Imperialist policy versus welfare state policy: 
A Theory of Political Competition over 
Military Policy and Income Redistribution

Naoki Yoshihara*
The Institute of Economic Research, Hitotsubashi University,†‡
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Abstract
We discuss political competition games between Left and Right parties, in which the policy space is two-dimensional. One issue is to choose a proportional tax rate, and the second is to allocate tax revenue between military policies and social welfare policies. In these political issues, the stylized fact of the two parties is that the Left prefers higher tax rates and lower military expenditures than the Right. We examine in what kinds of political environments this fact can be rationalized as the equilibrium outcome of a given political game. By adopting the notion of Party-Unanimity Nash Equilibrium (PUNE) [Roemer (2001)], not only voters’ economic motivations, but also their ideological positions are shown to be crucial factors explaining the stylized party behaviors.

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*Phone: (81)-42-580-8354, Fax: (81)-42-580-8333. e-mail: yosihara@ier.hit-u.ac.jp
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1 Introduction

Military policy, which is in principle to provide for the national security of citizens, has been considered as one of the most basic roles of modern nation-states. This is because the provision of national security is indispensable for citizens to enjoy their individual liberties in that country. In fact, the classical liberalists proposed the so-called night watchman state (or the minimal state) as the appropriate solution for protecting individual liberties from potential invaders. In contrast, the social welfare policy, which seeks to provide minimum levels of income, service or other support for disadvantaged people, has been one of the main programs that contemporary nation-states are expected to implement. A country that provides comprehensive social welfare programs is often identified as being a welfare state. In such a country, access to social welfare services is often considered a basic and inalienable right to those in need.

So far, since after World War II, most advanced countries have been expected to implement both military and social welfare policies. Thus, it has been one of the most controversial political issues as to what kind of tax system and budgetary allocation appropriately implement ‘optimal’ provisions of national security and social welfare services. Facing that issue, the Left-wing party usually places a higher priority on social welfare policies, which may involve the increase of the income tax rate or of the budgetary allocation to those policies, whereas the Right-wing party is not as sympathetic as the Left to those policies. In contrast, the Left often strongly criticizes the expansion of military expenditure, whereas the Right often justifies it by reason of the existence of a threat to national security.

However, the difference between the two parties in their social welfare policies may not have a game-theoretic reasoning when we consider the standard Downsian game [Downs (1957)] as a canonical model of political competition, because it is often the case that there is only one unique Downsian equilibrium, in which both parties propose the same policy. Moreover, the rational reason why the Left usually prefers a lower military expenditure than the Right seems to be ambiguous, because national security is a pure public good that every citizen collectively and simultaneously enjoys regardless of his/her income or social class.

In this paper, we discuss political competition games between Left and Right parties, in which the policy space is two-dimensional. One policy issue is to choose a tax rule (a proportional tax rate), and the second is to allo-
cate the tax revenue between military expenditure and the funding of social welfare provisions. In these games, an increasing tax rate and a decreasing share of the tax revenue for military expenditure indicates a strengthening of welfare-state policy, which the Left is expected to support. In contrast, if the proposed military expenditure exceeds a threshold — which is an optimal level of military forces to defend a country from foreign aggression —, and a higher share of tax revenue is allocated to military policy, such a policy proposal may be suitably called imperialism, to which the Right may be more sympathetic than the Left.

Since the policy space in this political model is multi-dimensional, the canonical political games such as Downsian party models and Wittman party models [Wittman (1973)] cannot reliably produce a Nash equilibrium (in pure strategies). Facing this difficulty, we adopt the notion of party-unanimity Nash equilibrium (PUNE) introduced by Roemer (1998; 1999; 2001). The model introduces the idea that the decision makers in parties have different interests, which implies that in each party, its activists are divided into three factions; the Opportunist, the Militant, and the Reformist. The Opportunist is a faction that is concerned solely with winning office, the Militant is a faction that is concerned solely with publicity of the party’s view, and the Reformist is a faction that is concerned with the expected welfare of the party’s members. Facing the structure of having three factions within a party, the three factions of each party should bargain on the policy proposal, given a policy proposal by its opponent party. If a policy proposal agreed on in this party is Pareto efficient for the three factions, this constitutes a solution for the bargaining problem within the party. Then, a PUNE is a pair of policy proposals, each of which is the result of an intra-party bargaining, when facing the other party’s proposal.

We examine under what competitive structure between the two parties the above-mentioned stylized party behaviors can be rationalized as a PUNE. 

\footnote{Here, the term ‘imperialism’ simply indicates a scheme of military policies that allows a nation the possibility of acting *hegemonically* against other nations by using the threat of a superior military force, which is beyond the necessary level for the national security of that nation. Such a usage of the term ‘imperialism’ may be far from the classical Marxian theories of imperialism which focus on the *economic relations* between countries (and within countries), rather than the more formal political and/or military relationships. For instance, Lenin (1916) argued that capitalism necessarily induced monopoly capitalism — which he also called “imperialism” — in order to find new markets and resources, representing the last and highest stage of capitalism.}
of the political game. We first start from the analysis of the two-dimensional political games with citizens’ quasi-linear utility functions, which is a typical assumption in public-goods economies. In such a politico-economic environment, we show that the stylized party behaviors cannot be generated as a PUNE of the political game. There are only two types of PUNEs in such a game: one is that both the Left and the Right propose the Left’s ideal policy, which is the combination of the highest tax rate and the ‘optimal’ military expenditure, whereas the other is that both parties propose their own ideal policies and the Left wins the election with probability one. Therefore, the ideal policy of the Left is always implemented. Moreover, the Right’s ideal military expenditure is less than the Left’s in equilibrium. Thus, all of these are far from real politics, which implies that the standard public-goods politico-economic model fails to explain the stylized party behaviors.

The above result may indicate that every citizen does not vote, according solely to his own economically motivated preference on military forces, and he may also have a non-economically motivated preference on those kinds of public goods. We introduce a simple form of non-economically motivated preference on military forces, which reflects each citizen’s political ideology. In such an extended model, each citizen has an economic preference over military forces and welfare services as well as a non-economic, ideological preference on the level of military forces. Then, under reasonable assumptions, we show that every PUNE of the political game in this extended model rationalizes the stylized party behaviors, whenever all citizens’ ideological concerns about alternative military policies are pronounced enough.

In the following, section 2 defines a basic model of the above two-dimensional political games, and also introduces the PUNE with exogenous party formations. Sections 3 and 4 discuss the existence and the characterization of PUNEs with exogenous party formations, respectively in non-ideological societies and in societies with ideological concerns. Section 5 defines the PUNE with endogenous party formations, and shows that basically the same existence and characterization results, as in the case of exogenous party formations, are obtained in the case of endogenous party formations. Finally, section 6 provides some concluding remarks.

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\[2\]There have been some recent developments in the literature on the theory of public goods provision with non-economically motivated preferences, which attempts to rationally explain why there are some voluntary associations like NPOs and/or NGOs which survive even under the threat of the free-rider problem and function as a part of public goods provisions. For instance, see Francois (2000; 2003; 2006).
2 Model

Let the set of voter types be $H$, the policy space be $\Upsilon$, a probability distribution of voter types in the polity be $F$ on $H$, and the utility function of type $h \in H$ over policies be $v(\cdot ; h)$. Let $v(\cdot ; h)$ be a non-negative real valued function for any $h \in H$. Let $(\tau^1, \tau^2) \in \Upsilon \times \Upsilon$ be a pair of policies. The set of voters who prefer $\tau^1$ to $\tau^2$ is denoted by $\Omega(\tau^1, \tau^2) \equiv \{ h \in H \mid v(\tau^1; h) \geq v(\tau^2; h) \}$.

Now, we impose the following assumption:

Assumption 1 (A1): For any $\tau, \tau^0 \in \Upsilon$ with $\tau^0 = \tau$, the set of voters who are indifferent between $\tau$ and $\tau^0$ is of $F$-measure zero.

Following Roemer (2001; Section 2.3), the fraction of voters who prefer policy $\tau^1$ to policy $\tau^2$ would be $F(\Omega(\tau^1, \tau^2))$. However, we assume that there is some aggregate uncertainty in how people will vote, so that the true fraction of the vote for $\tau^1$ will be $F(\Omega(\tau^1, \tau^2)) \pm \gamma$, where $\gamma > 0$ is an error term. Thus, the probability that $\tau^1$ defeats $\tau^2$ is:

$$\pi(\tau^1, \tau^2) = \begin{cases} 
0 & \text{if } F(\Omega(\tau^1, \tau^2)) + \gamma \leq \frac{1}{2} \\
\frac{F(\Omega(\tau^1, \tau^2)) + \gamma - \frac{1}{2}}{2\gamma} & \text{if } \frac{1}{2} \in (F(\Omega(\tau^1, \tau^2)) - \gamma, F(\Omega(\tau^1, \tau^2)) + \gamma) \\
1 & \text{if } F(\Omega(\tau^1, \tau^2)) - \gamma \geq \frac{1}{2}
\end{cases}$$

whenever $\tau^1 \neq \tau^2$, and $\pi(\tau^1, \tau^2) = \frac{1}{2}$ whenever $\tau^1 = \tau^2$. Thus, one political environment is generally denoted by a tuple $(H, F, \Upsilon, v, \gamma)$.

In this paper, each voter is characterized by his income $w \in \mathbb{R}_+$ and his ideological position $a \in [0, 1]$, where the ideological position indicates a person’s preference on the issue of how much the defense expenditure should be. Thus, the set of voters is specified by:

$$H = \{(w, a) \in W \times [0, 1] \mid W \equiv [w, \overline{w}] \subset \mathbb{R}_+\}.$$  

This population is characterized by two cumulative distribution functions $G(w)$ and $R(a)$.

The policy issue of this society is to choose a pair of proportional tax rate $t \in [0, 1]$ and a ratio of defense expenditure over tax revenue, $\alpha \in [0, 1]$. Thus, the policy space of this society is specified by:

$$\Upsilon = \{(t, \alpha) \mid t \in [0, 1] \text{ and } \alpha \in [0, 1]\}.$$  

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If the society chooses \((t, \alpha)\), then its tax revenue is \(t\mu\) per capita, where \(\mu\) is the average income of this society, and its defense expenditure is \(\alpha t\mu\) per capita. Then, \((1 - \alpha) t\mu\) is the subsidy which every citizen receives through an income redistribution policy. Thus, the choice of \((t, \alpha)\) implies the choice of income redistribution and military forces in this society. Any voter \((w, a)\) has the same utility function

\[
v(t, \alpha; w, a) = (1 - \beta) [(1 - t) w + (1 - \alpha) t\mu + \sigma (\alpha t\mu)] - \frac{\beta}{2} (\alpha t - a)^2 \tag{4}
\]

where \(\beta \in [0, 1]\), and \(a\) indicates this voter’s ideological position on the issue of defense expenditure. Also, the term \((1 - t) w + (1 - \alpha) t\mu\) represents voter’s after-tax income when the policy \((t, \alpha)\) is implemented; the term \(\sigma (\alpha t\mu)\) represents voter’s benefit from the national security supplied by the military forces; the term \(-\frac{\beta}{2} (\alpha t - a)^2\) represents voter’s satisfaction of political preference over the issue of military expenditure. Assume that the function \(\sigma\) is continuously differentiable, strictly concave, and monotonic. Finally, \(\beta\) is a weight of ideological views concerning military forces.

We impose the following additional condition for the function \(\sigma\):

**Assumption 2 (A2):**

\[
\text{lim}_{\lambda \to 0} \frac{\partial \sigma(\lambda \mu)}{\partial \lambda \mu} = +\infty, \text{ and for some } \lambda^* \in (0, 1), \frac{\partial \sigma(\lambda^* \mu)}{\partial \lambda^* \mu} = 1.
\]

The first component of A2 implies that the zero-defense expenditure is exclusively undesirable in terms of national security. The second component of A2 implies that if the national income is exhausted by defense expenditure, it would result in an excess supply of military forces. In other words, the optimal defense expenditure \(\lambda^* \mu\) does not require the exhaustion of the national income for military forces. Thus, \(\lambda^* \mu\) can be interpreted as a threshold, and so if a military expenditure exceeds \(\lambda^* \mu\), it implies that the main purpose of having this level of military force is not only to protect the citizens from foreign aggression, but also to act hegemonically against other nations by using the threat of this military force.

There are two political parties, Left \((L)\) and Right \((R)\), in the society. In the following sections 3 and 4, we suppose that the membership of both parties are exogenously fixed, and \(L\) represents a relatively poorer citizen \(w_L < \mu\), and its ideological position \(a_L\) indicates its preference to “peace,” or “antimilitarism.” In contrast, \(R\) represents a relatively richer citizen \(w_R > \mu\), and its ideological position \(a_R\) indicates its preference to “relatively stronger
military power.” Let $a^m$ denote the median ideological view: $R(a^m) = \frac{1}{2}$. We assume $a_L < a^m < a_R$. Let $v_L(t, \alpha) = v(t, \alpha; w_L, a_L)$ be $L$’s utility function, and $v_R(t, \alpha) = v(t, \alpha; w_R, a_R)$ be $R$’s utility function.

Let us define an equilibrium notion of this political game, that is, party-unanimity Nash equilibrium (PUNE), which was introduced by Roemer (1998; 1999; 2001; Chapter 8).

**Definition 1:** Given a political environment $\langle H, F, Y, v, \gamma \rangle$ specified above, and the two parties Left ($L$) and Right ($R$), a pair of policies $(\tau^L, \tau^R) \in Y \times Y$ with $\tau^i = (t_i, \alpha_i)$ $(i = L, R)$ constitutes a party-unanimity Nash equilibrium (PUNE) if $(\tau^L, \tau^R)$ satisfies the following:

(a) given $\tau_R^R$, there is no policy $\tau \in Y$ such that $\pi(\tau, \tau^R) \geq \pi(\tau^L, \tau^R)$ and $v_L(\tau) \geq v_L(\tau^L)$, with at least one strict inequality;

(b) given $\tau_L^R$, there is no policy $\tau \in Y$ such that $\pi(\tau^L, \tau) \leq \pi(\tau^L, \tau^R)$ and $v_R(\tau) \geq v_R(\tau^R)$, with at least one strict inequality.

In Definition 1, condition (a) says that facing the opponent’s proposal $\tau^R_R$, there is no policy in $Y$ that can improve the payoffs of all three factions in party $L$, and condition (b) makes the analogous statement for the factions of party $R$.\(^3\)

### 3 Political equilibrium in non-ideological society

We first consider a non-ideological society: $\beta = 0$. When $\beta = 0$, every voter’s utility function can be reduced to the following quasi-linear type:

$$v(t, \alpha; w, a) = (1 - t) w + (1 - \alpha) t \mu + \sigma (a \mu). \quad (5)$$

In this case, the relevant information of voter’s type is only his income level $w$, so that the voter space is uni-dimensional. However, the policy space is

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\(^3\)In this definition, there is no statement for the Reformists’ payoffs, since (2a) and (2b) describe the conditions for the Opportunists’ payoffs, $\pi(\cdot, \cdot)$ and $1 - \pi(\cdot, \cdot)$, and the Militants’ payoffs, $v_L$ and $v_R(\cdot)$, only. However, as Roemer (2001: Chapter 8; Theorem 8.1(3)) showed, the equilibrium set corresponding to this simpler definition of PUNE is equivalent to that of the rigorous definition of PUNE given in Roemer (2001: Chapter 8; Definition 8.1).
still two-dimensional. Given a pair of policies \((\tau^L, \tau^R) = ((t_L, \alpha_L), (t_R, \alpha_R))\), let us define:

\[
\Delta^{(w,a)}((t_L, \alpha_L), (t_R, \alpha_R)) \equiv v(t_L, \alpha_L; w, a) - v(t_R, \alpha_R; w, a). \tag{6}
\]

By definition, \(\Delta^{(w,a)}((t_L, \alpha_L), (t_R, \alpha_R)) > 0\) implies that the voter \((w, a)\) prefers \(L\) to \(R\) when \(L\) offers \((t_L, \alpha_L)\) and \(R\) offers \((t_R, \alpha_R)\).

Let \(\Delta t \equiv t_R - t_L\). Then, \(\Delta^{(w,a)}((t_L, \alpha_L), (t_R, \alpha_R)) > 0\) holds if and only if:

\[
w < \mu - \frac{\mu \cdot (\alpha_R t_R - \alpha_L t_L) + [\sigma(\alpha_L t_L \mu) - \sigma(\alpha_R t_R \mu)]}{\Delta t} \quad \text{if } \Delta t < 0, \tag{7a}
\]

\[
w > \mu - \frac{\mu \cdot (\alpha_R t_R - \alpha_L t_L) + [\sigma(\alpha_L t_L \mu) - \sigma(\alpha_R t_R \mu)]}{\Delta t} \quad \text{if } \Delta t > 0. \tag{7b}
\]

Note \(\Omega(\tau^L, \tau^R) = \{(w, a) \in H | \Delta^{(w,a)}(\tau^L, \tau^R) \geq 0\}\). Thus, the fraction of voters who prefer \(\tau^L = (t_L, \alpha_L)\) to \(\tau^R = (t_R, \alpha_R)\) is defined as:

\[
\mathbb{F}(\Omega(\tau^L, \tau^R)) = \int_{w}^{\mu-\Theta(\tau^L, \tau^R)} g(w)dw \quad \text{if } \Delta t < 0, \tag{8a}
\]

\[
\mathbb{F}(\Omega(\tau^L, \tau^R)) = \int_{\mu-\Theta(\tau^L, \tau^R)}^{w} g(w)dw \quad \text{if } \Delta t > 0, \tag{8b}
\]

where \(\Theta(\tau^L, \tau^R) = \frac{\mu \cdot (\alpha_R t_R - \alpha_L t_L) + [\sigma(\alpha_L t_L \mu) - \sigma(\alpha_R t_R \mu)]}{\Delta t}\)

and \(g(w)\) is a density on \(W\).

Now, we identify each party’s ideal policy. Let \((\bar{t}_L, \bar{\alpha}_L)\) be the ideal policy of \(L\). Then:

**Lemma 1:** Let A2 hold and \(\beta = 0\). Then, \((\bar{t}_L, \bar{\alpha}_L) = (1, \lambda^*), \) where \(\lambda^* = \lambda^*\).

Note that any voter whose income is lower than \(\mu\) has the same ideal policy as \(L\)’s in the non-ideological society \(\beta = 0\). This is because any voter receives the same level of benefit \(\sigma(\alpha t \mu)\) from the military policy, and the different character between \(L\) and \(R\) is due to their respective income positions.

Let \((\bar{t}_R, \bar{\alpha}_R)\) be the ideal policy of \(R\).

**Lemma 2:** Let A2 hold and \(\beta = 0\). Then, \((\bar{t}_R, \bar{\alpha}_R) = (t^* , 1), \) where \(t^*\) is the solution of \(\frac{\partial \sigma(t^* \mu)}{\partial \mu} = \frac{w_R}{\mu}\).
Note that any voter \( h \) whose income \( w_h \) is higher than \( \mu \) has the same ideal military policy as \( R \)'s in the non-ideological society \( \beta = 0 \). However, his ideal tax policy \( t^h \) has the property \( \frac{\partial \sigma(t^h \mu)}{\partial t \mu} = \frac{w_h}{\mu} > 1 \), which differs from \( R \)'s. Note if \( w_h = \mu \), then \( \frac{\partial \sigma(t^h \mu)}{\partial t \mu} = 1 \), which implies \( t^h = \lambda^* \). Moreover, this voter is indifferent between \((1, \lambda^*)\) and \((\lambda^*, 1)\). Since both \((1, \lambda^*)\) and \((\lambda^*, 1)\) are his own ideal policies, he prefers \( L \) to \( R \). Thus, some voter \( h \) with \( w_h > \mu \) may still prefer \( L \) to \( R \). In fact, any voter whose income is higher than \( \mu \), but lower than \( \mu - \Theta(\tau^L, \tau^R)(\mu) \), prefers \( L \) to \( R \).

By Lemmas 1 and 2, we also know \( \alpha^* \mu > t^* \mu \), that is, the ideal supply of military forces by \( L \) is higher than that by \( R \), since \( w_R > \mu \). This result might be reasonable in the non-ideological society, since in that model, \( L \) represents only the relatively poorer citizen, and \( R \) only the relatively richer citizen. Note that the poor need a higher level of national security than the rich, because the rich may easily flee their home country when its national security is worse off, whereas the poor may not have such an alternative.

We consider a society, in which more than half the voters have income less than the mean. This implies the following reasonable assumption:

**Assumption 3 (A3):** \( G(\mu) > \frac{1}{2} \).

We will characterize the set of PUNEs in the game \( \langle H, F, L, R, \Upsilon, v, \gamma \rangle \) with \( \beta = 0 \). Denote any PUNE in this game by \( (\tau^L(0), \tau^R(0)) = ((t_L(0), \alpha_L(0)), (t_R(0), \alpha_R(0))) \).

**Lemma 3:** Let \( A1, A2, \) and \( A3 \) hold. Let \( \gamma \) be small enough. Then, there is a unique PUNE such that \( t_L(0) \leq t_R(0) \). This PUNE is: \( (\tau^L(0), \tau^R(0)) = ((1, \alpha^*), (1, \alpha^*)) \).

**Theorem 1:** Let \( A1, A2, \) and \( A3 \) hold. Let \( \gamma \) be small enough. Then, in the game with \( \beta = 0 \), there are only two PUNEs: \( ((\overline{t}_L, \overline{\alpha}_L), (\overline{t}_R, \overline{\alpha}_R)) \) and \( ((\overline{t}_L, \overline{\alpha}_L), (\overline{t}_L, \overline{\alpha}_L)) \), where \( (\overline{t}_L, \overline{\alpha}_L) = (1, \alpha^*) \) and \( (\overline{t}_R, \overline{\alpha}_R) = (t^*, 1) \). Moreover, in the first PUNE, we have \( \pi((\overline{t}_L, \overline{\alpha}_L), (\overline{t}_L, \overline{\alpha}_L)) = 1 \).

**Remark 1:** In this political model with \( \beta = 0 \), there is no Wittman equilibrium. This is because any Wittman equilibrium is a PUNE, but in the game of Theorem 1, there are only two types of PUNEs, neither of which is a Wittman equilibrium. In contrast, we can show there is one unique Down-sian equilibrium in this game, which coincides with the the second PUNE \( ((\overline{t}_L, \overline{\alpha}_L), (\overline{t}_L, \overline{\alpha}_L)) \) of Theorem 1.
Theorem 1 says that in the political issue of budget assignment between income redistribution and defense expenditure, L’s ideal policy is always implemented when voters’ preferences on military forces are solely economic-motivated. There is no room for compromise to shift to a more Right wing policy; R may win with probability one-half in the second political equilibrium, but in this case, it should implement L’s ideal policy. This is exclusively implausible, since in real political competitions, R can often win and implement a more Right-wing policy. Also, we should mention that a typical phenomenon in the two party political system is the frequent change in office between the two parties. A plausible model should be able to explain the frequent change in office of real two-party systems.

4 Political equilibrium in the society with ideological concern

Here, let us consider a society with ideological concern: \( \beta > 0 \). When \( \beta > 0 \), every voter’s utility function is represented by (4). In this case, the relevant information of voter’s type is not only his income level \( w \), but also his ideological position \( a \). So, the voter space is also two-dimensional as well as the policy space.

Given a pair of policies \( (\tau^L, \tau^R) = ((t_L, \alpha_L), (t_R, \alpha_R)) \), \( \Delta^{(w,a)} ((t_L, \alpha_L), (t_R, \alpha_R)) \) is defined by (6) as in the case of \( \beta = 0 \). By definition, \( \Delta^{(w,a)} ((t_L, \alpha_L), (t_R, \alpha_R)) > 0 \) implies that the voter \( (w, a) \) prefers \( L \) to \( R \) when \( L \) offers \( (t_L, \alpha_L) \) and \( R \) offers \( (t_R, \alpha_R) \).

Let \( \Delta t \equiv t_R - t_L, \Delta \alpha t \equiv \alpha_R t_R - \alpha_L t_L, \) and \( \overline{\alpha t} \equiv \frac{1}{2} (\alpha_R t_R + \alpha_L t_L) \). Then, \( \Delta^{(w,a)} ((t_L, \alpha_L), (t_R, \alpha_R)) > 0 \) holds if and only if:

\[
a < \frac{\Delta t \cdot \Delta \alpha t}{\alpha t} + \frac{1 - \beta}{\beta} \left[ \Delta t \left( w - \mu \right) + \Delta \alpha t \cdot \left\{ \sigma \left( \alpha_L t_L \mu - \sigma \left( \alpha_R t_R \mu \right) \right) \right\} \right] \quad \text{if } \Delta \alpha t > 0, \quad (9a)
\]

\[
a > \frac{\Delta t \cdot \Delta \alpha t}{\alpha t} + \frac{1 - \beta}{\beta} \left[ \Delta t \left( w - \mu \right) + \Delta \alpha t \cdot \left\{ \sigma \left( \alpha_L t_L \mu - \sigma \left( \alpha_R t_R \mu \right) \right) \right\} \right] \quad \text{if } \Delta \alpha t < 0, \quad (9b)
\]

\[
w > \mu \quad \text{if } \Delta \alpha t = 0 \text{ and } \Delta t > 0, \quad (9c)
\]

\[
w < \mu \quad \text{if } \Delta \alpha t = 0 \text{ and } \Delta t < 0. \quad (9d)
\]

Note \( \Omega(\tau^L, \tau^R) = \{(w, a) \in H \mid \Delta^{(w,a)} (\tau^L, \tau^R) \geq 0\} \). Thus, the fraction of
voters who prefer $\tau^L = (t_L, \alpha_L)$ to $\tau^R = (t_R, \alpha_R)$ is defined as:

$$\mathbb{F}(\Omega(\tau^L, \tau^R)) = \frac{\int_{W} \int_{0}^{\alpha + \Phi_w(\tau^L, \tau^R; \beta)} g(w)r(a; w) \, da \, dw}{\beta \cdot \Delta \alpha}$$

if $\Delta \alpha > 0$, \hspace{1cm} (10a)

$$\mathbb{F}(\Omega(\tau^L, \tau^R)) = \frac{\int_{W} \int_{1}^{\alpha + \Phi_w(\tau^L, \tau^R; \beta)} g(w)r(a; w) \, da \, dw}{\beta \cdot \Delta \alpha}$$

if $\Delta \alpha < 0$, \hspace{1cm} (10b)

$$\mathbb{F}(\Omega(\tau^L, \tau^R)) = \int_{\mu}^\infty g(w) \, dw$$

if $\Delta \alpha = 0$ and $\Delta t > 0$, \hspace{1cm} (10c)

$$\mathbb{F}(\Omega(\tau^L, \tau^R)) = \int_{\mu}^\infty g(w) \, dw$$

if $\Delta \alpha = 0$ and $\Delta t < 0$, \hspace{1cm} (10d)

where $\Phi_w(\tau^L, \tau^R; \beta) = \frac{(1 - \beta) \left[ \Delta t (w - \mu) + \Delta \alpha \cdot \{\sigma(\alpha_L t_L \mu) - \sigma(\alpha_R t_R \mu)\} \right]}{\beta \cdot \Delta \alpha}$,

and $g(w)$ is a density on $W$, and $r(a; w)$ is a density of ideological position $a \in [0, 1]$ at the population with income level $w$.

Now, we are ready to analyze PUNEs in the political competition game $\langle H, \mathbb{F}, L, R, \Upsilon, v, \gamma \rangle$ with $\beta > 0$. Denote any PUNE in the game $\langle H, \mathbb{F}, L, R, \Upsilon, v, \gamma \rangle$ with $\beta > 0$ by $(\tau^L(\beta), \tau^R(\beta)) = ((t_L(\beta), \alpha_L(\beta)), (t_R(\beta), \alpha_R(\beta)))$. Denote a pair of ideal policies of the Militants of both parties in $\langle H, \mathbb{F}, L, R, \Upsilon, v, \gamma \rangle$ with $\beta > 0$ by $(\tau^L(\beta), \tau^R(\beta)) = ((\Upsilon_L(\beta), \Upsilon_L(\beta)), (\Upsilon_R(\beta), \Upsilon_R(\beta)))$. Note $(\tau^L(\beta), \tau^R(\beta))$ is also a PUNE in the game $\langle H, \mathbb{F}, L, R, \Upsilon, v, \gamma \rangle$ with $\beta > 0$. We would like to examine the existence of the following refinement of PUNE:

**Definition 2:** Given $\langle H, \mathbb{F}, L, R, \Upsilon, v, \gamma \rangle$ with $\beta \in [0, 1]$, and the two parties Left (L) and Right (R), a pair of policies $(\tau^L(\beta), \tau^R(\beta)) \in \Upsilon \times \Upsilon$ constitutes a non-trivial and non-pure PUNE if $(\tau^L(\beta), \tau^R(\beta))$ is a PUNE such that $0 < \pi(\tau^L(\beta), \tau^R(\beta)) < 1$, $\tau^L(\beta) \neq \tau^L(\beta)$, and $\tau^R(\beta) \neq \tau^R(\beta)$.

First, we will consider the polar case $\beta = 1$.

**Theorem 2:** Let $A1$ hold. Let $\beta = 1$. Then, for each $\gamma > 0$, there is some positive number $\epsilon(\gamma) > 0$ such that any pair of policies $(\tau^L, \tau^R) = ((t_L, \alpha_L), (t_R, \alpha_R))$ with $\frac{1}{2} (\alpha_R t_R + \alpha_L t_L) \in (a^m - \epsilon(\gamma), a^m + \epsilon(\gamma))$ and $\alpha_R > \alpha_R t_R \geq \alpha_L t_L > a_L$ constitutes a non-trivial and non-pure PUNE $(\tau^L(1), \tau^R(1))$ in the game $\langle H, \mathbb{F}, L, R, \Upsilon, v, \gamma \rangle$ with $\beta = 1$. Moreover, in this game, there is no other type of non-trivial and non-pure PUNE.

Theorem 2 shows there are many non-trivial PUNEs in another polar case $\beta = 1$. The set of non-trivial PUNEs when $\beta = 1$ is given by the Figure.
Figure 1 around here.

The result of the game with $\beta = 1$ still remains unreasonable. This is because the set of non-trivial and non-pure PUNEs entails the case $\alpha_R t_R \geq \alpha_L t_L$ with $(1 - \alpha_R) \cdot t_R > (1 - \alpha_L) \cdot t_L$, which implies $R$ proposes a stronger welfare policy than $L$. Since real politics indicate the opposite behavior, the polar case $\beta = 1$ is still inappropriate for our subject, and we will shift our attention to the case $0 < \beta < 1$.

We consider the following population:

**Assumption 4 (A4):** The mean income of the cohort of voters with the median ideological view $a^m$ is less than mean income, $\mu$, of the population:

$$\int_W g(w)r(a^m; w) \cdot (w - \mu) \, dw < 0.$$ 

Combining with A3, A4 seems to be a natural condition. Then:

**Theorem 3:** Let A1, A2, and A3 hold. Then, the strategy profile $(t^L(\beta), t^R(\beta)) = ((t_L(\beta), \alpha_L(\beta)), (t_R(\beta), \alpha_R(\beta)))$ constitutes a non-trivial and non-pure PUNE in the game $\langle H, F, L, R, T, v, \gamma \rangle$ with any $\beta > 0$ close enough to one, if and only if it holds that $((t_L(\beta), \alpha_L(\beta)), (t_R(\beta), \alpha_R(\beta))) = (1, \alpha^*_L, (t^*_R, 1))$ with $\frac{1}{2} (t^*_R + \alpha^*_L) \in (a^m - \epsilon(\gamma), a^m + \epsilon(\gamma))$ and $\alpha^*_R > t^*_R > \alpha^*_L > a_L$ for $\epsilon(\gamma) > 0$.

Moreover, for any $\beta > 0$ close enough to one, if A4 holds, then there is no other type of non-trivial and non-pure PUNE for this $\epsilon(\gamma) > 0$.

The non-emptiness and a characterization of the set of non-trivial and non-pure PUNEs are discussed in Theorem 3. These PUNEs seem to describe a natural political competition in the real world; in such a PUNE, $L$ proposes a stronger social welfare policy for the poor and a weaker military force, whereas $R$ proposes a stronger military force and a lower tax rate for the rich, and neither party can win the vote with probability one, which implies the possibility of frequent change in office between these two parties. These seem to capture typical phenomena of political competition in advanced democratic countries after World War II. By both Theorem 3 and Theorem 1, we may conjecture that the existence of ideological views over military policies gives us a game-theoretic reasoning of the stylized party behaviors in those countries.
5 In Case of Endogenous Party Formations

In this section, we consider the case that the memberships of both parties are endogenously formed. In this case, the two parties will each represent a coalition of voter types: thus, there will be a partition of the set of voter types

\[ H = L \cup R, \ L \cap R = \emptyset. \]

Each party will represent its members, in the sense that the party’s preferences will be the average preferences of its members; that is, we define the parties’ utility functions on \( \Upsilon \) by:

\[
v_L(\tau) = (1 - \beta) [(1 - t) w_L + (1 - \alpha) t \mu + \sigma (at \mu)] - \frac{\beta}{2} (at - a_L)^2;
\]

and

\[
v_R(\tau) = (1 - \beta) [(1 - t) w_R + (1 - \alpha) t \mu + \sigma (at \mu)] - \frac{\beta}{2} (at - a_R)^2,
\]

where

\[
w_L \equiv \int_{h \in L} w_h dF(h), \ a_L \equiv \int_{h \in L} a_h dF(h),
\]

\[
w_R \equiv \int_{h \in R} w_h dF(h), \ a_R \equiv \int_{h \in R} a_h dF(h).
\]

We are ready to define an equilibrium notion of this multi-dimensional political competition game, party-unanimity Nash equilibria with endogenous parties (PUNEEPs), which was introduced by Roemer (2001; Chapter 13).

**Definition 3:** A partition of voter types \( L, R \), and a pair of policies \((\tau^L, \tau^R)\) \( \in \Upsilon \times \Upsilon \) with \( \tau^i = (t_i, \alpha_i) \) \( (i = L, R) \) constitutes a party-unanimity Nash equilibrium with endogenous parties (PUNEEP) if:

1. \( H = L \cup R \) and \( L \cap R = \emptyset \);
2. \((\tau^L, \tau^R)\) satisfies Definition 1(a) and (b);
3. for all \( h \in L \), \( v(\tau^L; h) \geq v(\tau^R; h) \) and for all \( h \in R \), \( v(\tau^L; h) < v(\tau^R; h) \).

In Definition 3, condition (3) states that party membership is stable in the sense that every party member prefers his party’s policy to the opponent’s policy. By this condition, the coalition of those who vote for a party and the coalition whom the party represents are identical. Such a condition was used in Baron (1993) in the context of endogenous party formation, and was treated more generally in Caplin and Nalebuff (1997).
First, consider $\beta = 0$, which is the case of non-ideological society with endogenous party formation. Given a party membership $R$, let $t\left(\frac{w_R}{\mu}\right)$ be the tax rate satisfying $\frac{\partial \pi}{\partial \mu}(t\left(\frac{w_R}{\mu}\right)) = \frac{w_R}{\mu}$ for $w_R = \int_{h \in R} w_h d\mathcal{F}(h)$. Then:

**Theorem 5:** Let A1, A2, and A3 hold. Let $\gamma$ be small enough. Then, in the game with $\beta = 0$, there exists $h^* \in H$ with $w_{h^*} > \mu$ such that $L = \{h \in H \mid w_h \leq w_{h^*}\}$ and $R = \{h \in H \mid w_h > w_{h^*}\}$ associated with two PUNEEPs: $((\bar{t}_L, \bar{\alpha}_L), (\bar{t}_R, \bar{\alpha}_R))$ and $((\bar{t}_L, \bar{\alpha}_L), (\bar{t}_L, \bar{\alpha}_L))$, where $(\bar{t}_L, \bar{\alpha}_L) = (1, \alpha^*)$ and $(\bar{t}_R, \bar{\alpha}_R) = \left(t\left(\frac{w_R}{\mu}\right), 1\right)$. Moreover, in the first PUNEEP, we have $\pi((\bar{t}_L, \bar{\alpha}_L), (\bar{t}_R, \bar{\alpha}_R)) = 1$.

Second, consider the case of a society with ideological concern and endogenous party formation. We would like to examine the existence of a non-trivial and non-pure PUNEEP.

**Definition 4:** A partition of voter types $L$, $R$, and a pair of policies $(\tau^L(\beta), \tau^R(\beta)) \in \Upsilon \times \Upsilon$ with $\tau^i = (t_i, \alpha_i) (i = L, R)$ constitutes a non-trivial and non-pure PUNEEP in the game $\langle H, F, L, R, \Upsilon, \nu, \gamma \rangle$ with $\beta$ if it is a PUNEEP such that $\tau^L(\beta) \neq \tau^R(\beta)$, $0 < \pi(\tau^L(\beta), \tau^R(\beta)) < 1$, $\tau^L(\beta) \neq \tau^L(\beta)$, and $\tau^R(\beta) \neq \tau^R(\beta)$.

First, consider $\beta = 1$.

**Theorem 6:** Let A1 hold. Let $\beta = 1$. Then, for each $\gamma > 0$, there is a positive number $\epsilon(\gamma) > 0$ such that for any $\bar{\alpha} \in (a^m - \epsilon(\gamma), a^m + \epsilon(\gamma))$, any pair of policies $(\tau^L, \tau^R) = ((t_L, \alpha_L), (t_R, \alpha_R))$ with \( \frac{1}{2} (\alpha_R t_R + \alpha_L t_L) = \bar{\alpha} \) and \( \int_{h \in L} a_h d\mathcal{F}(h) < \alpha_L t_L < \alpha_R t_R < \int_{h \in R} a_h d\mathcal{F}(h) \), where $L = \{h \in H \mid a_h \leq \bar{\alpha}\}$ and $R = \{h \in H \mid a_h > \bar{\alpha}\}$, constitutes a non-trivial and non-pure PUNEEP $(\tau^L(1), \tau^R(1))$ in the game $\langle H, F, L, R, \Upsilon, \nu, \gamma \rangle$ with $\beta = 1$. Moreover, in this game, there is no other type of non-trivial and non-pure PUNEEP.

Second, consider $\beta > 0$ close to one. Then:

**Theorem 6:** Let A1, A2, and A3 hold. Then, in the game $\langle H, F, L, R, \Upsilon, \nu, \gamma \rangle$ with any $\beta > 0$ close enough to one, there exists a non-trivial and non-pure PUNEEP $(\tau^L(\beta), \tau^R(\beta)) = ((1, \alpha^*_L), (t^*_R, 1))$ with \( \frac{1}{2} (t_R + \alpha^*_L) \in (a^m - \epsilon(\gamma), a^m + \epsilon(\gamma)) \) and \( \int_{h \in R} a_h d\mathcal{F}(h) > t^*_R \geq \alpha^*_L > \int_{h \in L} a_h d\mathcal{F}(h) \), where $L = \Omega(\tau^L(\beta), \tau^R(\beta))$ and $R = H \setminus \Omega(\tau^L(\beta), \tau^R(\beta))$, for $\epsilon(\gamma) > 0$. Moreover, for any $\beta > 0$ close
enough to one, if $A4$ holds and $\int_0^\mu \int_0^1 ag(w) r(a; w) daw \leq a^m$, then there is no other type of non-trivial and non-pure PUNEEP for this $\epsilon(\gamma) > 0$.

Remark 2: In Theorem 6, if $\int_0^\mu \int_0^1 ag(w) r(a; w) daw \leq a^m$ does not hold, then $(\tau^L(\beta), \tau^R(\beta)) = ((\alpha^*_L, 1), (1, t^*_R))$ constitutes another type of non-trivial and non-pure PUNEEP, and except these two types, there is no other type.

6 Concluding Remarks

In this paper, we have discussed the existence and the characterizations of PUNEs in two-dimensional political games for military and social welfare policies. Through the whole discussion, we have shown that the existence of non-economically motivated, ideological views on military policies is a crucial factor giving us a game-theoretic reasoning of the stylized party behaviors. In fact, Theorems 3 and 6 in this paper indicate that if the median ideological view $a^m$ is in the neighborhood of the threshold level $\lambda^*$ of military expenditure, then we can find a PUNE in which the Right proposes an imperialistic military policy $t_R > \lambda^*$, regardless of whether party formations are exogenous or endogenous. Moreover, in the case that $a^m$ exceeds $\lambda^*$, we can find a PUNE in which even the Left proposes an imperialistic military policy $a_L > \lambda^*$. Such a phenomenon has been an historical fact, as the behaviors of Germany’s Social Democratic Party in the lead-up to World War I and of the Japanese Social Democratic Party in the lead-up to World War II, whereas the political games with pure economically motivated preferences in section 3 cannot produce such a phenomenon as an equilibrium.

Note that Roemer (1998) also discussed that the existence of non-economically motivated preferences is a crucial factor in explaining real politics in the U.S. Our model seems to be similar to Roemer (1998), but there are some significant differences, so the results also have different implications.\footnote{Roemer (1998) discussed that the poor do not expropriate the rich through democratic policy making, because he may vote for the Right-wing party according to his religious views. Roemer (1998) obtained this message by comparing a uni-dimensional space of redistribution policies with a two-dimensional space of redistribution and religious policies.}
7 Appendix

7.1 Proofs in Section 3

Proof of Lemma 1: To characterize $L$’s ideal policy, we have:

$$
\frac{\partial v_L}{\partial \alpha}(\bar{\tau}_L, \bar{\alpha}_L) = -\bar{\tau}_L \mu + \frac{\partial \sigma(\bar{\alpha}_L \bar{\tau}_L \mu)}{\partial \alpha \mu} \bar{\tau}_L \mu. \quad (11)
$$

From (11), we know that $\frac{\partial v_L}{\partial \alpha}(\bar{\tau}_L, \bar{\alpha}_L) \leq 0 \iff \frac{\partial \sigma(\bar{\alpha}_L \bar{\tau}_L \mu)}{\partial \alpha \mu} \leq 1$. Note if $\frac{\partial v_L}{\partial \alpha}(\bar{\tau}_L, \bar{\alpha}_L) > 0$, then $\bar{\alpha}_L = 1$, and if $\frac{\partial v_L}{\partial \alpha}(\bar{\tau}_L, \bar{\alpha}_L) < 0$, then $\bar{\alpha}_L = 0$. Also, we have:

$$
\frac{\partial v_L}{\partial \alpha}(\bar{\tau}_L, \bar{\alpha}_L) = -w_L + (1 - \bar{\alpha}_L) \mu + \frac{\partial \sigma(\bar{\alpha}_L \bar{\tau}_L \mu)}{\partial \alpha \mu} \bar{\alpha}_L \mu. \quad (12)
$$

Suppose $\frac{\partial v_L}{\partial \alpha}(\bar{\tau}_L, \bar{\alpha}_L) > 0$. Then, (12) implies $\frac{\partial v_L}{\partial \alpha}(\bar{\tau}_L, \bar{\alpha}_L) = -w_L + \frac{\partial \sigma(\bar{\alpha}_L \bar{\tau}_L \mu)}{\partial \alpha \mu} \mu > 0$, since $w_L < \mu$ and $\frac{\partial \sigma(\bar{\alpha}_L \bar{\tau}_L \mu)}{\partial \alpha \mu} > 1$. Thus, $(\bar{\tau}_L, \bar{\alpha}_L) = (1, 1)$ should hold. However, this implies $\frac{\partial \sigma(\bar{\alpha}_L \bar{\tau}_L \mu)}{\partial \alpha \mu} = \frac{\partial \sigma(\mu)}{\partial \alpha \mu} > 1$, which is a contradiction, due to A2. Thus, $\frac{\partial v_L}{\partial \alpha}(\bar{\tau}_L, \bar{\alpha}_L) > 0$ is impossible.

Suppose $\frac{\partial v_L}{\partial \alpha}(\bar{\tau}_L, \bar{\alpha}_L) < 0$. Then, (12) implies $\frac{\partial v_L}{\partial \alpha}(\bar{\tau}_L, \bar{\alpha}_L) = -w_L + \mu > 0$, so that $(\bar{\tau}_L, \bar{\alpha}_L) = (1, 0)$ should hold. However, this implies that $\frac{\partial \sigma(\bar{\alpha}_L \bar{\tau}_L \mu)}{\partial \alpha \mu} = \frac{\partial \sigma(0)}{\partial \alpha \mu} > 1$, which is a contradiction, due to $\frac{\partial v_L}{\partial \alpha}(\bar{\tau}_L, \bar{\alpha}_L) < 0 \iff \frac{\partial \sigma(\bar{\alpha}_L \bar{\tau}_L \mu)}{\partial \alpha \mu} < 1$.

Thus, $\frac{\partial v_L}{\partial \alpha}(\bar{\tau}_L, \bar{\alpha}_L) < 0$ is impossible.

Let $\frac{\partial \sigma(\bar{\alpha}_L \bar{\tau}_L \mu)}{\partial \alpha \mu} = 0$. Then, (12) implies $\frac{\partial v_L}{\partial \alpha}(\bar{\tau}_L, \bar{\alpha}_L) = -w_L + \mu > 0$, so that $\bar{\tau}_L = 1$. Then, $\bar{\alpha}_L$ should meet the condition $\frac{\partial \sigma(\bar{\alpha}_L \bar{\tau}_L \mu)}{\partial \alpha \mu} = 1$, which implies $\bar{\alpha}_L = \lambda^*$. Thus, $(\bar{\tau}_L, \bar{\alpha}_L) = (1, \alpha^*)$, where $\alpha^* = \lambda^*$.  

Proof of Lemma 2: Since $R$’s ideal policy is characterized by $\frac{\partial v_R}{\partial \alpha}(\bar{\tau}_R, \bar{\alpha}_R) = -\bar{\tau}_R \mu + \frac{\partial \sigma(\bar{\alpha}_R \bar{\tau}_R \mu)}{\partial \alpha \mu} \bar{\tau}_R \mu$, we obtain $\frac{\partial v_R}{\partial \alpha}(\bar{\tau}_R, \bar{\alpha}_R) \geq 0$ if and only if $\frac{\partial \sigma(\bar{\alpha}_R \bar{\tau}_R \mu)}{\partial \alpha \mu} \geq 1$.

Let us consider $\frac{\partial \sigma(\bar{\alpha}_R \bar{\tau}_R \mu)}{\partial \alpha \mu} = 1$. Then,

$$
\frac{\partial v_R}{\partial \alpha}(\bar{\tau}_R, \bar{\alpha}_R) = -w_R + (1 - \bar{\alpha}_R) \mu + \frac{\partial \sigma(\bar{\alpha}_R \bar{\tau}_R \mu)}{\partial \alpha \mu} \bar{\alpha}_R \mu = -w_R + \mu < 0. \quad (13)
$$
Thus, $\bar{t}_R = 0$, so that $\bar{\alpha}_R \bar{t}_R \mu = 0$, which implies $\frac{\partial \sigma(\bar{\pi}_R \bar{t}_R \mu)}{\partial \alpha \mu} > 1$, that being a desired contradiction. Let us consider $\frac{\partial \sigma(\bar{\pi}_R \bar{t}_R \mu)}{\partial \alpha \mu} < 1$. Then, since $\frac{\partial v_R(\bar{t}_R, \bar{\pi}_R)}{\partial \alpha} < 0$, it holds $\bar{\alpha}_R = 0$, which implies $\frac{\partial \sigma(\bar{\pi}_R \bar{t}_R \mu)}{\partial \alpha \mu} > 1$, also a contradiction.

Let us consider $\frac{\partial \sigma(\bar{\pi}_R \bar{t}_R \mu)}{\partial \alpha \mu} > 1$. Then, $\bar{\pi}_R = 1$, and the right hand side of (13) is reduced to $- \left( \frac{w_R}{\mu} - 1 \right) \mu + \left( \frac{\partial \sigma(\bar{\pi}_R \bar{t}_R \mu)}{\partial \alpha \mu} - 1 \right) \mu$. Thus, $\frac{\partial v_R(\bar{t}_R, \bar{\pi}_R)}{\partial t} = 0$ if and only if $\frac{w_R}{\mu} - 1 = \frac{\partial \sigma(\bar{\pi}_R \bar{t}_R \mu)}{\partial \alpha \mu} - 1$. Since $\lim_{\lambda \to 0} \frac{\partial \sigma(\lambda \mu)}{\partial \lambda \mu} = +\infty$ and $\frac{\partial \sigma(\lambda^* \mu)}{\partial \lambda^* \mu} = 1$ by A2, there exists $t^*$ such that $\frac{\partial \sigma(t^* \mu)}{\partial \mu} = \frac{w_R}{\mu}$. Thus, $(\bar{t}_R, \bar{\pi}_R) = (t^*, 1)$ constitutes the solution. \[\blacksquare\]

**Proof of Lemma 3:** (Case 1): Let us take a pair of policies $(\tau^L, \tau^R) = (t_L, \alpha_L, t_R, \alpha_R)$ such that $t_L < t_R$. Then, (7b) and (8b) are applied. Note that, in this case, $-\Theta(\tau^L, \tau^R) \geq 0$ if and only if $$\Xi((t_L, \alpha_L), (t_R, \alpha_R)) \equiv (\alpha_R t_L - \alpha_L t_L) \cdot \mu + \{\sigma(\alpha_L t_L \mu) - \sigma(\alpha_R t_R \mu)\} \leq 0.$$ 

(Case 1-1): Let $\Xi((t_L, \alpha_L), (t_R, \alpha_R)) \geq 0$. Then, consider a deviation from $(t_L, \alpha_L)$ to $(t'_L, \alpha'_L)$ with $\alpha'_L t'_L = \alpha_L t_L$, $\alpha'_L < \alpha_L$, and $t_R > t'_L > t_L$. Since $t_L < t_R$, we can always find such a strategy. Then, $\Xi((t'_L, \alpha'_L), (t_R, \alpha_R)) = \Xi((t_L, \alpha_L), (t_R, \alpha_R)) \geq 0$ and $t_R - t'_L < t_R - t_L$, which implies $0 \leq \Theta(\tau^L, \tau^R) \leq \Theta(\tau^L, \tau^R)$, so that $\mathbb{F}(\Omega(\tau^L, \tau^R)) \geq \mathbb{F}(\Omega(\tau^L, \tau^R))$. Moreover, we have that $v_L((t'_L, \alpha'_L), (t_R, \alpha_R)) - v_L((t_L, \alpha_L), (t_R, \alpha_R)) > 0,$ since $$v_L((t'_L, \alpha'_L), (t_R, \alpha_R)) - v_L((t_L, \alpha_L), (t_R, \alpha_R)) = - (t'_L - t_L) w_L + (t'_L - t_L) \mu - (\alpha'_L t'_L - \alpha_L t_L) \mu + \sigma(\alpha'_L t'_L \mu) - \sigma(\alpha_L t_L \mu) = (t'_L - t_L) (\mu - w_L) > 0,$$ since $w_L < \mu$, (14) which implies that this deviation can improve the L-militant’s payoff without hurting the L-opportunists.

(Case 1-2): Let $\Xi((t_L, \alpha_L), (t_R, \alpha_R)) < 0$. In this case, since $-\Theta(\tau^L, \tau^R) > 0$, we have $\mathbb{F}(\Omega(\tau^L, \tau^R)) < \frac{1}{2}$ by (8b) and A3, which implies $\pi(\tau^L, \tau^R) = 0$ for $\gamma > 0$ small enough. Thus, $L$ can improve its militant’s payoff by deviating to $(t'_L, \alpha'_L) = (1, \alpha^*)$ without hurting its opportunist.

In summary, any strategy profile $((t_L, \alpha_L), (t_R, \alpha_R))$ with $t_L < t_R$ cannot be a PUNE.
(Case2): Consider \( t_L = t_R \). Then, \( \Delta^{(w,a)}((t_L, \alpha_L), (t_R, \alpha_R)) > 0 \) holds if and only if \( \Xi((t_L, \alpha_L), (t_R, \alpha_R)) > 0 \). \( L \) would like to maximize \( \Xi((t_L, \alpha_L), (t_R, \alpha_R)) \), while \( R \) would like to minimize \( \Xi((t_L, \alpha_L), (t_R, \alpha_R)) \).

(Case2-1): Let \( 0 < t_L = t_R < 1 \). Then, \( L \) can choose \( \alpha'_Lt_L, \alpha'_L < \alpha_L \), and \( t'_L > t_L \), so that \( v_L((t'_L, \alpha'_L), (t_R, \alpha_R)) - v_L((t_L, \alpha_L), (t_R, \alpha_R)) > 0 \). Thus, the \( L \)-militant is better off by this deviation without hurting its opportunist.

(Case2-2): Let \( t_L = t_R = 0 \). Then, \( \pi((t_L, \alpha_L), (t_R, \alpha_R)) = \frac{1}{2} \). Note, when \( t_L = t_R = 0 \), any choice of \( \alpha \) is irrelevant to both parties’ militants and opportunists. Then, consider \( \tau^L_0 = (t'_L, \alpha'_L) \) as \( t'_L > 0 \) and \( \alpha'_L = 0 \). Then, since \( t'_L > t_R \), \( F(\Omega(\tau^L_0, \tau^R_0)) \) is determined by (8a) with \( -\Theta(\tau^L_0, \tau^R_0) = 0 \). Thus, by A3, \( F(\Omega(\tau^L_0, \tau^R_0)) > \frac{1}{2} \), which implies that \( \pi(\tau^L_0, \tau^R_0) \geq \frac{1}{2} \). Moreover, by (15), the \( L \)-militant’s payoff is also improved by deviation from \( \tau^L_0 \) to \( \tau^L_0 \). Thus, this case does not correspond to PUNE, either.

(Case2-3): Let \( t_L = t_R = 1 \). Note, if the tax rate is fixed by one, the optimal ratio of defense expenditure is \( \alpha^* > 0 \) for every individual, where the last strict inequality implies that

\[
\frac{\partial v_L(1, \alpha^*; w, a)}{\partial \alpha} = -\mu + \frac{\partial \sigma(\alpha^*\mu)}{\partial \alpha\mu} \mu = 0,
\]

which in turn implies \( \frac{\partial \sigma(\alpha^*\mu)}{\partial \alpha\mu} = 1 \). Even for the \( R \)-militant, \( \alpha_R = \alpha^* \) is his best strategy, given tax rate \( t_L = t_R = 1 \). That is, given tax rate equal to one, there is a unanimous single peak of satisfaction about the ratio of defense expenditure. Thus, for any \( (\tau^L, \tau^R) = ((1, \alpha_L), (1, \alpha_R)) \), if \( \alpha_R \neq \alpha^* \neq \alpha_L \) and \( \pi(\tau^L, \tau^R) < (\text{resp.} \geq) 1 \), then \( L \) (\text{resp.} \( R \)) can improve its militant’s and opportunist’s payoffs by moving to \((1, \alpha^*)\). If \( \alpha_R = \alpha^* \neq \alpha_L \), then \( \pi(\tau^L, \tau^R) = 0 \). Then, \( L \) can improve its militant’s and opportunist’s payoffs by moving to \((1, \alpha^*)\). If \( \alpha_R \neq \alpha^* \neq \alpha_L \), then \( \pi(\tau^L, \tau^R) = 1 \), and \( R \) can be better off by moving to \((1, \alpha^*)\).

In summary, except \((1, \alpha^*), (1, \alpha^*)\), no pair of policies \((t_L, \alpha_L), (t_R, \alpha_R)\) can be a PUNE whenever \( t_L \leq t_R \).

(Case 3): Consider \( t_L = t_R = 1 \) and \( \alpha_L = \alpha_R = \alpha^* \). Then, \( L \) offers its ideal policy. For \( R \), any deviation from \( \alpha_R \) to \( \alpha'_R \) with keeping \( t'_R = 1 \) makes its opportunist worse off, since \( \Xi((1, \alpha^*), (1, \alpha'_R)) > 0 \). Now, we will show that given \( \alpha_L t_L = \alpha^* \), any deviation from \( \alpha_R t_R = \alpha^* \) to \( \alpha'_R t'_R \neq \alpha^* \) leads to \( \Xi((t_L, \alpha_L), (t_R', \alpha'_R)) > 0 \).

If \( \alpha'_R t'_R < \alpha^* \), then \((\alpha'_R t'_R - \alpha_L t_L) \cdot \mu < 0 \) and \( |\sigma(\alpha_L t_L \mu) - \sigma(\alpha'_R t'_R \mu)| > 0 \). By the strict concavity of \( \sigma \), it holds \( \sigma(\alpha_L t_L \mu) - \sigma(\alpha'_R t'_R \mu) > |(\alpha'_R t'_R - \alpha_L t_L) \cdot \mu| \),
which implies $\Xi((t_L, \alpha_L), (t'_R, \alpha'_R)) > 0$. If $\alpha'_R t'_R > \alpha^*$, then $(\alpha'_R t'_R - \alpha_L t_L) \cdot \mu > 0$ and $\{\sigma (\alpha_L t_L) - \sigma (\alpha'_R t'_R)\} < 0$. Since $\frac{\partial \sigma (\alpha'_R t'_R)}{\partial \alpha'_R t'_R} - \frac{\partial \sigma (\alpha^*)}{\partial \alpha} = 1$ by A2, it holds $(\alpha'_R t'_R - \alpha_L t_L) \cdot \mu > |\sigma (\alpha_L t_L) - \sigma (\alpha'_R t'_R)|$, which implies $\Xi((t_L, \alpha_L), (t'_R, \alpha'_R)) > 0$.

Let $t'_R < t_R = t_L = 1$. Then, $\mathbb{F} (\Omega(\tau^L, \tau^R))$ is defined by (8a), and $\Xi((t_L, \alpha_L), (t'_R, \alpha'_R)) > 0$ implies $-\Theta(\tau^L, \tau^R) > 0$, so that $\mathbb{F} (\Omega(\tau^L, \tau^R)) > \frac{1}{2}$ by A3. Thus, if $t_L = t_R = 1$ and $\alpha_L = \alpha_R = \alpha^*$, any deviation from $\alpha_R t_R = \alpha^*$ to $\alpha'_R t'_R \neq \alpha^*$ makes the $R$-opportunist worse off. Finally, consider a deviation strategy $\tau^R = (t'_R, \alpha'_R)$, where $\alpha'_R t'_R = \alpha^*$, $\alpha'_R > \alpha^*$, and $t'_R < t_R$. Then, $\Xi((t_L, \alpha_L), (t'_R, \alpha'_R)) = 0$, but since $t'_R < t_L$, $\mathbb{F} (\Omega(\tau^L, \tau^R))$ is defined by (8a), and it becomes

$$
\mathbb{F} (\Omega(\tau^L, \tau^R)) = \int_{\omega}^{\mu - \Theta(\tau^L, \tau^R)} g(w) dw = \int_{\omega}^{\mu} g(w) dw > \frac{1}{2},
$$

by A3. Hence, $\pi(\tau^L, \tau^R) > \frac{1}{2} = \pi(\tau^L, \tau^R)$. Therefore, such a deviation makes the $R$-opportunist worse off, although it makes the $R$-militant better off. This implies that if $t_L = t_R = 1$ and $\alpha_L = \alpha_R = \alpha^*$, $\tau^R = (t_R, \alpha_R)$ is a best reply for $R$ to $(t_L, \alpha_L)$. In summary, $(\tau^L(0), \tau^R(0)) = ((1, \alpha^*), (1, \alpha^*))$ constitutes a PUNE.

**Proof of Theorem 1:** By Lemma 3, we have already seen that $\Xi((t_L, \alpha_L), (t_R, \alpha_R))$ is the unique PUNE whenever $t_L \leq t_R$. Thus, we only need to check $t_L > t_R$. Let take such a pair of policies $(\tau^L, \tau^R) = ((t_L, \alpha_L), (t_R, \alpha_R))$.

Then, $\Delta t < 0$, so $\mathbb{F} (\Omega(\tau^L, \tau^R))$ is defined by (8a), and $-\Theta(\tau^L, \tau^R) \geq 0$ if and only if $\Xi((t_L, \alpha_L), (t_R, \alpha_R)) > 0$.

**Case 1:** Consider $\Xi((t_L, \alpha_L), (t_R, \alpha_R)) > 0$. In this case, since $-\Theta(\tau^L, \tau^R) > 0$, we have $\mathbb{F} (\Omega(\tau^L, \tau^R)) > \frac{1}{2}$ by A3, which implies $\pi(\tau^L, \tau^R) = 1$ for small enough $\gamma > 0$. Then, if $\tau^R \neq (\bar{t}_R, \bar{\alpha}_R)$, then $R$ can improve its militant’s payoff by moving to $(\bar{t}_R, \bar{\alpha}_R)$ without hurting its opportunist. If $\tau^R = (\bar{t}_R, \bar{\alpha}_R)$ and $\tau^L \neq (\bar{t}_L, \bar{\alpha}_L)$, then $L$ can improve its militant’s payoff by moving to $(\bar{t}_L, \bar{\alpha}_L)$ without hurting its opportunist, since $\pi((\bar{t}_L, \bar{\alpha}_L), (\bar{t}_R, \bar{\alpha}_R)) = 1$ as we will discuss below.

**Case 2:** Consider $\Xi((t_L, \alpha_L), (t_R, \alpha_R)) \leq 0$. Note that if $\alpha_R t_R \neq \alpha^* = \alpha_L t_L$, then $\Xi((t_L, \alpha_L), (t_R, \alpha_R)) > 0$.

**Case 2-1:** If $\alpha_R t_R = \alpha^* = \alpha_L t_L$, then $\Xi((t_L, \alpha_L), (t_R, \alpha_R)) < 0$. In this case, $L$ can improve its militant’s and opportunist’s payoffs by moving to $\tau^L = (\bar{t}_L, \bar{\alpha}_L)$, since $\Xi((\bar{t}_L, \bar{\alpha}_L), (t_R, \alpha_R)) = 0$, which implies $\mathbb{F} (\Omega(\tau^L, \tau^R)) > \frac{1}{2}$.
\( \frac{1}{2} \) by A3, and so \( \pi(\tau^L, \tau^R) = 1 \) for small enough \( \gamma > 0 \). Moreover, \((\bar{t}_L, \bar{\tau}_L)\) is the ideal policy of \( L \).

(Case 2-2): If \( \alpha_R t_R \neq \alpha^* \neq \alpha_L t_L \) with \( \Xi((t_L, \alpha_L), (t_R, \alpha_R)) \leq 0 \), then \( L \) can improve its militant’s and opportunist’s payoffs by moving to \( \tau^{L'} = (\bar{t}_L, \bar{\tau}_L) \), since \( \Xi((\bar{t}_L, \bar{\tau}_L), (t_R, \alpha_R)) > 0 \), so the same scenario as Case 2-1 can be applied.

(Case 2-3): If \( \alpha_R t_R = \alpha^* = \alpha_L t_L \), then \( \Xi((t_L, \alpha_L), (t_R, \alpha_R)) = 0 \) which implies \( F(\Omega(\tau^L, \tau^R)) > \frac{1}{2} \) by A3, and so \( \pi(\tau^L, \tau^R) = 1 \) for small enough \( \gamma > 0 \). Then, if \( (t_L, \alpha_L) \neq (\bar{t}_L, \bar{\tau}_L) \), then \( L \) can improve its militant by moving to \((\bar{t}_L, \bar{\tau}_L)\) without hurting its opportunist. If \( (t_L, \alpha_L) = (\bar{t}_L, \bar{\tau}_L) \), and \( (t_R, \alpha_R) \neq (\bar{t}_R, \bar{\alpha}_R) \), then \( R \) can improve its militant’s payoff by moving to \((\bar{t}_R, \bar{\alpha}_R)\) without hurting its opportunist.

In summary, there exists no PUNE with \( t_L > t_R \), except \( (\tau^L, \tau^R) = ((\bar{t}_L, \bar{\tau}_L), (\bar{t}_R, \bar{\alpha}_R)) \).

(Case 3): Note that \((\bar{t}_L, \bar{\tau}_L), (\bar{t}_R, \bar{\alpha}_R)\) is a PUNE, and it is the unique PUNE such that \( \tau^L \neq \tau^R \) with \( t_L > t_R \) in the game with \( \beta = 0 \) by the above argument. In this case,

\[
\Xi((t_L, \alpha_L), (t_R, \alpha_R)) = \Xi((1, \alpha^*), (t^*, 1)) = (t^* - \alpha^*) \cdot \mu + \{\sigma(\alpha^* \mu) - \sigma(t^* \mu)\}.
\]

Note that \( t^* < \alpha^* \) by \( \frac{\partial \Omega(t^*)}{\partial \mu} = \frac{w}{\mu} > 1 \). Thus, \( |t^* - \alpha^*| \cdot \mu < \sigma(\alpha^* \mu) - \sigma(t^* \mu) \), and so \( \Xi((\bar{t}_L, \bar{\tau}_L), (\bar{t}_R, \bar{\alpha}_R)) > 0 \), which implies \( \pi((\bar{t}_L, \bar{\tau}_L), (\bar{t}_R, \bar{\alpha}_R)) = 1 \) for small enough \( \gamma > 0 \). Combining with Lemma 3, we can conclude that there are only two PUNES, \((\bar{t}_L, \bar{\tau}_L), (\bar{t}_R, \bar{\alpha}_R)\) and \((\bar{t}_L, \bar{\alpha}_L), (\bar{t}_R, \bar{\tau}_L)\), in the game with \( \beta = 0 \) for small enough \( \gamma > 0 \).

### 7.2 Proofs in Section 4

Note, when \( \Delta \alpha t > 0 \), we have for the \( L \)-opportunist:

\[
\frac{\partial \mathbb{E}(\Omega(\tau^L, \tau^R))}{\partial t_L} = \int_W g(w) r(\alpha t + \Phi_w(\tau^L, \tau^R; \beta); w) \cdot \left[ \frac{\alpha_L}{2} + \frac{1 - \beta}{\beta \Delta \alpha t} \Lambda^t (\tau^L, \tau^R; w) \right] dw \quad (15)
\]

where

\[
\Lambda^t (\tau^L, \tau^R; w) = \left\{ (\mu - w) + \alpha_L + \frac{\partial \sigma(\alpha_L \mu \tau^L)}{\partial \alpha t \mu} \alpha_L \mu + \frac{\alpha_L}{\Delta \alpha t} \sum (\tau^L, \tau^R; w) \right\},
\]

and
\[ \Sigma (\tau^L, \tau^R; w) \equiv \left[ \Delta t (w - \mu) + \Delta \alpha t \cdot \{ \sigma (\alpha_L t_L \mu) - \sigma (\alpha_R t_R \mu) \} \right], \]

and

\[
\frac{\partial \mathcal{F} \left( \Omega(\tau^L, \tau^R) \right)}{\partial \alpha_L} = \int_W g(w) r \left( \overline{\alpha t} + \Phi_w (\tau^L, \tau^R; \beta); w \right) \cdot \left[ \frac{t_L}{2} + \frac{1 - \beta}{\beta \Delta \alpha t} \Lambda^{\alpha_L} (\tau^L, \tau^R; w) \right] \, dw \quad (16)
\]

where

\[ \Lambda^{\alpha_L} (\tau^L, \tau^R; w) \equiv -t_L + \frac{\partial \sigma (\alpha_L t_L \mu)}{\partial \alpha_L \mu} t_L \mu + \frac{t_L}{\Delta \alpha t} \Sigma (\tau^L, \tau^R; w). \]

Also, we have for the L-militant:

\[
\frac{\partial v_L (\tau^L, \tau^R)}{\partial t_L} = (1 - \beta) \left[ -w_L + (1 - \alpha_L) \mu + \frac{\partial \sigma (\alpha_L t_L \mu)}{\partial \alpha_L \mu} \alpha_L \mu \right] - \beta \alpha_L (\alpha_L t_L - a_L), \quad (17)
\]

\[ \frac{\partial v_L (\tau^L, \tau^R)}{\partial \alpha_L} = (1 - \beta) \left[ -t_L \mu + \frac{\partial \sigma (\alpha_L t_L \mu)}{\partial \alpha_L \mu} t_L \mu \right] - \beta t_L (\alpha_L t_L - a_L). \quad (18)
\]

In the same way, when \( \Delta \alpha t > 0 \), we have for the R-opportunist:

\[
\frac{\partial \mathcal{F} \left( \Omega(\tau^L, \tau^R) \right)}{\partial \alpha_R} = \int_W g(w) r \left( \overline{\alpha t} + \Phi_w (\tau^L, \tau^R; \beta); w \right) \cdot \left[ \frac{\alpha_R}{2} + \frac{1 - \beta}{\beta \Delta \alpha t} \Lambda^{\alpha_R} (\tau^L, \tau^R; w) \right] \, dw \quad (19)
\]

where

\[ \Lambda^{\alpha_R} (\tau^L, \tau^R; w) \equiv \left\{ (w - \mu) + \alpha_R - \frac{\partial \sigma (\alpha_R t_R \mu)}{\partial \alpha_R \mu} \alpha_R \mu - \frac{\alpha_R}{\Delta \alpha t} \Sigma (\tau^L, \tau^R; w) \right\}, \]

and

\[
\frac{\partial \mathcal{F} \left( \Omega(\tau^L, \tau^R) \right)}{\partial \alpha_R} = \int_W g(w) r \left( \overline{\alpha t} + \Phi_w (\tau^L, \tau^R; \beta); w \right) \cdot \left[ \frac{t_R}{2} + \frac{1 - \beta}{\beta \Delta \alpha t} \Lambda^{\alpha_R} (\tau^L, \tau^R; w) \right] \, dw \quad (20)
\]

where

\[ \Lambda^{\alpha_R} (\tau^L, \tau^R; w) \equiv \left\{ t_R - \frac{\partial \sigma (\alpha_R t_R \mu)}{\partial \alpha_R \mu} t_R \mu - \frac{t_R}{\Delta \alpha t} \Sigma (\tau^L, \tau^R; w) \right\}. \]
Also, we have for the $R$-militant:

$$
\frac{\partial v_R (\tau^L, \tau^R)}{\partial t_R} = (1 - \beta) \left[ -w_R + (1 - \alpha_R) \mu + \frac{\partial \sigma (\alpha_R t_R \mu)}{\partial \alpha t_R \mu} \alpha_R \mu \right] - \beta \alpha_R (\alpha_R t_R - a_R), \quad (21)
$$

$$
\frac{\partial v_R (\tau^L, \tau^R)}{\partial \alpha_R} = (1 - \beta) \left[ -t_R \mu + \frac{\partial \sigma (\alpha_R t_R \mu)}{\partial \alpha t_R \mu} t_R \mu \right] - \beta \alpha_R (\alpha_R t_R - a_R). \quad (22)
$$

Proof of Theorem 2: Let us take a pair of policies $(\tau^L, \tau^R) = ((t_L, \alpha_L), (t_R, \alpha_R))$ with $a_R > \alpha_R t_R \geq \alpha_L t_L > a_L$. In this case, since $\beta = 1$, the fraction of voters who prefer $L$ to $R$ can be reduced to:

$$
\mathbb{F} \left( \Omega (\tau^L, \tau^R) \right) = \int_0^{\alpha_l} r(a) \, da, \quad (23)
$$

where $\alpha_l = \frac{1}{2} (\alpha_R t_R + \alpha_L t_L)$. Given $\gamma > 0$, we can identify the maximal number $\epsilon (\gamma) > 0$ such that:

$$
\int_0^{a^m - \epsilon (\gamma)} r(a) \, da \geq \max \left\{ \frac{1}{2} - \gamma, 0 \right\} \quad \text{and} \quad \int_0^{a^m + \epsilon (\gamma)} r(a) \, da \leq \min \left\{ \frac{1}{2} + \gamma, 1 \right\}.
$$

Note that $\epsilon (\gamma)$ is an increasing function.

Suppose that $\alpha_l \in (a^m - \epsilon (\gamma), a^m + \epsilon (\gamma))$. Then, we shall show that this $(\tau^L, \tau^R)$ is a non-trivial and non-pure PUNE. By (23), we know that $\mathbb{F} \left( \Omega (\tau^L, \tau^R) \right) \in \left( \frac{1}{2} - \gamma, \frac{1}{2} + \gamma \right)$, so that $0 < \pi (\tau^L, \tau^R) < 1$. By looking (23) and (4) with $\beta = 0$, we can see that the deviation from $\alpha_L t_L$ to $\alpha_L t_L + \epsilon$ (resp. $\epsilon$) for any $\epsilon > 0$ can improve the $L$-opportunist (resp. the $L$-militant), but make the $L$-militant (resp. the $L$-opportunist) worse off. The same argument is also applied for $R$, so that we can see that $(\tau^L, \tau^R)$ is a non-trivial and non-pure PUNE.

Suppose that $\alpha_l \notin (a^m - \epsilon (\gamma), a^m + \epsilon (\gamma))$. Then, $\mathbb{F} \left( \Omega (\tau^L, \tau^R) \right) < \max \left\{ \frac{1}{2} - \gamma, 0 \right\}$ or $\mathbb{F} \left( \Omega (\tau^L, \tau^R) \right) > \min \left\{ \frac{1}{2} + \gamma, 1 \right\}$, which implies that $\pi (\tau^L, \tau^R) = 0$ or $\pi (\tau^L, \tau^R) = 1$. Thus, this case does not correspond to any non-trivial non-pure PUNE.

Suppose that $\alpha_R \leq \alpha_R t_R$. If $a_R = \alpha_R t_R$, then it is the ideal policy of $R$ when $\beta = 1$. Thus, it does not correspond to any non-trivial non-pure PUNE. If $a_R < \alpha_R t_R$, then deviation from $\alpha_R t_R$ to $\alpha_R t_R - \epsilon$ can make both the militant and the opportunist better off. In a similar way, we can see that $\alpha_L t_L \leq a_L$ does not correspond to any non-trivial and non-pure PUNE.
Suppose that $\alpha_R t_R < \alpha_L t_L$. Then, at least one of these two parties can have a deviation to make its militant and opportunist better off. Thus, this case does not correspond to any non-trivial and non-pure PUNE. ■

**Proof of Theorem 3:** By Theorem 2, we know that, any pair $((1, \alpha^*_L), (t^*_R, 1))$ with $\frac{1}{2} (t^*_R + \alpha^*_L) \in (a^m - \epsilon(\gamma), a^m + \epsilon(\gamma))$ and $a_R > t^*_R \geq \alpha^*_L > a_L$ for some $\epsilon(\gamma) > 0$ is a non-trivial and non-pure PUNE in the game $\langle H, F, L, R, T, v, \gamma \rangle$ with $\beta = 1$. Note that $\epsilon(\gamma)$ is determined in Theorem 2. We shall show this pair would be a non-trivial and non-pure PUNE in the game $\langle H, F, L, R, T, v, \gamma \rangle$ with $\beta > 0$ close to one. Note that for $\beta > 0$ close to one, the optimal policy for the $L$-militant is $T_L(\beta, \sigma_L(\beta))$ with $\sigma_L(\beta) = 1$ and $\sigma_L(\beta)$ is close to $a_L$, whereas the optimal policy for the $R$-militant is $T_R(\beta, \sigma_R(\beta))$ with $T_R(\beta)$ is close to $a_R$ and $\sigma_R(\beta) = 1$. Thus, w. l. o. g, we still keep the condition $T_R(\beta) > t^*_R \geq \alpha^*_L > \sigma_L(\beta)$.

(1) $(1, \alpha^*_L)$ is a best response for Left to $(t^*_R, 1)$ in the game $\langle H, F, L, R, T, v, \gamma \rangle$ with $\beta > 0$ close to one.

Suppose that there is $\beta > 0$, which is close enough to one, such that $(1, \alpha^*_L)$ is not a best response for $L$ to $(t^*_R, 1)$ in the game $\langle H, F, L, R, T, v, \gamma \rangle$ with this $\beta$. Then, there exists a vector $(-1, \delta(\beta))$ such that $\nabla^\beta v_L ((1, \alpha^*_L), (t^*_R, 1)) \cdot (-1, \delta(\beta)) > 0$ and $\nabla^\beta F_L ((1, \alpha^*_L), (t^*_R, 1)) \cdot (-1, \delta(\beta)) \geq 0$, where

$$
\nabla^\beta v_L ((1, \alpha^*_L), (t^*_R, 1)) = \left( \frac{\partial v_L ((1, \alpha^*_L), (t^*_R, 1); \beta)}{\partial t_L}, \frac{\partial v_L ((1, \alpha^*_L), (t^*_R, 1); \beta)}{\partial \alpha_L} \right)
$$

and

$$
\nabla^\beta F_L ((1, \alpha^*_L), (t^*_R, 1)) = \left( \frac{\partial F_L (\Omega(\tau_L, \tau_R); \beta)}{\partial t_L}, \frac{\partial F_L (\Omega(\tau_L, \tau_R); \beta)}{\partial \alpha_L} \right).
$$

Since $\nabla^\beta v_L ((1, \alpha^*_L), (t^*_R, 1)) \cdot (-1, \delta(\beta)) > 0$, we have to extrapolate from (17) and (18):

$$
\delta(\beta) > \frac{(1 - \beta) \left[ -w_L + (1 - \alpha^*_L) \mu + \frac{\partial \sigma (\alpha^*_L \mu)}{\partial \alpha_L} \alpha^*_L \mu \right] - \beta \alpha^*_L (\alpha^*_L - a_L)}{(1 - \beta) \left[ -\mu + \frac{\partial \sigma (\alpha^*_L \mu)}{\partial \alpha_L} \mu \right] - \beta (\alpha^*_L - a_L)}. \tag{24}
$$

By the choice of $\alpha^*_L$, $\nabla^1 v_L ((1, \alpha^*_L), (t^*_R, 1)) \cdot (-1, \delta(1)) > 0$, so that $\delta(1) > \alpha^*_L$ holds. This implies that

$$
\delta(\beta) > \alpha^*_L \text{ for } \beta \text{ close enough to one.}
$$
Now, let us consider \( \nabla^\beta \mathbb{F}_L ((1, \alpha^*_L), (t^*_R, 1)) \cdot (-1, \alpha^*_L) \). Then,

\[
\nabla^\beta \mathbb{F}_L ((1, \alpha^*_L), (t^*_R, 1)) \cdot (-1, \alpha^*_L) = -\frac{1 - \beta}{\beta \Delta \alpha t} \int_W g(w) r(\alpha t + \Phi_w(\tau^L, \tau^R; \beta); w) \cdot (w - \mu) \, dw.
\]

Let us rewrite \( \delta(\beta) = \alpha^*_L + \varepsilon(\beta) \), where \( \varepsilon(\beta) > 0 \) for \( \beta > 0 \) close enough to one. Then,

\[
\nabla^\beta \mathbb{F}_L ((1, \alpha^*_L), (t^*_R, 1)) \cdot (-1, \delta(\beta)) = \varepsilon(\beta) \frac{1}{2} \int_W g(w) r(\alpha t + \Phi_w(\tau^L, \tau^R; \beta); w) \, dw
\]

\[
+ \varepsilon(\beta) \frac{1 - \beta}{\beta \Delta \alpha t} \int_W g(w) r(\alpha t + \Phi_w(\tau^L, \tau^R; \beta); w) \cdot (w - \mu) \, dw
\]

\[
\geq 0 \quad \text{(by supposition)}. \quad (25)
\]

By repeating the argument to derive (24) and (25), we can see that for any \( \beta' \in [\beta, 1] \), there exists a vector \((-1, \delta(\beta')) \equiv (-1, \delta(\beta)) \) such that \( \nabla^\beta v_L ((1, \alpha^*_L), (t^*_R, 1)) \cdot (-1, \delta(\beta')) > 0 \) and \( \nabla^\beta \mathbb{F}_L ((1, \alpha^*_L), (t^*_R, 1)) \cdot (-1, \delta(\beta')) \geq 0 \). In particular, we see that \( \nabla^1 v_L ((1, \alpha^*_L), (t^*_R, 1)) \cdot (-1, \delta(1)) > 0 \) and \( \nabla^1 \mathbb{F}_L ((1, \alpha^*_L), (t^*_R, 1)) \cdot (-1, \delta(1)) \geq 0 \), which is a contradiction, since \(((1, \alpha^*_L), (t^*_R, 1))\) is a non-trivial and non-pure \textbf{PUNE} when \( \beta = 1 \). Thus, \((1, \alpha^*_L)\) is a best response for \( L \) to \((t^*_R, 1)\) in the game \( \langle H, \mathbb{F}, L, R, \Upsilon, v, \gamma \rangle \) with \( \beta > 0 \) close enough to one.

\( (2) \) \((t^*_R, 1)\) is a best response for Right to \((1, \alpha^*_L)\) in the game \( \langle H, \mathbb{F}, L, R, \Upsilon, v, \gamma \rangle \) with \( \beta > 0 \) close to one.

Suppose that there is \( \beta > 0 \), which is close enough to one, so that \((t^*_R, 1)\) is not a best response for \( R \) to \((1, \alpha^*_L)\) in the game \( \langle H, \mathbb{F}, L, R, \Upsilon, v, \gamma \rangle \) with this \( \beta \). Then, there exists a vector \((\delta(\beta), -1)\) such that \( \nabla^\beta v_R ((1, \alpha^*_L), (t^*_R, 1)) \cdot (\delta(\beta), -1) > 0 \) and \( \nabla^\beta \mathbb{F}_R ((1, \alpha^*_L), (t^*_R, 1)) \cdot (\delta(\beta), -1) \leq 0 \), where

\[
\nabla^\beta v_R ((1, \alpha^*_L), (t^*_R, 1)) = \left( \frac{\partial v_R ((1, \alpha^*_L), (t^*_R, 1); \beta)}{\partial t_R}, \frac{\partial v_R ((1, \alpha^*_L), (t^*_R, 1); \beta)}{\partial \alpha_R} \right)
\]

and \( \nabla^\beta \mathbb{F}_R ((1, \alpha^*_L), (t^*_R, 1)) = \left( \frac{\partial \mathbb{F} (\Omega(\tau^L, \tau^R); \beta)}{\partial t_R}, \frac{\partial \mathbb{F} (\Omega(\tau^L, \tau^R); \beta)}{\partial \alpha_R} \right) \).

24
Since $\nabla^\beta v_R ((1, \alpha^*_L), (t^*_R, 1)) \cdot (\delta(\beta), -1) > 0$, we have to extrapolate from (21) and (22):

$$\delta(\beta) > \frac{(1 - \beta) \left[-t^*_R \mu + \frac{\partial \sigma(t^*_R \mu)}{\partial \alpha^*_L} t^*_R \mu \right] - \beta t^*_R (t^*_R - a_R)}{(1 - \beta) \left[-w_R + \frac{\partial \sigma(t^*_R \mu)}{\partial \alpha^*_L} \mu \right] - \beta (t^*_R - a_R)}.$$  

By the choice of $t^*_R$, $\nabla^1 v_R ((1, \alpha^*_L), (t^*_R, 1)) \cdot (\delta(1), -1) > 0$, so that $\delta(1) > t^*_R$ holds. This implies that

$$\delta(\beta) > t^*_R \text{ for } \beta > 0 \text{ close enough to one.} \quad (26)$$

Let $\delta(\beta) = t^*_R + \varepsilon(\beta)$ where $\varepsilon(\beta) > 0$. Consider

$$\nabla^\beta F_R ((1, \alpha^*_L), (t^*_R, 1)) \cdot (\delta^*, -1)$$

$$= t^*_R \frac{1 - \beta}{\beta \Delta \alpha^*_L} \int_W g(w)r \left( \overline{\alpha t} + \Phi_w(\tau^L_+, \tau^R_+; \beta); w \right) \cdot (w - \mu) dw.$$  

So, we have:

$$\nabla^\beta F_R ((1, \alpha^*_L), (t^*_R, 1)) \cdot (\delta(\beta), -1)$$

$$= \varepsilon(\beta) \frac{1}{2} \int_W g(w)r \left( \overline{\alpha t} + \Phi_w(\tau^L_+, \tau^R_+; \beta); w \right) dw$$

$$+ \varepsilon(\beta) \frac{1 - \beta}{\beta \Delta \alpha^*_L} \int_W g(w)r \left( \overline{\alpha t} + \Phi_w(\tau^L_+, \tau^R_+; \beta); w \right) \cdot \Lambda^w (\tau^L_+, \tau^R_+; w) dw$$

$$+ t^*_R \frac{1 - \beta}{\beta \Delta \alpha^*_L} \int_W g(w)r \left( \overline{\alpha t} + \Phi_w(\tau^L_+, \tau^R_+; \beta); w \right) \cdot (w - \mu) dw$$

$$\leq 0 \quad \text{(by supposition).} \quad (27)$$

This implies

$$\lim_{\beta \to 1} \nabla^\beta F_R ((1, \alpha^*_L), (t^*_R, 1)) \cdot (\delta(\beta), -1)$$

$$= \varepsilon(1) \frac{\alpha_R}{2} \int_W g(w)r \left( \overline{\alpha t} + \Phi_w(\tau^L_+, \tau^R_+; \beta); w \right) dw$$

$$\leq 0 \quad \text{(from (27)).} \quad (28)$$

Note that (27) and (28) imply $\varepsilon(\beta) \leq 0$ for $\beta > 0$ close enough to one. This is a contradiction, since $\varepsilon(\beta) > 0$ by the fact that $\nabla^\beta v_R ((1, \alpha^*_L), (t^*_R, 1)) \cdot$
(δ(β), −1) > 0 and (26). Thus, we may conclude that \((t^*_R, 1)\) is a best response for \(R\) to \((1, α_L^*)\) in the game \(<H, F, L, R, T, v, γ>\) with any \(β > 0\) close to one.

(3) Any other type of strategy profile cannot constitute a non-trivial and non-pure PUNE in the game \(<H, F, L, R, T, v, γ>\) with \(β > 0\) close to one.

First, let us consider any strategy \((t_L, α_L)\) with \(α_L t_L = α_L^*\) and \(t_L < 1\), and any strategy \((t_R, α_R)\) with \(α_R t_R = t_R^*\) and \(α_R < 1\). Then, for \(L\), it can improve both its militant and its opportunist by deviating to \((1, α_L^*)\). For \(R\), it can improve both its militant and its opportunist by deviating to \((t_R^*, 1)\). This is easy to verify for the militants of both parties. Let us examine for the opportunist of both parties.

For the \(L\)-opportunist, we need to examine whether \(∇βF_L((t_L, α_L), (t_R, α_R))\cdot(1 − t_L, α_L^* − α_L) > 0\) holds or not. Note

\[(1 − t_L) α_L + (α_L^* − α_L) t_L = (1 − t_L) α_L + (t_L α_L − α_L) t_L = (1 − t_L) α_L − (1 − t_L) α_L t_L > 0\]

Then, by using (15) and (16), we obtain

\[\begin{align*}
∇βF_L((t_L, α_L), (t_R, α_R)) \cdot (1 − t_L, α_L^* − α_L) &= \int_W g(w) r(\overline{αt} + Φ_w(τ^L, τ^R; β); w) \cdot \left[ Ψ_L + \frac{1 − β}{βΔαt} (1 − t_L) (μ − w) \right] dw,
\end{align*}\]

where \(Ψ_L = ((1 − t_L) α_R + (α_L^* − α_L) t_L) \left[ \frac{1}{2} + \frac{1 − β}{βΔαt} \left( \frac{∂σ(α_L t_L, μ)}{∂μ} + \frac{Σ(τ^L, τ^R, w)}{Δαt} \right) \right]\)

\[\frac{(1 − β) \cdot |(1 − t_L)α_R − (α_L^* − α_L) t_L|}{βΔαt}.\]

Note that \(Ψ_L > 0\), since \((1 − t_L) α_L + (α_L^* − α_L) t_L > 0\) and \(α_L^* − α_L < 0\). By construction of this strategy profile, \(\overline{αt} + Φ(τ^L, τ^R; β)\) is close to \(\alpha^m\). Thus, by A4, we can see that

\[\frac{1 − β}{βΔαt} (1 − t_L) \int_W g(w) r(\overline{αt} + Φ_w(τ^L, τ^R; β); w) (μ − w) dw > 0.\]

Thus, \(∇βF_L((t_L, α_L), (t_R, α_R)) \cdot (1 − t_L, α_L^* − α_L) > 0\) holds.

For the \(R\)-opportunist, we need to examine whether \(∇βF_R((t_L, α_L), (t_R, α_R))\cdot(t_R^* − t_R, 1 − α_R) > 0\) holds or not. Note

\[(t_R^* − t_R) α_R + (1 − α_R) t_R = (t_R α_R − t_R) α_R + (1 − α_R) t_R = (1 − α_R) t_R − (1 − α_R) t_R α_R > 0.\]
Then, by using (19) and (20), we obtain
\[
\nabla^\beta \mathbb{F}_R \left( (t_L, \alpha_L), (t_R, \alpha_R) \right) \cdot (t_R^* - t_R, 1 - \alpha_R)
\]
\[
= \int_W g(w) \left( \frac{1 - \beta}{\beta \Delta \alpha t} \right) \left( t_R^* - t_R \right) (w - \mu) \, dw,
\]
where \( \Psi_R = \left( (t_R^* - t_R) \alpha_R + (1 - \alpha_R) t_R \right) \left[ \frac{1}{2} + \frac{1 - \beta}{\beta \Delta \alpha t} \left( 1 - \frac{\partial \sigma(\alpha_L, \alpha_R, \mu)}{\partial \alpha} \right) \right]. \)

Note that \( t_R^* - t_R < 0 \), we can see, by (19) and (20), that
\[
\frac{1 - \beta}{\beta \Delta \alpha t} (t_R^* - t_R) \int_W g(w) \left( \frac{1 - \beta}{\beta \Delta \alpha t} \right) \left( t_R^* - t_R \right) (w - \mu) \, dw > 0.
\]
Thus, \( \nabla^\beta \mathbb{F}_R \left( (t_L, \alpha_L), (t_R, \alpha_R) \right) \cdot (t_R^* - t_R, 1 - \alpha_R) > 0 \) holds.

Consider any strategy profile \( (\tau^L, \tau^R) = ((t_L, \alpha_L), (t_R, \alpha_R)) \) with \( t_L = 1, \alpha_R = 1, \) and \( \frac{1}{2} (t_R + \alpha_L) \notin (a^m - \epsilon(\gamma), a^m + \epsilon(\gamma)) \). Then, for any \( \beta > 0 \) close enough to one, \( \tau \left( \tau^L, \tau^R \right) \in \{0, 1\} \) and so the profile does not correspond to any non-trivial and non-pure PUNE.

Consider any strategy profile \( (\tau^L, \tau^R) = ((t_L, \alpha_L), (t_R, \alpha_R)) \) with \( t_L = 1, \alpha_R = 1, \frac{1}{2} (t_R + \alpha_L) \in (a^m - \epsilon(\gamma), a^m + \epsilon(\gamma)) \), and \( \alpha_L < t_R \). Then, \( \tau \) can improve both its militant and opportunist by deviating from \( (t_R, \alpha_R) \) to \( (t_R - \varepsilon, \alpha_R) \) whenever \( \beta > 0 \) is close enough to one.

Consider any strategy profile \( (\tau^L, \tau^R) = ((t_L, \alpha_L), (t_R, \alpha_R)) \) with \( t_L = 1, \alpha_R = 1, \frac{1}{2} (t_R + \alpha_L) \in (a^m - \epsilon(\gamma), a^m + \epsilon(\gamma)) \), and \( \alpha_L < a_L \). Then, \( \tau \) can improve both its militant and opportunist by deviating from \( (t_L, \alpha_L) \) to \( (t_L, \alpha_L + \varepsilon) \) whenever \( \beta > 0 \) is close enough to one.

Consider any strategy profile \( (\tau^L, \tau^R) = ((t_L, \alpha_L), (t_R, \alpha_R)) \) with \( t_L = 1, \alpha_R = 1, \frac{1}{2} (t_R + \alpha_L) \in (a^m - \epsilon(\gamma), a^m + \epsilon(\gamma)) \), and \( t_R < \alpha_L \). Then, at least one of these two parties can have a deviation to make its militant and opportunist better off, whenever \( \beta > 0 \) is close enough to one.

In summary, there is no other type of non-trivial and non-pure PUNE than the type \( ((1, \alpha^*_L), (t^*_{R}, 1)) \) with \( \frac{1}{2} (t^*_{R} + \alpha^*_L) \in (a^m - \epsilon(\gamma), a^m + \epsilon(\gamma)) \) and \( a_R \geq t^*_{R} \geq \alpha^*_L \geq a_L \) for some \( \epsilon(\gamma) > 0 \), whenever \( \beta > 0 \) is close enough to one.

**7.3 Proofs for Section 5**

**Proof of Theorem 4:** As discussed in section 3, every voter \( h \in H \) with \( w_h \leq \mu \) has the same ideal policy \( (\overline{t}_h, \overline{\alpha}_h) = (1, \alpha^*) \), whereas every voter
\( h \in H \) with \( w_h > \mu \) has the ideal policy \( (\overline{t}_h, \overline{\omega}_h) = \left( t \left( \frac{w_h}{\mu} \right), 1 \right) \). Given \( \tau_L = (1, \alpha^*) \), and given \( w_{h'} > \mu \), consider

\[
\max_{(t_R, \alpha_R) \in \Upsilon} \left[ (1 - t_R) \int_{w_{h'} > \mu} w_h d\Pi(h) + (1 - \alpha_R) t_R \mu + \sigma (\alpha_R t_R \mu) \right].
\]

Since the above objective function is strictly concave, there is a unique optimal solution \( \tau_R (w_{h'}) = (t_R (w_{h'}), \alpha_R (w_{h'})) \in \Upsilon \) for \( w_{h'} > \mu \). Then, \( \Omega (\tau^L, \tau_R (w_{h'})) \) is identified. We want to find \( h' \in H \) with \( w_{h'} = \mu - \Theta (\tau^L, \tau_R (w_{h'})) \). Note that if \( w_{h'} = \mu \), then \( w_{h'} < \mu - \Theta (\tau^L, \tau_R (w_{h'})) \). Also if \( w_{h'} = \overline{w} \), then \( \tau_R (w_{h'}) = \arg \max_{(t_R, \alpha_R) \in \Upsilon} (1 - \alpha_R) t_R \mu + \sigma (\alpha_R t_R \mu) \), which implies \( w_{h'} > \mu - \Theta (\tau^L, \tau_R (w_{h'})) \). When \( w_{h'} \) increases from \( \mu \) to \( \overline{w} \), then \( -\Theta (\tau^L, \tau_R (w_{h'})) \geq 0 \) decreases. Thus, there exists \( h^* \in H \) with \( w_{h^*} > \mu \) such that \( w_{h^*} = \mu - \Theta (\tau^L, \tau_R (w_{h^*})) \).

Define \( L = \{ h \in H \mid w_h \leq w_{h^*} \} \) and \( R = \{ h \in H \mid w_h > w_{h^*} \} \). By construction, \( w_R = \int_{h \in R} w_h d\Pi(h) > \mu \), so that \( w_L = \int_{h \in L} w_h d\Pi(h) < \mu \). Since \( R = H \setminus \Omega (\tau^L, \tau_R (w_{h^*})) \) and \( L = \Omega (\tau^L, \tau_R (w_{h^*})) \) by construction, for all \( h \in R, v(\tau^L; h) \leq v(\tau^R; h) \), and for all \( h \in L, v(\tau^L; h) > v(\tau^R; h) \). Since \( w_R > \mu \), we have, by following the same argument as presented in the proof of Lemma 2, \( \tau_R (w_{h^*}) = \left( t \left( \frac{w_{h^*}}{\mu} \right), 1 \right) \), which is the \( R \)-militant’s ideal policy. Since \( w_L < \mu \), we have, by following the same argument as presented in the proof of Lemma 1, \( \tau^L = (1, \alpha^*) \), which is still the \( L \)-militant’s ideal policy. Then, following the same arguments as in the proofs of Lemma 3 and Theorem 1, we have the desired result.

**Proof of Theorem 5:** As shown in the proof of Theorem 2, given \( \gamma > 0 \), we can identify the maximal number \( \epsilon (\gamma) > 0 \) such that

\[
\int_{0}^{a^m - \epsilon (\gamma)} r(a) da \geq \max \left\{ \frac{1}{2} - \gamma, 0 \right\} \quad \text{and} \quad \int_{0}^{a^m + \epsilon (\gamma)} r(a) da \leq \min \left\{ \frac{1}{2} + \gamma, 1 \right\}.
\]

Take any number \( \overline{a} \in (a^m - \epsilon (\gamma), a^m + \epsilon (\gamma)) \), and define \( L \equiv \{ h \in H \mid a_h < \overline{a} \} \) and \( R \equiv \{ h \in H \mid a_h > \overline{a} \} \). Consider any pair of policies \( (\tau^L, \tau^R) = ((t_L, \alpha_L), (t_R, \alpha_R)) \) such that \( \frac{1}{2} (\alpha_R t_R + \alpha_L t_L) = \overline{a} \) and \( \alpha_L t_L < \alpha_R t_R \). Then, for any \( h \in L, v(\tau^L; h) \geq v(\tau^R; h) \) and for all \( h \in R, v(\tau^L; h) < v(\tau^R; h) \). Based on this partition, we can identify \( a_L = \int_{h \in L} a_h d\Pi(h) \) and \( a_R = \int_{h \in R} a_h d\Pi(h) \). By definition, \( a_L < \overline{a} < a_R \). Then, we can appropriately choose a pair of policies \( (\tau^L, \tau^R) = ((t_L, \alpha_L), (t_R, \alpha_R)) \) such that \( \frac{1}{2} (\alpha_R t_R + \alpha_L t_L) = \overline{a}, \alpha_L t_L < \alpha_R t_R, \)
and \( a_L < \alpha_L t_L < \pi < \alpha_R t_R < a_R \). Then, as shown in the proof of Theorem 2, we can see that this \((\tau^L, \tau^R)\) constitutes a non-trivial and non-pure PUNEEP, and there is no other type of non-trivial PUNEEP. ■

**Proof of Theorem 6:** Let us consider two cases of population distribution.

(Case 1) \( \int_0^\mu \int_0^1 ag(w) r(a; w) dw \leq a^m \) holds.

Given \( \epsilon(\gamma) > 0 \) which is obtained in Theorem 2, take any number \( \pi \in (a^m - \epsilon(\gamma), a^m + \epsilon(\gamma)) \), and consider any pair of policies \((\tau^L, \tau^R) = ((t_L, \alpha_L), (t_R, \alpha_R)) = ((1, \alpha^*), (t^*, 1))\) such that \( \frac{1}{2}(\alpha_R t_R + \alpha_L t_L) = \pi \) and 
\[
\int_{h: a_h < \pi} a_h dF(h) < \alpha_L t_L < \alpha_R t_R < \int_{h: a_h \geq \pi} a_h dF(h) \].

We can define \( L \equiv \Omega(\tau^L, \tau^R) \) and \( R \equiv H(\Omega(\tau^L, \tau^R) \). Then, \( a_L = \int_W \int_{\pi + \Phi_w(\tau^L, \tau^R; \beta)} ag(w) r(a; w) dw \)
and \( a_R = \int_W \int_1 ag(w) r(a; w) dw \). Since \( \beta > 0 \) is close enough to one, \( \Phi_w(\tau^L, \tau^R; \beta) \) is very small for each \( w \in W \), which implies that \( a_L \) (resp. \( a_R \)) is very close to \( \int_{h: a_h < \pi} a_h dF(h) \) (resp. \( \int_{h: a_h \geq \pi} a_h dF(h) \)). Thus, \( a_L < \alpha_L t_L < \pi < \alpha_R t_R < a_R \) still holds.

Note that \( \int_0^\mu \int_0^1 ag(w) r(a; w) dw \leq a^m \) implies \( \int_W \int_0^a wg(w) r(a; w) dw < \mu \). Since \( \pi \) is very close to \( a^m \), this also implies \( w_L = \int_W \int_0^{\pi + \Phi_w(\tau^L, \tau^R; \beta)} wg(w) r(a; w) dw < \mu \). In like a manner, \( w_R = \int_W \int_1 \Phi_w(\tau^L, \tau^R; \beta) \) constitutes a non-trivial and non-pure PUNEEP, and there is no other type of non-trivial PUNEEP.

(Case 2) \( \int_0^\mu \int_0^1 ag(w) r(a; w) dw > a^m \) holds.

Let us show that the above strategy profile \((\tau^L, \tau^R) = ((1, \alpha^*), (t^*, 1))\) can be a PUNEEP even in this Case 2. Note that \( \int_0^\mu \int_0^1 ag(w) r(a; w) dw > a^m \) implies \( \int_W \int_0^a wg(w) r(a; w) dw > \mu \), so that \( w_L > \mu \). Consider a deviation from \( \tau^L \) to another strategy \( \tau^{L'} = (\alpha^*, 1) \). Then, though the level of defense expenditure does not change, the after-tax income \((1 - \alpha^*) w_L + \alpha^* \mu \) of \( L \) under \( \tau^{L'} \) is larger than \( \mu \) which is the after-tax income of \( L \) under \( \tau^L \). Since \( \beta < 1 \), it implies that the L-militant is improved by this deviation. However, as discussed in the proof of Theorem 3, we can see that \( \nabla^\beta \mathbb{E}_L((1, \alpha^*), (t^*, 1)) \cdot (t'L - 1, \alpha'_L - \alpha^*) < 0 \) by A4 whenever \( t'_L, \alpha'_L = \alpha^* \) holds. Thus, the L-opportunist is worse off by this deviation. Note that for any other type of deviation, we can follow the same argument as the proof for Theorem 3. Thus, \((1, \alpha^*) \) is still a best reply to \((t^*, 1) \) for \( L \). In like a manner, \((t^*, 1) \) is still a best reply to \((1, \alpha^*) \) for \( R \).
Note that if \((\tau_L, \tau_R) = ((1, \alpha^*), (t^*, 1))\) is a PUNEEP in Case 2, then it is easy to check that \((\tau_L', \tau_R') = ((\alpha^*, 1), (1, t^*))\) is also a PUNEEP in Case 2.

8 References


Figure 1: The set of non-trivial and non-pure PUNEs when $\beta=1$:

$$\{(t_L, \alpha_L, t_R, \alpha_R) \in \Upsilon \mid a^L < \alpha_L t_L \leq \alpha_R t_R < a^R, \ a^m - \varepsilon(\gamma) \leq (1/2) \cdot (\alpha_L t_L + \alpha_R t_R) \leq a^m + \varepsilon(\gamma) \}$$