Equity and Efficiency: A Method of Finding All Envy-free and Efficient Allocations when Preferences are Single-peaked

Hidekazu Anno
Graduate School of Economics, Waseda University
and

Hiroo Sasaki
School of Commerce, Waseda University

September 30, 2007 (This version: February 9, 2008)

Very preliminary and incomplete.

1 Introduction

Equity is a notion to which economists pay greatest attention as well as *efficiency*. Especially *envy-freeness*, which requires that each agent preferes own consumption bundle to others', has been one of the most important equity concept of the fair allocation of goods since Foley (1967) and Varian (1974) (1976). Varian (1974) (1976) showed that there is a tension between the envy-freeness and the efficiency in pure exchange economies. As shown in Varian (1976) this tension becomes tighter under the environment of pure exchange economies with continum set of consumers ; that is, the only envy-free and efficient competitive equilibrium is the equal income Walrasian allocation. While this result showed that it is difficult to cope with both envy-freeness and efficiency in very general environment, it is natural and interesting to inquire into whether the tight tension still occurs in another environment or not. Sprumont (1991) investigated properties of a *rule* or an allocation mechanism defined on the set of all single-peaked preferences of a single commodity. He proved that, the *uniform rule*, where the same amount of a single divisible good is basically alloted to everyone except people whose peaks are small enough if excess demand exists or large enough if excess supply exists, is the unique rule satisfying envy-freeness, efficiency and strategy-proofness.

Although Sprumont (1991) imposed three conditions on a rule, to fit with our purpose, we focus on the first two condiions, that is, the envy-freeness and the efficiency. In general, it is easy to show that there may be (even uncountably) many envy-free and efficient allocations in Sprumont's (1991) environment, and it is very interesting to study the structure of the set of envy-free and efficient allocations.

Recently Sakai and Wakayama (2007) showed that the set of allocations which is assigned by a rule satisfying peak-onliness [1] and envy-freeness can be linearly ordered by the Pareto dominance relation, and the allocation obtained by the uniform rule is the most efficient allocation among them. Generally speaking, whether an allocation is envy-free with respect to a preference profile depends not only on the position of peaks but also on the overall shape of a preference relations, especially on the shape of indifference curves. Because of the peak-onliness, in Sakai and Wakayama (2007) the set of envy-free allocations obtained by their rule is in a very restricted subset of the set of all envy-free allocations. However, if we remove the assumption of peak-onliness, the set of envy-free allocations becomes much larger and more complicated.

There may be some equity concepts different from envy-freeness in Sprumont's (1991) environment. Schummer and Thomson (1997) investigated the allocation obtained by the uniform rule, and they showed that it has the smallest variance of the distribution of agents' allotments among all of efficient allocations. Sprumont (1991) also showed that uniform rule can be characterized by *anonymity*, which requires that rule doesn't depend on the name of agent, efficiency and strategy-proofness. Ching (1994) showed that in Sprumont's theorem envy-freeness (anonymity) can be replaced by weaker condition *equal treatment of equals*, which requires that if agent i and j have the same preference, then allotments for them are indifferent for them.

The main purpose of this paper is to introduce the family of algorithms with two kinds of parameters that reaches any envy-free and efficient allocations if preferences are single-peaked and there is only one good. In addition, our result can be interpreted as a characterization of the correspondence from the set of preference profiles to the set of allocations which assigns the set of all envy-free and efficient allocations for each profile.[2]

The rest of this paper goes as follows. In section 2 we set notations and some properties of envy-free and efficient allocations in Sprumont's (1991) environment are presented. **Proposition 1** will offer a necessary and sufficient condition for envy-freeness under efficiency. In section 3 we explain our method by using some figures. Section 4 offers the formal definition of our algorithm. To do this, **Proposition 1** will play a very important role. And the main **theorem** which characterizes the envy-free and efficient correspondence is presented in section 4. Section 5 concludes this paper.

2 Setup

2.1 Notations

Let $N = \{1, 2, \ldots, n\}$ be the set of agents. We assume that $\#N \geq 2$ and n is finite. And there is a perfectly devisible good whose amount we can be available is $\Omega \in \mathbb{R}_{++}$. For each $i \in N, x_i \in [0, \Omega]$ denotes the amount of allotment that agent i receives. And if $\mathbf{x} = (x_1, \ldots, x_n) \in [0, \Omega]^N$ satisfies $\sum_{i \in \mathbf{N}} x_i = \Omega$, $\mathbf{x}$ is called *a feasible allocation*, or

[1] This condition requires that rule depends only on the positions of agents' peak. The requirement is implied by efficiency and strategy-proofness (See Lemma 2 in Sprumont (1991)). So if we pay great attention to strategic aspect, then this requirement has much importance.

[2] Formal definition of this correspondence will be given in section 2.

simply *an allocation*. Let $\mathbf{X}$ be the set of allocations.

For each $i \in N$, $R_i \subseteq [0, \Omega] \times [0, \Omega]$ denotes the preference of agent i. We assume that R_i satisfies *completeness* [3], *transitivity* [4] and *continuity* [5]. And $\mathcal{P}(R_i)$ [6] and $\mathcal{I}(R_i)$ [7] denote the strict part and the indifferent part of R_i, respectively. Furthermore we assume that R_i satisfies *single-peakedness*; there exists a point called the *peak* of R_i $x^*(R_i) \in [0, \Omega]$ such that for all $y, z \in [0, \Omega]$

$$\begin{cases} x^*(R_i) < y < z \Rightarrow x^*(R_i)\ \mathcal{P}(R_i)\ y\ \mathcal{P}(R_i)\ z \\ z < y < x^*(R_i) \Rightarrow x^*(R_i)\ \mathcal{P}(R_i)\ y\ \mathcal{P}(R_i)\ z. \end{cases}$$

Let $\mathbb{S}$ stand the set of all complete, transitive, continuous and single-peaked preferences on $[0, \Omega]$. And we call $R = (R_1, \dots, R_n) \in \mathbb{S}^N$ *a preference profile*, or simply *a profile*. For the convenience we set

$$\begin{aligned} \mathcal{S}_d &= \{R \in \mathbb{S}^N \mid \textstyle\sum_{i \in N} x^*(R_i) > \Omega\}, \\ \mathcal{S}_0 &= \{R \in \mathbb{S}^N \mid \textstyle\sum_{i \in N} x^*(R_i) = \Omega\}, \\ \mathcal{S}_s &= \{R \in \mathbb{S}^N \mid \textstyle\sum_{i \in N} x^*(R_i) < \Omega\}. \end{aligned}$$

Sprumont (1991) presented an notion that catches the character of single-peaked preferences well. That is the equivalent of $x \in [0, \Omega]$ with respect to R_i, denoted by $e_{R_i}(x)$, and he expressed it as "closest substitute on the other side of the peak of R_i." It denotes the indifferent point to x in R_i if it exists. Here we give the formal definition of it by following him. Given $R_i \in \mathbb{S}$ and $x \in [0, \Omega]$, $Y_{R_i}(x)$ is the set defined by the following;

$$Y_{R_i}(x) = \begin{cases} \{y \in [0, \Omega] \mid x^*(R_i) \le y\} & \text{if } x < x^*(R_i) \\ \{x^*(R_i)\} & \text{if } x = x^*(R_i) \\ \{y \in [0, \Omega] \mid y \le x^*(R_i)\} & \text{if } x^*(R_i) < x, \end{cases}$$

and $e_{R_i}(x)$ is the point in $[0, \Omega]$ that satisfies following three conditions;

$$\begin{cases} (1) e_{R_i}(x) \in Y_{R_i}(x) \\ (2) e_{R_i}(x)\ R_i\ x \\ (3) \nexists y \in Y_{R_i}(x) \text{ s.t. } e_{R_i}(x)\ \mathcal{P}(R_i)\ y\ \wedge\ y\ R_i\ x. \end{cases} \quad [8]$$

[3] For all $x, y \in [0, \Omega]$, xR_iy or yR_ix.

[4] For all $x, y, z \in [0, \Omega]$, xR_iy and yR_iz imply xR_iz.

[5] For all $y \in [0, \Omega]$, $\{x \in [0, \Omega] \mid xR_iy\}$ and $\{x \in [0, \Omega] \mid yR_ix\}$ are closed sets in $[0, \Omega]$.

[6] $\mathcal{P}(R_i) = \{(x, y) \in [0, \Omega] \times [0, \Omega] \mid xR_iy \text{ and } \neg(yR_ix)\}$.

[7] $\mathcal{I}(R_i) = \{(x, y) \in [0, \Omega] \times [0, \Omega] \mid xR_iy \text{ and } yR_ix\}$.

[8] It is clear that $e_{R_i}(x)$ always exists and determined uniquely. Furthermore

$$e_{R_i}(x) = x \Leftrightarrow x = x^*(R_i)$$

and

$$e_{R_i}(x) \in (0, \Omega) \Rightarrow e_{R_i}(x)\ \mathcal{I}(R_i)\ x.$$

2.2 Envy-freeness and efficiency

As we stated in introduction, this paper is concerned with envy-free and efficient allocations. Envy-freeness requires that every agent doesn't strictly prefer others' allotments to one's own allotment. Formally, it is defined by the following.

Envy-freeness: Given $R = (R_1, \ldots, R_n) \in \mathbb{S}^N$. $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}$ is envy-free with respect to R if and only if for all $i, j \in N$, $x_i \, R_i \, x_j$.

For each $R \in \mathbb{S}^N$, $EF(R)$ denotes the set of allocations that are envy-free with respect to R.

Efficiency requires that no one can be better off without making someone be worse off. But in the present environment, it is well-known that the requirement is equivalent to the *same-sidedness* that requires everyone's allotment is on the same side of one's peak. We adopt it as the definition of efficiency.

Efficiency: Given $R = (R_1, \ldots, R_n) \in \mathbb{S}^N$. $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}$ is efficient with respect to R if and only if

$$\begin{cases} \sum_{i \in N} x^*(R_i) \geq \Omega \Rightarrow \forall i \in \mathbf{N} : x_i \leq x^*(R_i) \\ \sum_{i \in N} x^*(R_i) \leq \Omega \Rightarrow \forall i \in \mathbf{N} : x^*(R_i) \leq x_i. \end{cases}$$

For each $R \in \mathbb{S}^N$, $\mathit{eff}(R)$ denotes the set of allocations that are efficient with respect to R. [9]

If a function $F : \mathbb{S}^N \to 2^{\mathbf{X}}$ satisfies $F(R) = EF(R) \cap \mathit{eff}(R)$ for each $R \in \mathbb{S}^N$, we call F the *envy-free and efficient correspondence*.

2.3 A Property of Envy-free allocations under efficiency

As efficiency can be represented by handy notion same-sidedness in this environment, envy-freeness also has a character that helps us understand the requirement graphycally under efficiency. To see this, we prepare a notation. For each $R_i \in \mathbb{S}^N$, $x \in [0, \Omega]$, $SU(R_i, x)$ denotes the strict part of the upper contour set of x with respect to R_i. That is, $SU(R_i, x) = \{y \in [0, \Omega] \mid y \, \mathcal{P}(R_i) \, x\}$. And

[9] Note the following fact; for each $R \in \mathcal{S}_0$,

$$\mathit{eff}(R) = \{(x^*(R_1), \ldots, x^*(R_n))\}.$$

And obviously the allocation in the set is envy-free with respect to R because all agents get the most desirebale allotment. After here we will mainly deal with profiles in $\mathcal{S}_d$ or $\mathcal{S}_s$.

$$SU^*(R_i, x) = \begin{cases} SU(R_i, x) & \text{if } x \neq x^*(R_i), \\ \{x\} & \text{if } x = x^*(R_i). \end{cases}$$ [10] [11]

Let's consider the situation in which $SU^*(R_i, x_i)$ and $SU^*(R_j, x_j)$ are separated for all $i, j (i \neq j)$. The situation is described in figure 2-1.

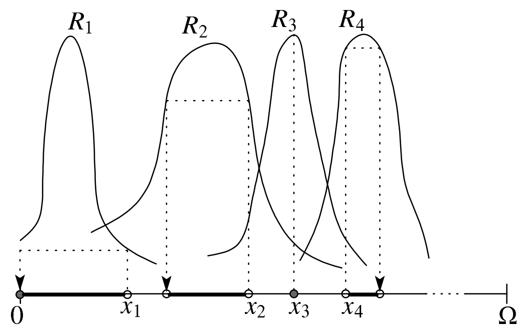

figure2-1

In this situation, **x** satisfies envy-freeness because the situation implies others' allotments are in the lower contour set. So we have the following fact.

Lemma 1. Given $R \in \mathbb{S}^N$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}$. If **x** satisfies the following condition $(*)$, then **x** is envy-free with respect to R.

$$(*) \quad \forall i, j \in N : x_i = x_j \vee SU^*(R_i, x_i) \cap SU^*(R_j, x_j) = \emptyset$$

Proof. Obvious.

Although the second half of the condition $(*)$ roughly requires that each agent's strict upper contour set is separated, this condition is due to our single-peaked environment. [12]

Next we consider the converse of **Lemma 1**. The following example implies that the converse of **Lemma 1** doesn't hold in general.

---

[10] Note that $x^*(R_i) \in SU^*(R_i, x)$.

[11] This is equivalent to the following;

$$SU^*(R_i, x) = \begin{cases} (x, e_{R_i}(x)) & \text{if } x < x^*(R_i) \ \wedge \ e_{R_i}(x) \, \mathcal{I}(R_i) \, x, \\ (x, e_{R_i}(x)] & \text{if } x < x^*(R_i) \ \wedge \ e_{R_i}(x) \, \mathcal{P}(R_i) \, x, \\ \{x\} & \text{if } x = x^*(R_i), \\ (e_{R_i}(x), x) & \text{if } x^*(R_i) < x \ \wedge \ e_{R_i}(x) \, \mathcal{I}(R_i) \, x, \\ [e_{R_i}(x), x) & \text{if } x^*(R_i) < x \ \wedge \ e_{R_i}(x) \, \mathcal{P}(R_i) \, x. \end{cases}$$

[12] For example, let's consider the ordinary environment such that the consumption set for each agent is a subset of $\mathbb{R}^\ell_+$ and each agent is described as a monotonic preference on it. Let $x_i, x_j \in \mathbb{R}^\ell_+$ be the allotments for agent i and j. Suppose that one agent cannot obtain all feasible resources. Then we can easily construct y in the consumption set such that $y \, \mathcal{P}(R_i) \, x_i$ and $y \, \mathcal{P}(R_j) \, x_j$. So strict upper contour sets usually intersect.

Example 1.
Let $n = 2$. R_1 and R_2 are preferences in $\mathbb{S}$ that satisfy $0 \; \mathcal{I}(R_1) \; \Omega$ and $0 \; \mathcal{I}(R_2) \; \Omega$. Let $\mathbf{x} = (x_1, x_2) = (\Omega, 0)$. Then $\mathbf{x}$ is envy-free with respect to (R_1, R_2). But

$$SU^*(R_1, x_1) \cap SU^*(R_2, x_2) = (0, 1) \neq \emptyset.$$

But the converse of **Lemma 1** holds if allocation $\mathbf{x}$ is efficient.

Proposition 1. Given $R \in \mathbb{S}^N$. Suppose that $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{X}$ is efficient with respect to R. $\mathbf{x}$ is envy-free with respect to R *if and only if* $\mathbf{x}$ satisfies the following condition $(*)$.

$$(*) \quad \forall i, j \in N : x_i = x_j \vee SU^*(R_i, x_i) \cap SU^*(R_j, x_j) = \emptyset$$

Proof. See appendix.

Corollary 1. Given $R \in \mathbb{S}^N$. Suppose that $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{X}$ is efficient with respect to R and envy-free with respect to R. Then

$$\forall i, j \in N : \left[x^*(R_i) \leq x^*(R_j) \Rightarrow x_i \leq x_j \right].$$

Proof. Obvious.

2.4 Uniform rule

In this subsection, we give the formal definition of uniform rule. $f : \mathbb{S}^N \to \mathbf{X}$ is called *a rule* . And **uniform rule** f^u is the rule defined by the following; for each $R = (R_1, \dots, R_n) \in \mathbb{S}^N$ and $i \in N$,

$$f_i^u(R) = \begin{cases} \min\{x^*(R_i), \lambda(R)\} & \text{if } \sum_{i \in N} x^*(R_i) \geq \Omega, \\ \max\{x^*(R_i), \mu(R)\} & \text{if } \sum_{i \in N} x^*(R_i) \leq \Omega, \end{cases}$$

where $\lambda(R)$ solves the equation $\sum_{i \in N} \min\{x^*(R_i), \lambda(R)\} = \Omega$ and $\mu(R)$ solves the equation $\sum_{i \in N} \max\{x^*(R_i), \mu(R)\} = \Omega$.

It is well-known that f^u is the only rule that satisfies *envy-freeness, efficiency* and *strategy-proofness.* [13]

[13] Note that three axioms are imposed on the rule here while envy-freeness and efficiency defined in 2.2 said the relationship between a profile and an allocation. So envy-freeness here requires that for each $R \in \mathbb{S}^N$, $f(R)$ satisfies envy-freeness with respect to R in a manner defined in 2.2. Efficiency here requires that for each $R \in \mathbb{S}^N$, $f(R)$ satisfies efficiency with respect to R in a manner defined in 2.2. And strategy-proofness requires that $\forall i \in N, R \in \mathbb{S}^N, R'_i \in \mathbb{S} : f(R) R_i f(R'_i, R_{-i})$.

3 Intuitive Explanation of a Method of Finding any Envy-free and Efficient Allocations

In this section we consider a procedure that reaches an envy-free and efficient allocation. **Proposition 1** and **Corollary 1** offer a clue to do that. We take up only the case $R \in \mathcal{S}_d$, but the case $R \in \mathcal{S}_s$ can be dealt with dually. For the simplicity, we suppose $x^*(R_1) \leq x^*(R_2) \leq \cdots \leq x^*(R_n)$. Roughly speaking, our procedure is composed of two parts. In the first part, we set the allotment w_i for each i from the agent who has the smallest peak in R to the largest one. And consequently we obtain a candidate of allocation $\mathbf{w} = (w_1, \ldots, w_n) \in [0, \Omega]^N$. In the second part, we check the feasibility of $\mathbf{w}$. Here, we explain the first part in detail. Suppose that utility representations of R_1, R_2, R_3 and R_4 in R are described in figure 3-1.

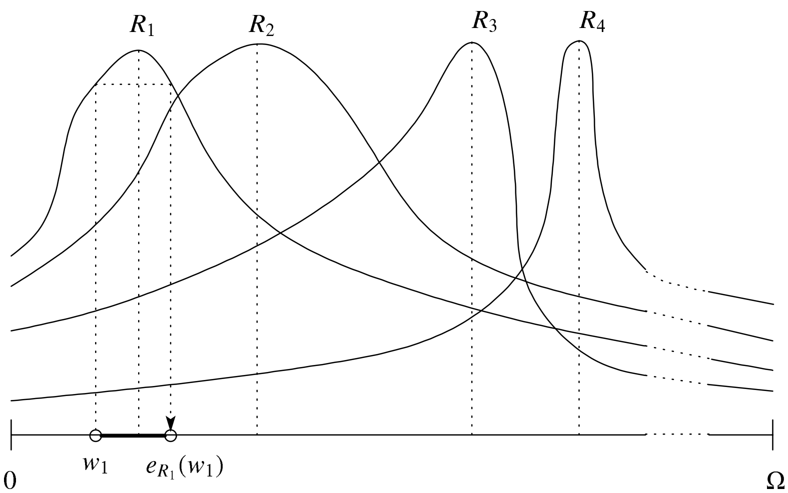

figure 3-1

First of all, our method sets the allotment w_1 for agent 1 who has the smallest peak. Since we want to construct an efficient allocation, w_1 must be a point in $[0, x^*(R_1)]$. Formally, the point in $[0, x^*(R_1)]$ that is chosen by our method is determined by the parameter that is given. Here assume that w_1 is determined on the point described in figure 3-1. Next we consider the allotment w_2 for agent 2. Before determine it, let's consider $SU^*(R_1, w_1)$. This set is denoted by the bold-faced open interval. w_2 must be in $[0, x^*(R_2)]$ if the allocation we are constructing is efficient. But if $w_2 \in [0, e_{R_1}(w_1))$, then obviously

$$SU^*(R_1, w_1) \cap SU^*(R_2, w_2) \neq \emptyset,$$

and according to **Proposition 1**, envy-freeness for the allocation that we are constructing is guaranteed only if $w_2 = w_1$. So the only point in $[0, e_{R_1}(w_1))$ that can be w_2 is w_1. And if $w_2 \in [e_{R_1}(w_1), x^*(R_2)]$, then obviously

$$SU^*(R_1, w_1) \cap SU^*(R_2, w_2) = \emptyset.$$

Hence w_2 must be a point in $\{w_1\} \cup [e_{R_1}(w_1), x^*(R_2)]$. Formally, the point in $\{w_1\} \cup [e_{R_1}(w_1), x^*(R_2)]$ that is chosen by our method is determined by the parameter that is given. And we define $w_3, \ldots, w_n$ successively. This is the basic idea of the first part of our procedure.

Note that w_1 in figure 3-1 was an allotment that satisfies $x^*(R_2) \notin SU^*(R_1, w_1)$. But the allotment for agent1 determined by another parameter may not satisfy this condition. Let's consider about the point w'_1 that are described in figure 3-2.

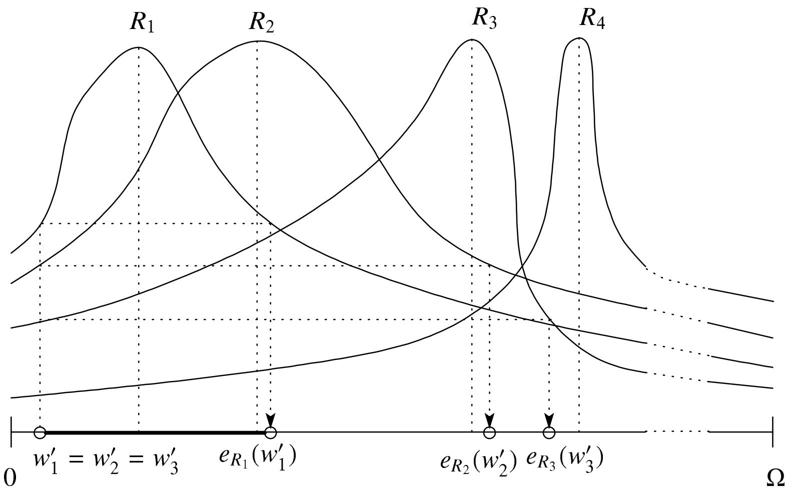

figure 3-2

The question we want to consider here is the following; which allotment can agent 2 take if we want to construct an envy-free and efficient allocation? By efficiency, agent 2's allotment w'_2 must be in $[0, x^*(R_2)]$. But obviously

$$SU^*(R_1, w'_1) \cap SU^*(R_2, w'_2) \neq \emptyset$$

for any $w'_2 \in [0, x^*(R_2)]$. So by **Proposition 1**, w'_2 must be equal to w'_1 if we want to construct $\mathbf{w} = (w_1, \ldots, w_n)$ to be envy-free. Summing up the above, when we have set the allotment of agent 1 w'_1, if $x^*(R_2) \in SU^*(R_1, w'_1)$, then w'_2 is determined to be equal to w'_1 automatically. Furthermore, $x^*(R_3) \in SU^*(R_2, w'_2)$ in figure 3-2. So the same argument induces the fact w'_3 must be equal to w'_2. We continue to the same argument till $x^*(R_{i+1}) \notin SU^*(R_i, w_i)$ for some i. [14] In figure 3-2, $x^*(R_4) \notin SU^*(R_3, w_3)$. So we return to define w'_4 to be

$$w'_4 \in \{w'_3\} \cup [e_{R_3}(w'_3), x^*(R_4)],$$

depending on the parameter that is given.

Now assume that we have finished to define $\mathbf{w} = (w_1, \ldots, w_n) \in [0, \Omega]^N$. This is the end of the first part of our procedure. We go to the second part, and check the feasibility of $\mathbf{w}$. That is, we check whether

[14] If $x^*(R_{i+1}) \in SU^*(R_i, w_i)$ for all i, then the first part of our procedure is finished. And we go to the second part.

$$\sum_{i \in N} w_i = \Omega$$

or not. [15] The next section offers the formal definition of our method we have seen in this section.

4 A Method of Finding Any Envy-free and Efficient Allocations

4.1 Definition of the Method

As we saw in figure 3-2, agents 1 to 3 receive the same amount of good. This is because if we want to construct an efficient and envy-free allocation, any agents whose preferences are located "near" must receive the same amount. What is the rigorous meaning of "near" in our setup? The following is the definition of it.

Definition. Given $R \in \mathbb{S}^N$. For each $i_0 \in N,\ x \in [0, \Omega]$, we define $\mathcal{N}(i_0, x) \subseteq N$ inductively.

Step1
Let $\mathcal{N}_1(i_0, x) = \{i \in N \mid x^*(R_i) \in SU^*(R_{i_0}, x)\}$. [16]
If $SU^*(R_i, x) \subseteq SU^*(R_{i_0}, x)$ for all $i \in \mathcal{N}_1(i_0, x)$, then $\mathcal{N}(i_0, x) = \mathcal{N}_1(i_0, x)$. [17]

Suppose that $SU^*(R_i, x) \not\subseteq SU^*(R_{i_{\ell-2}}, x)$ for some $i \in \mathcal{N}_{\ell-1}(i_0, x)$ in *step*$(\ell - 1)$ for $\ell \geq 2$. Then pick $i_{\ell-1} \in \mathcal{N}_{\ell-1}(i_0, x)$ such that $SU^*(R_i, x) \subseteq SU^*(R_{i_{\ell-1}}, x)$ for all $i \in \mathcal{N}_{\ell-1}(i_0, x)$ and go to *step* ℓ.[18]

stepℓ
Let $\mathcal{N}_\ell(i_0, x) = \{i \in N \mid x^*(R_i) \in SU^*(R_{i_{\ell-1}}, x)\}$. [19]
If $SU^*(R_i, x) \subseteq SU^*(R_{i_{\ell-1}}, x)$ for all $i \in \mathcal{N}_\ell(i_0, x)$, then $\mathcal{N}(i_0, x) = \mathcal{N}_\ell(i_0, x)$.

Because N is finite, this procedure stops in some steps. [20] [21]

[15] If $\mathbf{w}$ doesn't satisfy feasibility, we consider our method reaches the uniform allocation $f^u(R)$.

[16] Because $i_0 \in \mathcal{N}_1(i_0, x)$, $\mathcal{N}_1(i_0, x)$ is not empty.

[17] Note that $\subseteq$ is complete on $\{SU^*(R_i, x) \mid i \in \mathcal{N}_1(i_0, x)\}$.

[18] Note that $\subseteq$ is complete on $\{SU^*(R_i, x) \mid i \in \mathcal{N}_{\ell-1}(i_0, x)\}$. And since $\{SU^*(R_i, x) \mid i \in \mathcal{N}_{\ell-1}(i_0, x)\}$ is finite, there exists an maximal element with respect to $\subseteq$.

[19] Because $i_{\ell-1} \in \mathcal{N}_\ell(i_0, x)$, $\mathcal{N}_\ell(i_0, x)$ is not empty.

[20] Obviously if $x = x^*(R_{i_0})$, then $\mathcal{N}(i_0, x) = \{i \in N \mid x^*(R_i) = x\}$.

[21] Note that the following fact which plays an important role in lemmas later. Given $R \in \mathbb{S}^N$, $i_0 \in N$, $x \in [0, \Omega]$. Then

$$(1) x < x^*(R_{i_0}) \Rightarrow \left[\forall j \in \mathcal{N}(i_0, x) : x^*(R_{i_0}) \leq x^*(R_j) \Rightarrow \mathcal{N}(j, x) = \mathcal{N}(i_0, x)\right],$$
$$(2) x = x^*(R_{i_0}) \Rightarrow \left[\forall j \in \mathcal{N}(i_0, x) : \mathcal{N}(j, x) = \mathcal{N}(i_0, x)\right],$$
$$(3) x^*(R_{i_0}) < x \Rightarrow \left[\forall j \in \mathcal{N}(i_0, x) : x^*(R_j) \leq x^*(R_{i_0}) \Rightarrow \mathcal{N}(j, x) = \mathcal{N}(i_0, x)\right].$$

The following lemma guarantees that $\mathcal{N}(i_0, x)$ is the set of agents whose preferences are near to agent i_0's in the sense allotment for them must be equal to x if i_0 obtains x under the given profile.

Lemma 2. Given $R \in \mathbb{S}^N$. Suppose that $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{X}$ is envy-free and efficient with respect to R. Then

$$\forall i \in N, \forall j \in \mathcal{N}(i, x_i) : x_j = x_i.$$

Proof. See appendix.

Now we define the algorithm that is composed of two parts. The first part sets the allotment $w_i \in [0, \Omega]$ for each agent $i \in N$, and the second part check the feasibility condition of $(w_1, \dots, w_n)$ obtained in first part.

Definition. Given $R \in \mathcal{S}_d$. We define $\alpha^R : [0,1]^N \times \{0,1\}^N \to \mathbf{X}$ inductively. For each $(\mathbf{p}, \mathbf{q}) \in [0,1]^N \times \{0,1\}^N$, $\alpha^R(\mathbf{p}, \mathbf{q})$ is the allocation obtained by the following procedure. Without loss of generality, suppose that $x^*(R_{i_1}) \le \cdots \le x^*(R_{i_n})$. [22]

Part A
step 1
Let $y_1 = p_{i_1} \cdot 0 + (1 - p_{i_1}) x^*(R_{i_1})$. [23] [24] And let $N_1^{\mathbf{p},\mathbf{q}}(R) = \mathcal{N}(i_1, y_1)$.
If $N \backslash N_1^{\mathbf{p},\mathbf{q}}(R) = \emptyset$, then go to *Part B*.

Suppose that $N \backslash \Big(\cup_{m=1}^{\ell-1} N_m^{\mathbf{p},\mathbf{q}}(R) \Big) \neq \emptyset$ in *step*$(\ell - 1)$ for $\ell \ge 2$. Then pick $j_{\ell-1} \in N_{\ell-1}^{\mathbf{p},\mathbf{q}}(R)$ such that $SU^*(R_i, y_1) \subseteq SU^*(R_{j_{\ell-1}}, y_1)$ for all $i \in N_{\ell-1}^{\mathbf{p},\mathbf{q}}(R)$ and go to *step* ℓ.

step ℓ
Since $N \backslash \Big(\cup_{m=1}^{\ell-1} N_m^{\mathbf{p},\mathbf{q}}(R) \Big) \neq \emptyset$, there exists $i_{k(\ell)} \in N$ such that $N \backslash \Big(\cup_{m=1}^{\ell-1} N_m^{\mathbf{p},\mathbf{q}}(R) \Big) = \{i_{k(\ell)}, i_{k(\ell)+1}, \dots, i_n\}$. Let

$$y_\ell = \begin{cases} y_{\ell-1} & \text{if } q_{i_{k(\ell)}} = 0, \\ p_{i_{k(\ell)}} \cdot e_{R_{j_{\ell-1}}}(y_{\ell-1}) + (1 - p_{i_{k(\ell)}}) \cdot x^*(R_{i_{k(\ell)}}) & \text{if } q_{i_{k(\ell)}} = 1. \end{cases}$$ [25]

Let $N_\ell^{\mathbf{p},\mathbf{q}}(R) = \mathcal{N}(i_{k(\ell)}, y_\ell)$.
If $N \backslash \Big(\cup_{m=1}^{\ell} N_m^{\mathbf{p},\mathbf{q}}(R) \Big) = \emptyset$, then go to *Part B*.
Because N is finite, *Part A* stops in some steps.

Part B
Let ℓ be the number of step *Part A* stops. And $\mathbf{w} = (w_1, \dots, w_n) \in [0, \Omega]^N$ is the point defined by the following; for each $i \in N$,

[22] Suppose also that if $x^*(R_{i_j}) = x^*(R_{i_{j'}})$, then $i_j < i_{j'} \Leftrightarrow j < j'$.
[23] Note that $y_1 \le x^*(R_i)$ for all $i \in N$.
[24] Because $i_1 \in N_1^{\mathbf{p},\mathbf{q}}(R)$, $N_1^{\mathbf{p},\mathbf{q}}(R)$ is not empty.
[25] Note that $y_\ell \le x^*(R_i)$ for all $i \in N \backslash \Big(\cup_{m=1}^{\ell-1} N_m^{\mathbf{p},\mathbf{q}}(R) \Big)$ because $y_{\ell-1} \le e_{R_{j_{\ell-1}}}(y_{\ell-1}) \le x^*(R_{i_{k(\ell)}}) \le x^*(R_i)$.

$$w_i = \begin{cases} y_1 & \text{if } i \in N_1^{\mathbf{p},\mathbf{q}}(R), \\ \vdots & \vdots \\ y_\ell & \text{if } i \in N_\ell^{\mathbf{p},\mathbf{q}}(R). \end{cases}$$

And let

$$\alpha^R(\mathbf{p}, \mathbf{q}) = \begin{cases} \mathbf{w} & \text{if } \mathbf{w} \in \mathbf{X}, \\ f^u(R) & \text{otherwise.} \end{cases}$$

Example 2.
Let $n = 4$ and $R = (R_1, \cdots, R_4) \in \mathbb{S}^N$ be the profile composed of the following preferences. $x^*(R_1) = \frac{\Omega}{8}, x^*(R_2) = \frac{\Omega}{4}, x^*(R_3) = \frac{\Omega}{2}$ and $x^*(R_4) = \frac{3\Omega}{4}$. And R_i $(i = 1, \cdots, 4)$ satisfies

$\forall x, x' \in [0, \Omega] : \left|x^*(R_i) - x\right| = \left|x^*(R_i) - x'\right| \Rightarrow x\mathcal{I}(R_i)x'$.

It is easy to check that if we chose parameters $\mathbf{p} = (1, 1, \frac{3}{4}, \frac{3}{4})$ and $\mathbf{q} = (1, 1, 1, 0)$, then the algorithm $\alpha^R(\mathbf{p}, \mathbf{q})$ reaches $f^u(R)$ non-trivially.

Furthermore, our method reaches efficient and envy-free allocation that cannot be captured by uniform rule. Let's consider the parameters $\mathbf{p}' = (\frac{1}{2}, 1, \frac{2}{3}, \frac{2}{3})$ and $\mathbf{q}' = (1, 1, 1, 0)$. Then

$y_1 = \frac{1}{2} \cdot 0 + (1 - \frac{1}{2})\frac{\Omega}{8} = \frac{\Omega}{16}$. $N_1^{(\mathbf{p}',\mathbf{q}')}(R) = \mathcal{N}(1, \frac{\Omega}{16}) = \{1\}$. And $e_{R_1}(\frac{\Omega}{16}) = \frac{3\Omega}{16}$.

$y_2 = 1 \cdot \frac{3\Omega}{16} + (1 - 1)\frac{\Omega}{8} = \frac{3\Omega}{16}$. $N_2^{\mathbf{p}',\mathbf{q}')}(R) = \mathcal{N}(2, \frac{3\Omega}{16}) = \{2\}$. And $e_{R_2}(\frac{3\Omega}{16}) = \frac{5\Omega}{16}$.

$y_3 = \frac{2}{3} \cdot \frac{5\Omega}{16} + (1 - \frac{2}{3})\frac{\Omega}{2} = \frac{6\Omega}{16}$. And $y_4 = y_3 = \frac{6\Omega}{16}$ because $q'_4 = 0$.

Obviously $\alpha^R(\mathbf{p}', \mathbf{q}') = (\frac{\Omega}{16}, \frac{3\Omega}{16}, \frac{6\Omega}{16}, \frac{6\Omega}{16})$ satisfies efficiency and envy-freeness. Note that the allocation is different from equal division and no agent obtains her own peak. So it is clear that our method reaches efficient and envy-free allocation that cannot be assigned by uniform rule.

In example 2, the allocations assigned by α^R was envy-free and efficient. The next lemma shows that this is true for all $(\mathbf{p}, \mathbf{q}) \in [0, 1]^N \times \{0, 1\}^N$.

Lemma 3. Given $R \in \mathcal{S}_d$. For all $(\mathbf{p}, \mathbf{q}) \in [0, 1]^N \times \{0, 1\}^N$, $\alpha^R(\mathbf{p}, \mathbf{q})$ satisfies envy-freeness with respect to R and efficiency with respect to R.

Proof. See appendix.

In example 2, we saw two kinds of envy-free and efficient allocations were reached by α^R. Next lemma guarantees that there is no envy-free and efficient allocation with respect to R which cannot be reached by α^R.

Lemma 4. Given $R \in \mathcal{S}_d$. Suppose that $\mathbf{x}$ is an allocation that satisfies envy-freeness with respect to R and efficiency with respect to R. Then there exists a parameter $(\mathbf{p}, \mathbf{q}) \in [0, 1]^N \times \{0, 1\}^N$ such that $\alpha^R(\mathbf{p}, \mathbf{q}) = \mathbf{x}$.

Proof. See appendix.

For each $R \in \mathcal{S}_s$, it is easy to define $\beta^R : [0,1]^N \times \{0,1\}^N \to \mathbf{X}$ by imitating the definition of α. And this definition results in the analogue of **Lemma 3** and **Lemma 4**.

Lemma 5. Given $R \in \mathcal{S}_s$. For all $(\mathbf{p}, \mathbf{q}) \in [0,1]^N \times \{0,1\}^N$, $\beta^R(\mathbf{p}, \mathbf{q})$ satisfies envy-freeness with respect to R and efficiency with respect to R.

Proof. See appendix.

Lemma 6. Given $R \in \mathcal{S}_s$. Suppose that $\mathbf{x}$ is an allocation that satisfies envy-freeness with respect to R and efficiency with respect to R. Then there exists a parameter $(\mathbf{p}, \mathbf{q}) \in [0,1]^N \times \{0,1\}^N$ such that $\beta^R(\mathbf{p}, \mathbf{q}) = \mathbf{x}$.

Proof. See appendix.

Now we are ready to state our main theorem.The theorem shows that any envy-free and efficient allocations can be reached by the algorithms if the parameters$(\mathbf{p}, \mathbf{q}) \in [0,1]^N \times \{0,1\}^N$ are chosen appropriately.

Theorem. Given $R \in \mathbb{S}^N$.
(1)$R \in \mathcal{S}_d \Rightarrow EF(R) \cap \mathit{eff}(R) = \alpha^R([0,1]^N \times \{0,1\}^N)$,
(2)$R \in \mathcal{S}_0 \Rightarrow EF(R) \cap \mathit{eff}(R) = \{(x^*(R_1), \ldots, x^*(R_n))\}$,
(3)$R \in \mathcal{S}_s \Rightarrow EF(R) \cap \mathit{eff}(R) = \beta^R([0,1]^N \times \{0,1\}^N)$.

Proof. **Lemma 3** and **Lemma 4** proves (1). (2) is obvious. **Lemma 5** and **Lemma 6** proves (3). *Q.E.D.*

5 Conclusion

We have characterized the set of allocations that is envy-free and efficient under the environment described in section 2. But the environment we have dealt was very particular. First although we assumed that there's only one good, whether our method could have insight to establish a method of finding all envy-free and efficient allocations in the environment that have more than two goods or not will be an interesting question to be studied next. Second, we have assumed that all agents had continuous preferences. Whether this assumption can be dropped or not is worth while investigating.

Topological properties of the set of envy-free and efficient allocations are not dealt in this paper. But whether it is connected or not is an interesting subject.

Appendix : Proofs

Proposition 1. Given $R \in \mathbb{S}^N$. Suppose that $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}$ is efficient with respect to R. $\mathbf{x}$ is envy-free with respect to R *if and only if* $\mathbf{x}$ satisfies the following condition $(*)$.

$$(*) \quad \forall i, j \in N : x_i = x_j \vee SU^*(R_i, x_i) \cap SU^*(R_j, x_j) = \emptyset$$

Proof. Thanks to **Lemma 1**, *if* part is trivial. We prove that envy-freeness implies the condition $(*)$. Let $i, j \in N$ be arbitrary. And suppose that $x_i \neq x_j$. We show that $SU^*(R_i, x_i) \cap SU^*(R_j, x_j) = \emptyset$. Without loss of generality, assume that $x_i < x_j$.

Case 1 $R \in \mathcal{S}_0$
By efficiency, $\mathbf{x} = (x^*(R_1), \ldots, x^*(R_n))$. So $SU^*(R_i, x_i) = \{x_i\}$ and $SU^*(R_j, x_j) = \{x_j\}$. So $SU^*(R_i, x_i) \cap SU^*(R_j, x_j) = \emptyset$ because $x_i < x_j$.

Case 2 $R \in \mathcal{S}_d$
Subcase 2.1 $SU^*(R_i, x_i) = (x_i, e_{R_i}(x_i))$
Envy-freeness implies $e_{R_i}(x_i) \leq x_j$. By efficiency,

$$SU^*(R_j, x_j) = \begin{cases} (x_j, e_{R_j}(x_j)) & \text{if } x_j < x^*(R_j) \ \wedge \ e_{R_j}(x_j)\ \mathcal{I}(R_j)\ x_j, \\ (x_j, e_{R_j}(x_j)] & \text{if } x_j < x^*(R_j) \ \wedge \ e_{R_j}(x_j)\ \mathcal{P}(R_j)\ x_j, \\ \{x_j\} & \text{if } x_j = x^*(R_j). \end{cases}$$

So obviously $SU^*(R_i, x_i) \cap SU^*(R_j, x_j) = \emptyset$.
Subcase 2.2 $SU^*(R_i, x_i) = (x_i, e_{R_i}(x_i)]$
This case cannot occur because $e_{R_i}(x_i)\ \mathcal{P}(R_i)\ x_i$ and envy-freeness implies $e_{R_i}(x_i) < x_j$. But $e_{R_i}(x_i) = \Omega$ in this case.
Subcase 2.3 $SU^*(R_i, x_i) = \{x_i\}$
By efficiency,

$$SU^*(R_j, x_j) = \begin{cases} (x_j, e_{R_j}(x_j)) & \text{if } x_j < x^*(R_j) \ \wedge \ e_{R_j}(x_j)\ \mathcal{I}(R_j)\ x_j, \\ (x_j, e_{R_j}(x_j)] & \text{if } x_j < x^*(R_j) \ \wedge \ e_{R_j}(x_j)\ \mathcal{P}(R_j)\ x_j, \\ \{x_j\} & \text{if } x_j = x^*(R_j). \end{cases}$$

And $x_i < x_j$ implies $SU^*(R_i, x_i) \cap SU^*(R_j, x_j) = \emptyset$

Case 2 $R \in \mathcal{S}_d$
A similar argument to **Case 2** yields the conclusion. *Q.E.D.*

Lemma 2. Given $R \in \mathbb{S}^N$. Suppose that $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}$ is envy-free and efficient with respect to R. Then

$$\forall i \in N, \forall j \in \mathcal{N}(i, x_i) : x_j = x_i.$$

Proof. We prove by contradiction. Suppose that there exist $i' \in N$ and $j' \in \mathcal{N}(i', x_{i'})$ such that $x_{j'} \neq x_{i'}$.

Case 1 $x_{i'} = x^*(R_{i'})$
$x^*(R_{j'}) = x_{i'}$ since $j' \in \mathcal{N}(i', x_{i'})$. So envy-freeness of $\mathbf{x}$ implies $x_{j'} = x^*(R_{j'})$. But this contradicts the hypothesis.

Case 2 $x_{i'} < x^*(R_{i'})$[26]
Let ℓ be the number of steps the procedure of finding $\mathcal{N}(i', x_{i'})$ stops. First we show that $x_j = x_{i'}$ for all $j \in \mathcal{N}_1(i', x_{i'})$. Since $j \in \mathcal{N}_1(i', x_{i'})$, $x^*(R_j) \in SU^*(R_{i'}, x_{i'})$. And this implies that $x^*(R_j) \in SU^*(R_{i'}, x_{i'}) \cap SU^*(R_j, x_j)$. So by **Proposition 1**, $x_j = x_{i'}$. Suppose that $x_j = x_{i'}$ for all $j \in \mathcal{N}_{k-1}(i', x_{i'})$ [27] and we prove $x_j = x_{i'}$ for all $j \in \mathcal{N}_k(i', x_{i'})$. By definition of $\mathcal{N}_k(i', x_{i'})$, there exists $i_{k-1} \in \mathcal{N}_{k-1}(i', x_{i'})$ such that $\mathcal{N}_k(i', x_{i'}) = \{i \in N \mid x^*(R_i) \in SU^*(R_{i_{k-1}}, x_{i'})\}$. So $x^*(R_j) \in SU^*(R_{i_{k-1}}, x_{i'})$. And this implies that $x^*(R_j) \in SU^*(R_{i_{k-1}}, x_{i'}) \cap SU^*(R_j, x_j)$. So by **Proposition 1**, $x_j = x_{i'}$.

Case 3 $x^*(R_{i'}) < x_{i'}$ [28]
We can prove this case by similar way to *Case 2*. *Q.E.D.*

Lemma 3. Given $R \in \mathcal{S}_d$. For all $(\mathbf{p}, \mathbf{q}) \in [0,1]^N \times \{0,1\}^N$, $\alpha^R(\mathbf{p}, \mathbf{q})$ satisfies envy-freeness with respect to R and efficiency with respect to R.

Proof. If $\alpha^R(\mathbf{p}, \mathbf{q}) = f^u(R)$, then the conclusion is obvious. So we suppose that $\mathbf{w}$ belongs to $\mathbf{X}$ in *Part B* of the procedure which finds $\alpha^R(\mathbf{p}, \mathbf{q})$. Efficiency of $\alpha^R(\mathbf{p}, \mathbf{q})$ is obvious. So we only show envy-freeness of $\alpha^R(\mathbf{p}, \mathbf{q})$.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ denote $\alpha^R(\mathbf{p}, \mathbf{q})$. Let $i, j \in N$ be arbitrary. And let ℓ and ℓ' be numbers such that $i \in N_\ell^{\mathbf{p},\mathbf{q}}(R)$ and $j \in N_{\ell'}^{\mathbf{p},\mathbf{q}}(R)$. If $\ell = \ell'$, then $x_i = y_\ell = y_{\ell'} = x_j$. So obviously $x_i R_i x_j$. We suppose that $\ell \neq \ell'$ below. And there's cases that allows $y_\ell = y_{\ell'}$ and $\ell \neq \ell'$ depending on parameters. But in this case obviously $x_i R_i x_j$. So suppose that $y_\ell \neq y_{\ell'}$ below.

Case 1 $\ell' < \ell$
In general, if $\ell' < \ell$, then $y_{\ell'} \leq y_\ell$. So $y_{\ell'} < y_\ell$.

$$\begin{aligned} x_j &= y_{\ell'} \\ &< y_\ell \\ &= x_i \\ &\leq x^*(R_i) \ (\because \mathbf{x} \text{ is efficient with respect to } R). \end{aligned}$$

So single-peakedness of R_i implies $x_i \, \mathcal{P}(R_i) \, x_j$.

Case 2 $\ell < \ell'$

[26] Efficiency of $\mathbf{x}$ implies $R \in \mathcal{S}_d$.
[27] k is a number in $\{2, \ldots, \ell - 1\}$
[28] Efficiency of $\mathbf{x}$ implies $R \in \mathcal{S}_s$.

Let i_ℓ be an agent such that $SU^*(R_h, y_\ell) \subseteq SU^*(R_{i_\ell}, y_\ell)$ for all $h \in N_\ell^{\mathbf{p},\mathbf{q}}(R)$. Then

$$\begin{aligned}
x_i &\le x^*(R_i) \ (\because \mathbf{x} \text{ is efficient with respect to } R)\\
&\le e_{R_i}(x_i)\\
&= e_{R_i}(y_\ell) \ (\because y_\ell = x_i)\\
&\le e_{R_{i_\ell}}(y_\ell) \ (\because i \in N_\ell^{\mathbf{p},\mathbf{q}}(R))\\
&\le y_{\ell'} \ (\because y_\ell \neq y_{\ell'})\\
&= x_j\\
&\le x^*(R_j) \ (\because \mathbf{x} \text{ is efficient with respect to } R).
\end{aligned}$$

Subcase 2.1 $e_{R_{i_\ell}}(y_\ell) = \Omega$

If $e_{R_{i_\ell}}(y_\ell) \ \mathcal{P}(R_{i_\ell}) \ y_\ell$, then because $SU^*(R_{i_\ell}, y_\ell) = (y_\ell, \Omega]$,

$$N_\ell^{\mathbf{p},\mathbf{q}}(R) = \{h \in N \mid x^*(R_{i_\ell}) \in (y_\ell, \Omega]\}.$$

But this contradicts that $\ell < \ell'$ and $j \in N_{\ell'}^{\mathbf{p},\mathbf{q}}(R)$. So $e_{R_{i_\ell}}(y_\ell) \ \mathcal{I}(R_{i_\ell}) \ y_\ell$. By the definition of α^R, $y_{\ell'} = \Omega$. Since $\alpha^R(\mathbf{p},\mathbf{q}) = \mathbf{x} \in \mathbf{X}$, $x_i = y_\ell = 0$. And if $\Omega \ \mathcal{P}(R_i) \ 0$, then $SU^*(R_{i_\ell}, y_\ell) \subsetneq SU^*(R_i, y_\ell)$. But this contradicts the definition of i_ℓ. So $0 \ \mathcal{I}(R_i) \ \Omega$ and this implies $x_i R_i x_j$.

Subcase 2.2 $e_{R_{i_\ell}}(y_\ell) < \Omega$

By the definition of i_ℓ, $e_{R_i}(y_\ell) < \Omega$. So $x_i \ \mathcal{I}(R_i) \ e_{R_i}(y_\ell)$. And single-peakedness of R_i implies $e_{R_i}(y_\ell) R_i x_j$. So transitivity of R_i implies $x_i R_i x_j$. *Q.E.D.*

Lemma 4. Given $R \in \mathcal{S}_d$. Suppose that $\mathbf{x}$ is an allocation that satisfies envy-freeness with respect to R and efficiency with respect to R. Then there exists a parameter $(\mathbf{p},\mathbf{q}) \in [0,1]^N \times \{0,1\}^N$ such that $\alpha^R(\mathbf{p},\mathbf{q}) = \mathbf{x}$.

Proof. Without loss of generality, we suppose that $x^*(R_1) \le \cdots \le x^*(R_n)$.

Step 1

Efficiency implies that $x_1 \le x^*(R_1)$. So there exists $p_1 \in [0,1]$ such that

$$x_1 = p_1 \cdot 0 + (1 - p_1) \cdot x^*(R_1).$$

Let $u_1 = x_1$ and $M_1(R,\mathbf{x}) = \mathcal{N}(1, u_1)$. Because $1 \in M_1(R,\mathbf{x})$, $M_1(R,\mathbf{x})$ is not empty. By **Lemma 2**, $x_i = u_1$ for all $i \in M_1(R,\mathbf{x})$.

For each $i \in M_1(R,\mathbf{x})$, we put

$$p_i = p_1 \text{ and } q_i = 1.$$

If $N \backslash M_1(R,\mathbf{x}) = \emptyset$, then obviously for $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ we have

$$\alpha^R(\mathbf{p},\mathbf{q}) = (u_1, \ldots, u_1) = (\frac{\Omega}{n}, \ldots, \frac{\Omega}{n}) = \mathbf{x}.$$

Suppose that $N \backslash \left(\cup_{m=1}^{\ell-1} M_m(R,\mathbf{x}) \right) \neq \emptyset$ in *step* ℓ and $\ell \ge 2$. Pick $i_{\ell-1} \in M_{\ell-1}(R,\mathbf{x})$ such that $SU^*(R_i, u_{\ell-1}) \subseteq SU^*(R_{i_{\ell-1}}, u_{\ell-1})$ for all $i \in M_{\ell-1}(R,\mathbf{x})$. And note that $e_{R_{i_\ell}}(u_{\ell-1}) \ \mathcal{I}(R_{i_{\ell-1}}) \ u_{\ell-1}$. If not, then $SU^*(R_{i_{\ell-1}}, u_{\ell-1}) = (u_{\ell-1}, \Omega]$. But this contradicts the hypothesis $N \backslash \left(\cup_{m=1}^{\ell-1} M_m(R, \mathbf{x}) \right) \neq \emptyset$.

Step ℓ
Since $N \backslash \left(\cup_{m=1}^{\ell-1} M_m(R, \mathbf{x}) \right) \neq \emptyset$, there exists $k(\ell) \in \{2, \ldots, n\}$ such that

$$N \backslash \left(\cup_{m=1}^{\ell-1} M_m(R, \mathbf{x}) \right) = \{k(\ell), k(\ell) + 1, \ldots, n\}.^{29}$$

By **Corollary 1**,

$$u_{\ell-1} = x_{k(\ell-1)} \leq x_{k(\ell)}.$$

Case 1 $u_{\ell-1} = x_{k(\ell)}$
Let $u_\ell = u_{\ell-1}$ and $M_\ell(R, \mathbf{x}) = \mathcal{N}(k(\ell), u_\ell)$. Since $k(\ell) \in M_\ell(R, \mathbf{x})$, $M_\ell(R, \mathbf{x})$ is not empty. And **Lemma 2** implies $x_i = u_\ell$ for all $i \in M_\ell(R, \mathbf{x})$. For each $i \in M_\ell(R, \mathbf{x})$, we put

$$p_i = 1 \text{ and } q_i = 0.$$

Case 2 $u_{\ell-1} < x_{k(\ell)}$
First, we show $e_{R_{i_{\ell-1}}}(u_{\ell-1}) \leq x_{k(\ell)} \leq x^*(R_{k(\ell)})$. By efficiency, the right side of this inequality is obvious. So we only show that $e_{R_{i_{\ell-1}}}(u_{\ell-1}) \leq x_{k(\ell)}$. But this is obvious by the hypothesis of **Case 2** combined with **Proposition 1**. [30]
So there exists $p_{k(\ell)} \in [0, 1]$ such that

$$x_{k(\ell)} = p_{k(\ell)} \cdot e_{R_{i_{\ell-1}}}(u_{\ell-1}) + (1 - p_{k(\ell)}) \cdot x^*(R_{k(\ell)}).$$

Let $u_\ell = x_{k(\ell)}$ and $M_\ell(R, \mathbf{x}) = \mathcal{N}(k(\ell), u_\ell)$. Since $k(\ell) \in M_\ell(R, \mathbf{x})$, $M_\ell(R, \mathbf{x})$ is not empty. And **Lemma 2** implies $x_i = u_\ell$ for all $i \in M_\ell(R, \mathbf{x})$. For each $i \in M_\ell(R, \mathbf{x})$, we put

$$p_i = p_{k(\ell)} \text{ and } q_i = 1.$$

Because N is finite, this procedure stops in finite steps. Let ℓ be the number; that is,

$$N \backslash \left(\cup_{m=1}^{\ell} M_m(R, \mathbf{x}) \right) = \emptyset.$$

Then we have obtained $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$. And obviously

$$N_1^{\mathbf{p},\mathbf{q}}(R) = M_1(R, \mathbf{x}), \ldots, N_\ell^{\mathbf{p},\mathbf{q}}(R) = M_\ell(R, \mathbf{x})$$

and

$$y_1 = u_1, \ldots, y_\ell = u_\ell.$$

[29] Let $k(1) = 1$.
[30] Note that since $i_{\ell-1} \in M_{\ell-1}(R, \mathbf{x})$, $x_{i_\ell} = u_{\ell-1}$.

So $\alpha^R(\mathbf{p}, \mathbf{q}) = \mathbf{x}$. *Q.E.D.*

Lemma 5. Given $R \in \mathcal{S}_s$. For all $(\mathbf{p}, \mathbf{q}) \in [0, 1]^N \times \{0, 1\}^N$, $\beta^R(\mathbf{p}, \mathbf{q})$ satisfies envy-freeness with respect to R and efficiency with respect to R.

Proof. A similar argument to the proof of **Lemma 3** yields the conclusion. *Q.E.D.*

Lemma 6. Given $R \in \mathcal{S}_s$. Suppose that $\mathbf{x}$ is an allocation that satisfies envy-freeness with respect to R and efficiency with respect to R. Then there exists a parameter $(\mathbf{p}, \mathbf{q}) \in [0, 1]^N \times \{0, 1\}^N$ such that $\beta^R(\mathbf{p}, \mathbf{q}) = \mathbf{x}$.

Proof. A similar argument to the proof of **Lemma 4** yields the conclusion. *Q.E.D.*

References

[1] Ching,S.(1994),"An Alternative Characterization of the Uniform Rule," *Social Choice and Welfare*,**11**,131-136.

[2] Ching,S. and Serizawa,S.(1998),"A Maximal Domain for the Existence of Strategy-Proof Rules," *Journal of Economic Theory*,**78**,157-166.

[3] Foley,D.(1967),"Resource Allocation and the Public Sector," *Yale Economic Essays*,**7**,45-98.

[4] Sakai,T. and Wakayama,T(2007),"Uniforn, Equal Division and Other Envy-Free Rules between the Two," *mimeo*.

[5] Schummer,J. and Thomson,W.(1997),"Two Derivations of the Uniform Rule and an Application to Bankruptcy," *Economics Letters*,**55**,333-337.

[6] Sprumont,Y.(1991),"The Division Problem with Single-Peaked Preferences: A Characterization of the Uniform Allocation Rule," *Econometrica*,**59**,509-519.

[7] Varian,H.R.(1974),"Equity, Envy, and Efficiency," *Journal of Economic Theory*,**9**,63-91.

[8] Varian,H.R.(1976),"Two Problems in the Theory of Fairness," *Journal of Public Economics*,**5**,249-260.