indirect tax revenue accruing to the government is 0.1138 enabling it to produce 0.1116 units of the public good, the shadow supply price being 1.0198. In this initial equilibrium, neither the tax rates nor the level of production of the public good are optimal.

At the reference equilibrium, the tax rate on good 2 is significantly lower than the tax rate on good 3. The optimal tax solution reverses this divergence of tax rates and, due to the asymmetries in the technology and preferences between private goods, the optimal tax rates are different and are given by \( t_{02}^0 = 0.2290 \) and \( t_{03}^0 = 0.2080 \). The consumer responds by reducing consumption of the good taxed at a higher rate (good 3) and increasing consumption of the other two private goods (good 2’s tax rate being lowered) compared to the reference situation. The optimal solution calls for a higher level of public good provision \( (g^0 = 0.1265) \) than at the reference equilibrium, and hence a greater tax revenue (in terms of the numeraire) \( (0.1341) \) is required. Of course, the level of utility is higher at the optimal solution \( (u^0 = 1.0609) \).

Calculating the Tax Optimality Index we find that \( TOI = 0.7972 \). Thus, we can achieve the reference utility \( u^1 \) by using tax rates that are precisely 0.7972 times the optimal taxes \( t^0 \). Thus, we say that the reference point taxes \( t^1 \) are 79.7% efficient. In other words, a 20.3% proportionate reduction in the optimal tariffs would achieve the reference utility level \( u^1 \).

Figure 3 provides an illustration of the \( TOI \) in tax space. The figure shows the reference tax point \( t^1 \), the optimal tax point \( t^0 \) and the constructed tax point \( \tilde{t}^1 = TOI \cdot t^0 = (0, 0.1826, 0.1658)' \). By construction, this tax point lies on the ray through the optimal tax point.

Figure 4 illustrates the REUT rate for our numerical example. The figure shows the tax revenue contours in tax space along with the reference tax point, the optimal tax point and the revenue-equivalent uniform tax vector. The reference tax rates are \( t_{12}^1 = 0.16 \) and \( t_{13}^1 = 0.21 \) and public good provision is \( g^1 = 0.1165 \). The same public good provision may be attained using a uniform tax rate of \( T_g = 0.1812 \). This provides a readily interpreted measure of the tax burden implied by the public sector: the public sector corresponds to an 18.1% uniform tax rate.

It is useful to compare our REUT calculations in this numerical example with traditional
Figure 3: The Tax Optimality Index: Numerical Example

Figure 4: Revenue Equivalent Uniform Tax: Numerical Example (labels rounded to two decimal places)
public finance measures used in the literature. Calculating the average effective trade tax at the reference equilibrium as the ratio of tax revenues over tax base, gives us a $TB^1 = 0.1814$. That is not exactly the same as our REUT, indicating that the two methods are different.\footnote{As is well known, the effective average tax has no normative value. Still, we could induce a normative assessment if we were: (i) calculating the average effective tax at the optimal equilibrium $TB^0$, and (ii) comparing $TB^1$ and $TB^0$. Step (i) gives us a $TB^0 = 0.2197$, implying a tax burden of 21.97%. Step (ii) reveals that the average reference trade tax does not differ substantially from the optimal average effective tax ($TB^1/TB^0 = 0.8258$). However, the tax burden indices were not designed to measure the optimality of the tax structure. Our TOI is designed for measuring optimality and, as indicated above, gives a result $TOI = 0.7972$, indicating a somewhat bigger tax inefficiency than the one derived using relative tax burden measures.}

6. Extensions

While the development of the tax optimality index above was undertaken in the context of a model with just one representative household, the index may be readily applied in the more general context of a many-household economy in which the government has a well defined social welfare function. The development above also assumes, as is common in the literature, that taxes are linear. However, it is possible to extend our analysis to deal with nonlinear tax structures. We now outline how the extensions of our method to these more general contexts may be undertaken.

6.1. Many Households. Let the consumption sector of the economy be inhabited by a set $H$ of households with expenditure functions $e^h(q, g, u^h)$ for $h \in H$. These expenditure functions may be summed to obtain the aggregate expenditure function $e(q, g, u) \equiv \sum_{h \in H} e^h(q, g, u^h)$, where $u$ is now a vector of utility levels. Households have income functions $\rho^h(p, g)$ that satisfy the identity $\sum_{h \in H} \rho^h(p, g) \equiv r(p, g) - r_g(p, g)g$, which simply requires household incomes to sum to the aggregate income generated by the private and public sector. Finally, the government is assumed to have a social welfare function $W = W(u)$ defined over the vector of household utilities.

Equations (5) and (6) are now replaced by

$$e^h(q, g, u^h) = \rho^h(p, g), \ h \in H,$$

$$e_{q}(q, g, u) = -r_g(p, g)g.$$  \hspace{1cm} (14)  \hspace{1cm} (15)

Proceeding as previously, we can solve (15) for $g = g(t, u)$ and substitute into (14) to obtain

$$B^h(t, u) \equiv e^h((1 + t) \cdot p, g(t, u), u^h) - \rho^h(p, g(t, u)) = 0, \ h \in H,$$

$$= 19$$
which may be expressed as the vector equation \( B(t, u) = 0 \). This set of equations determines the utility vector as a function of the tax rates, i.e., \( u = U(t) \).

The initial tax vector \( t^1 \) generates the utility vector \( u^1 \) and social welfare level \( W^1 = W(U(t^1)) \). The optimal tax vector \( t^0 \) is obtained by maximizing \( W(U(t)) \) and generates utility vector \( u^0 \) and social welfare level \( W^1 = W(U(t^0)) \). Accordingly, the Tax Optimality Index may now be defined by the distance function

\[
T(t^0, W^1) \equiv \min \{ \delta : W(U(\delta t^0)) = W^1, \ \delta > 0 \}.
\] (17)

Figure 1 continues to apply if we now interpret the indifference curves as applying to the social welfare function \( W(U(t)) \).

Thus, our tax optimality index may be readily applied in the case where the economy comprises many households and the government has a social welfare function that may be used to provide a scalar measure of the social benefit of tax policy.

### 6.2. Nonlinear Tax Structures.

To extend our model to handle nonlinear tax structures, let the ad valorem tax rates be given by the ad valorem tax function or schedule \( t_i = t_i(\theta_i) \), where \( \theta_i \) is a vector of parameters. The simplest interpretation is that \( t_i(\theta_i) \) is a segmented linear function with the elements of \( \theta_i \) being the marginal tax rates on each segment.\(^{27}\) We simply replace the vector \( t \) in the theoretical section (section 2) by the vector function \( t(\theta) \), where \( \theta \) is the vector of all marginal tax rates. The nation’s budget constraint may now (following the same approach as in section 2) be written as \( B(\theta, u) = 0 \) and the solution for utility as \( U(\theta) \). The initial tax parameter vector \( \theta^1 \) generates the utility vector \( u^1 \). The optimal tax structures are those for which \( U(\theta) \) is maximized with respect to \( \theta \), yielding the optimal tax schedule parameter (marginal tax rates) vector \( \theta^0 \).

The Tax Optimality Index may be now redefined as the distance function

\[
T(\theta^0, u^1) \equiv \min \{ \delta : B(\delta \theta^0, u^1) = 0, \ \delta > 0 \}.
\] (18)

\(^{27}\)In this interpretation, for simplicity of exposition, we subsume the base for the tax and assume that the positions of the rates changes on that base are exogenously given. The most common base would be the level or value of consumption of the product. In the case of labour, for example, a value base (labour income) together with a segmented linear tax schedule models the standard progressive income tax structure used by many countries. Accordingly, our model is readily able to be applied in models involving progressive direct tax structures as well as linear, indirect (consumption, production and trade) taxes.
In this context, the TOI is interpreted as the proportion by which all marginal tax rates (hence all tax schedules) can be reduced to still have the economy yield the initial utility level. This interpretation is in the same spirit as that provided further above in the context of linear taxes. There, the index measures the proportion by which (average = marginal) tax rates can be reduced to obtain the initial utility level; here it measures the proportion by which marginal tax rates can be reduced to obtain the initial utility level.

Thus, our Tax Optimality Index may be used to measure the degree of optimality of nonlinear tax structures. Two points are worthy of note at this juncture. First, while we have exposited the extension of our index to an economy with nonlinear tax structures for simplicity, it should be evident that it could also have been done in a model with many households. Second, while this extension uses segmented linear schedules as the backdrop, more general structures may be considered. For example, we can re-interpret the $\theta$ parameters as parameters of general nonlinear tax structures in which lower values of the parameters imply lower tax schedules.\textsuperscript{28}

7. Concluding Remarks

We have proposed a new measure of tax inefficiency, namely the Tax Optimality Index. Its advantage is its intuitive and informative interpretation: it tells immediately how efficient are the current taxes. For example, a TOI equal to 0.8 implies that current taxes are 80% efficient, or in other words, that a 20% reduction of the optimal taxes will bring welfare down to the current (distorted) welfare level. We argued that the type of information that this index conveys can be used to compare situations between countries and/or time without confronting the normalization problems inherent in the excess burden of taxation measure.

The methodology that we propose in this paper is based on one main assumption. We assume that the economy can be represented by a single social utility function which the government can maximize in deriving the optimal level of taxes, i.e., the standard representative agent interpretation of an economy. Of course, whether this utility function exists when consumers are heterogenous and issues of redistribution enter the government’s preferences, is an open issue that we say nothing about. In that sense, we make the same assumptions as any study that uses the excess burden of taxation measure of tax inefficiency.

We presented our ideas using a simple model of linear consumption taxes that finance the provision of a public good within the context of a single household model. However, as demon-

\textsuperscript{28}However, depending on the nature of even more general tax schedule models, the simplicity of our TOI’s interpretation might become blurred.
Strated in the paper, provided that we continue to assume that there exists a social utility function and, hence, that we can define an optimal tax structure, our method can easily be applied to more complicated environments where taxes are non-linear and there are heterogeneous consumers. Clearly, if the derivation of the single social utility function is precluded, then our method, as well as the excess burden of taxation method, cannot be used. Measuring the distortionary effect of taxes by using Harberger triangles then remains the only possibility.

Finally, while in this paper we have only illustrated our methods by considering a numerical example, our tax optimality index and the revenue-equivalent uniform tax may be computed for real economies. This does require, however, the availability of a computable general equilibrium model for the country in question that is sufficiently well constructed to incorporate public goods and to allow the computation of optimal taxes. That there are many computable general equilibrium models for various countries makes this a feasible task. Accordingly, the appropriate application of our method is to take it to the data and compute TOIs for different countries and time periods. That remains to be done.
Appendix: Specification of Numerical Example

In order to illustrate the use of the TOI, we present a numerical example. The model we use has four commodities - three private and one public good. As in our theoretical model, the economy is a small open economy with the prices of the private goods being given by world market conditions. Commodity 1 is taken to be the numeraire.

Production: The production side of the economy is described by the revenue function $R(p_1, p_2, p_3, p_4)$, where $p_4$ is the shadow supply price for the public good and the price ($p_1$) of the numeraire is now made an explicit argument. There is just one resource endowment in the numerical model, its value is set to unity and so the resource endowment notation is suppressed. The revenue functional form is

$$R(p_1, p_2, p_3, p_4) = \sum_{i=1}^{4} l_i p_i + \left( \sum_{i=1}^{4} b_i(p_i)^2 \right)^{0.5} = l(p) + \tilde{R}(p),$$

where $b = (b_1, b_2, b_3, b_4) > 0$ and $l = (l_1, l_2, l_3, l_4) \leq 0$ are vectors of parameters. This functional form is linearly homogeneous and convex in prices (assumed positive, of course). It is increasing in prices over a cone of prices; if $l = 0$ then it is increasing for all positive prices. The output supply functions are

$$y_i(p_1, p_2, p_3, p_4) = l_i + \frac{b_i p_i}{R(p)}.$$

The parameter vector $l$ is introduced to allow an output to be zero. Specifically, if $l_4 < 0$ then there exists a shadow supply price $p^S_4$ such that a zero output of the public good is feasible.

More generally, setting $g = y_4(p_1, p_2, p_3, p_4)$ we can solve for the shadow supply price for the public good $p^S_4$, which can then be eliminated from the revenue function to get the restricted revenue function that we define in section 2, i.e., $r(p_1, p_2, p_3, g) = R(p_1, p_2, p_3, p^S_4(p_1, p_2, p_3, g))$. We do not have to perform this solution and substitution analytically as it can be done numerically in the example.

The reason for specifying the functional form for the revenue function in this indirect way is that it is easy to specify a functional from for $R$ that satisfies all of the requirement of economic theory, whereas it is more difficult to do so for $r$ itself.

Consumption: The functional form for the expenditure function is

$$E(q_1, q_2, q_3, q_4, u) = \sum_{i=1}^{4} k_i q_i + \left( d \prod_{i=1}^{4} (q_i)^{a_i} \right) u = k(q) + \epsilon(q)u,$$
where \(a = (a_1, a_2, a_3, a_4) > 0\) and \(k = (k_1, k_2, k_3, k_4) \leq 0\) are vectors of parameters, \(d\) is a scalar parameter and \(u > 0\) is a scalar representing utility. The consumer price vector \((q_1, q_2, q_3, q_4)\) now includes \(q_4\) as the shadow demand price for the public good and now makes the price \((q_1)\) of the numeraire an explicit argument. This functional form is linearly homogeneous and concave in domestic prices (assumed positive, of course). It is increasing in prices over a cone of prices; if \(k = 0\) then it is increasing for all positive prices.

The compensated demand functions are

\[
c_i(q_1, q_2, q_3, q_4, u) = k_i + \frac{a_i e(q)}{q_i} u,
\]

which is the linear expenditure system. The parameter vector \(k\) is introduced to allow an output to be zero. Specifically, if \(k_4 < 0\) then there exists a shadow demand price \(q_4\) such that a zero demand for the public good is feasible.

More generally, setting \(g = c_4(q_1, q_2, q_3, q_4, u)\) we can solve for the shadow demand price for the public good \(q_4^D\), which can then be eliminated from the expenditure function to get the expenditure that we define in section 2, i.e., \(e(q_1, q_2, q_3, g, u) = E(q_1, q_2, q_3, q_4^D(p_1, p_2, p_3, g, u), u)\). Again, we do not have to perform this solution and substitution analytically as a numerical solution suffices.

Again, the reason for specifying the functional form for the expenditure function in this indirect way is that it is easy to specify a functional from for \(E\) that satisfies all of the requirement of economic theory, whereas it is more difficult to do so for \(e\) itself.

**Parameter Values:** The parameter values chosen are given in Table 2.

<table>
<thead>
<tr>
<th>Table 2: Parameter values</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Revenue:</strong> (l = \begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; -0.5 \end{bmatrix}^\top), (b = \begin{bmatrix} 1 &amp; 1 &amp; 1 \end{bmatrix}^\top)</td>
</tr>
<tr>
<td><strong>Expenditure:</strong> (k = \begin{bmatrix} 0 &amp; 0 &amp; -0.15 &amp; -0.5 \end{bmatrix}^\top), (a = \begin{bmatrix} 0.2 &amp; 0.2 &amp; 0.2 &amp; 0.4 \end{bmatrix}^\top), (d = 2)</td>
</tr>
<tr>
<td><strong>Other:</strong> (p = \begin{bmatrix} 1 &amp; 0.7 &amp; 0.5 \end{bmatrix}^\top), (v = 1)</td>
</tr>
</tbody>
</table>

As seen from the parameter values, we treat private goods (goods 1-3) symmetrically in our revenue function specification, but asymmetry is introduced into preferences via the choice of the parameter \(k_3\) for private good 3. Good 4, the public good, enters the expenditure function with a larger coefficient \((a_4)\), indicating a higher marginal willingness to pay for the public good than for the private goods. Finally, we normalize the endowment of the single factor \((v)\) to unity.
and we set the world prices of the private goods \((p)\) so that the country exports the numeraire good (good 1) and imports the other two goods (goods 2 and 3) (although the trade pattern does not matter here).
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