An Efficient Valuation and Hedging of Constant Maturity Swap Products under BGM Model

Abstract

The derivatives of a constant maturity swap (CMS) almost are evaluated by Monte Carlo methods in a LIBOR market model for previous researches. In this paper, we derive an approximated dynamic process of the swap rate under one-factor LIBOR market model. Based on the approximated dynamics for the swap rate, CMS spread options and CMS ratchet options are valued by the no-arbitrage method in approximated analytic formulas. In numerical analyses, the relative errors between the Monte Carlo simulations and the approximated closed-form formulas are very small for CMS spread options and CMS ratchet options. Finally, we provide an efficient way for hedging CMS products.

Keywords: CMS spread option, lognormal forward LIBOR model, CMS ratchet cap.
1. Introduction

Due to the fact that interest rates are volatile over the past decade, the risk management of assets and liabilities has attracted greater attention by financial institutions and risk managers. Therefore, the derivatives of swap rates have become increasingly popular. For example, the CMS spread option, whose payoff is based on the difference between long and short maturity swap rates; the CMS ratchet option, whose payoff is based on the difference between present and ahead period swap rates.

There are two types of the market model, the LIBOR market model and the swap rate market model providing the arbitrage-free pricing framework and the pricing formulas. HJM model (Heath et al., 1992) is the first model describing the evolution of forward rate term structure, the drawback of the mode is that data can not be observed from market. In fact, they depict the behavior of the term structure of zero coupon bonds (ZCB’s) rather than interest rates. Brace et al. (1997) developed the LIBOR market model to extend the behavior of discrete tenor forward rate, and to overcome some technical existence problems associated with the lognormal version of the HJM model. Musiela and Rutkowski (1997b) proposed that the forward LIBOR rate is martingale by taking the zero coupon bond as the numeraire to evaluate the caps.

The swap rate market model is the popular model for forward swap rate. Jamshidian (1997) investigated that the forward swap rate is martingale by taking the “present value for basis point” as the numeraire inducing a Black-type pricing formula to value the swaption. Galluccio and Hunter (2004) proposed a co-initial swap market model for the assumption of multi-dimensional forward swap rate to avoid the complex calibration techniques. Based on this assumption, the correlation of the forward swap rates is estimated by historical swap rates. Mercurio and Pallavicini
(2005) suggested the mixing Gaussian model and calibrating correlation parameters of
the forward swap rate using CMS spread options.

However, for products of swap rates there are no analytical formulas in general,
and for the correlations of different swap rates there are no methods to calibrate them.
Therefore, LIBOR market model is the unifying model of the interest rates and
capable of encompassing the global properties of the swap rate model, because the
forward LIBOR curve, volatility, and correlation are conveniently calibrated by the
interest rate swap, floor, and swaption (Brace et al., 1998). Therefore, we select the
lognormal forward LIBOR rates to describe the yield curve in this paper rather than
forward swap rates.

Brigo et al. (2003) proposed the dynamics of the forward rate to value an
approximated CMS spread option by changing forward swap measure to forward
measure. In this paper different from Brigo et al. (2003), we investigate the
approximated dynamic process of the swap rate under lognormal LIBOR market
model, which the forward swap rate is approximated by the linear combination of the
forward LIBOR rate. Based on the approximated dynamics of the swap rate, we
derive the analytic approximation formulas for the valuation of CMS spread options
and CMS ratchet options. For empirical studies, the covariance matrix of forward
rates is often rank one, and this permit us to adopt one-factor Libor market model to
price CMS products involving more than two reference rates. Under the framework,
all correlations of different forward swap rates are one in one factor lognormal
forward LIBOR market model. It can avoid the complicated numerical computation
for spread options. From the numerical analyses, the errors would be small between
the Monte Carlo simulation and the approximated formula.

The rest of this paper is organized as follows. In section 2, we propose the
approximated dynamic process of the forward swap rate under LIBOR market model, and the contracts of the CMS spread option and CMS ratchet option are introduced. Following the no-arbitrage theorem, we derive the approximated closed-form formulas of the contracts for the CMS derivatives in the section 3. In Section 4, we compute the errors between the Monte Carlo simulation and the approximation. The conclusion remarks are in Section 5.
2. Model and contracts

2.1 Model

The class of the ZCBs is the popular numeraires for pricing interest rate derivatives in European type while the reference interest rate possess log-normal process. In this section, we give an approximated dynamics of forward swap rate under forward measure by exploiting the technique of the change measure.

Assume there exists a probability space \((\Omega, F, P)\), where \(\Omega\) denotes the state pace, \(F\) represents the filtration, and \(P\) is the objective probability measure on \((\Omega, P)\). Consider a forward start interest rate swap (FSIRS) with \(c\) periods from the reset date \(T_\alpha\) and the maturity date \(T_{\alpha+c}\). The tenors are defined by \(0 \leq T_\alpha < T_{\alpha+1} < \cdots < T_{\alpha+c}\), the space of the tenor represents \(\tau_k = T_k - T_{k-1}\) (e.g., a quarter year or a half year), and ZCBs are denoted by \(P(., T_k)\) with the expiration date \(T_k\), where \(k = \alpha, \alpha+1, ..., \alpha+c\) in the FSIRS convention. Under the arbitrage-free circumstance, the forward swap rate \(S_{\alpha,\alpha+c}(t)\) at time \(t\) corresponding to this FSIRS can be expressed as the combination of ZCBs:

\[
S_{\alpha,\alpha+c}(t) = \frac{P(t, T_\alpha) - P(t, T_{\alpha+c})}{\sum_{i=\alpha+1}^{\alpha+c} \tau_i P(t, T_i)}.
\]

The forward rate agreement (FRA) is a FSIRS contract with a single period. The forward swap rate at time \(t\) associated with FRA can be reduced to forward LIBOR rate \(F(t; U_1, U_2)\) where \(U_1\) and \(U_2\) denote the reset date and the payment date, respectively. Suppose \(c\) forward LIBOR rates as \(F(t, T_{k-1}, T_k)\), where \(k = \alpha + 1, ..., \alpha + c\). To simplify notation, the forward LIBOR rate \(F(t, T_{k-1}, T_k)\) is denoted as \(F_k(t)\) and the relation between \(F(t, T_{k-1}, T_k)\) and ZCB is denoted as follows:
\[ F_k(t) = \frac{P(t, T_{k+1}) - P(t, T_k)}{\tau_k P(t, T_k)} , \quad k = \alpha + 1, \ldots, \alpha + c. \]

Therefore, to characterize the evolution of forward swap rate \( S_{\alpha,\alpha+c}(t) \) under \( T_\alpha \)-forward measure space \((\Omega,\mathcal{F},Q^{T_\alpha})\), forward swap rate can be expressed in alternative form as

\[ S_{\alpha,\alpha+c}(t) = \sum_{k=\alpha+1}^{\alpha+c} w^j_{\alpha,\alpha+c}(t) F_j(t), \quad (1) \]

where the weight is defined as

\[ w^j_{\alpha,\alpha+c}(t) = \frac{\tau_j P(t, T_j)}{\sum_{j'=\alpha+1}^{\alpha+c} \tau_{j'} P(t, T_{j'})} , \quad k = \alpha + 1, \ldots, \alpha + c. \]

From the previous equation, the forward swap rate can be interpreted as weighted averages of spanning forward rates. Let \( \nu_{\alpha,\alpha+c}(t) \) and \( \sigma_k(t) \) represent the volatilities of forward swap rate \( S_{\alpha,\alpha+c}(t) \) and forward LIBOR rate \( F_k(t) \), respectively. In practice, we assume that volatility of forward swap rate \( \nu_{\alpha,\alpha+c}(t) \) and forward LIBOR \( \sigma_k(t) \) are deterministic in \( t \) so as to calibrate simultaneously cap/floor and swaption. However, based on the lognormal forward LIBOR rate, Brigo et al. (2003) demonstrated the dynamic process of forward LIBOR rate under forward measure \( Q^{T_\alpha} \) follows

\[ dF_k(t) = F_k(t)\sigma_k(t) \left[ \sum_{j=\alpha+1}^{k} \frac{\tau_j \sigma_j(t) F_j(t)}{1 + \tau_j F_j(t)} dt + d\tilde{W}^{T_\alpha}(t) \right] , \quad \alpha < k , \quad t \leq T_\alpha , \quad (2) \]

where \( \tilde{W}^{T_\alpha}(t) \) denotes an n-dimensional Brownian motion defined on the measure space \((\Omega,\mathcal{F},Q^{T_\alpha})\). Due to empirical studies showing the variability of the weights to be small compared to the variability of the forward rates (cf. Brigo and Mercurio, 2001), one can approximate the weights by their (deterministic) initial value
\[ w_{a,a+c}^k(t) \approx w_{a,a+c}^k(0) \] for all \( k \), and equation (1) can be written as
\[ S_{a,a+c}(t) \approx \sum_{k=a+1}^{a+c} w_{a,a+c}^k(0)F_k(t). \] (3)

This will be helpful in estimating the absolute volatility of swap rates from the absolute volatility of forward rates and in valuing the derivatives the swap rate, swaptions and so on (cf. Rebonato, 2004). By Ito’s lemma, the forward swap rate in (3) can be derived as
\[ dS_{a,a+c}(t) = \sum_{k=a+1}^{a+c} w_{a,a+c}^k(0)dF_k(t). \] (4)

Substituting (3) to (4) and divide \( S_{a,a+c}(t) \) to both sides in (4)
\[ \frac{dS_{a,a+c}(t)}{S_{a,a+c}(t)} = \sum_{k=a+1}^{a+c} \frac{w_{a,a+c}^k(0)F_k(t)}{S_{a,a+c}(t)} \sigma_i(t) \left[ \sum_{j=a+1}^{k} \frac{\tau_j \sigma_j(t)F_j(t)}{1+\tau_j F_j(t)} dt + d\tilde{W}_k^\tau(t) \right]. \] (5)

For the sake of acquiring Black-Scholes type process, we can further freeze the forward LIBOR rate \( F \) at time 0 (cf. Hunter, 2001 and Jamshidian, 1997) on the right hand size in (5), then the drift term and volatility term will be deterministic (in fact, they depend only on the \( \sigma(t)'s \)). Let

\[ G_k(t) = \sigma_i(t) \sum_{j=a+1}^{k} \frac{\tau_j \sigma_j(t)F_j(t)}{1+\tau_j F_j(t)} \]

and the new weight \( \tilde{w}_{a,a+c}^k = \frac{w_{a,a+c}^k(0)F_k(t)}{S_{a,a+c}(0)} = \frac{P(0,T_{a+c})-P(0,T_a)}{P(0,T_a)-P(0,T_{a+c})} \), then equation (5) can be rewritten as
\[ \frac{dS_{a,a+c}(t)}{S_{a,a+c}(t)} = \tilde{m}_{a,a+c}(t)dt + \sum_{k=a+1}^{a+c} \tilde{w}_{a,a+c}^k \sigma_i(t)d\tilde{W}_k^\tau(t), \] (6)

where the drift term of the swap rate dynamics is
\[ \tilde{m}_{a,a+c}(t) = \sum_{k=a+1}^{a+c} \tilde{w}_{a,a+c}^k G_k(t). \]

Moreover, in order to capture volatility the of swap rate, we convert (6) into
\[ \frac{dS_{a,a+c}(t)}{S_{a,a+c}(t)} = \tilde{m}_{a,a+c}(t)dt + \nu_{a,a+c}(t)d\tilde{W}^\nu(t) \]
by making use of Gaussian distribution properties, where $\tilde{W}_{T}^{\alpha}(t)$ is a new Brownian motion under $Q_{T}^{\alpha}$, and the relative volatility of the swap rate is represented as

$$ v_{a,a+\epsilon}(t) = \left( \sum_{k=a+1}^{a+\epsilon} \tilde{W}_{a,a+\epsilon}^k \sigma_k(t) \right), $$

Therefore, the approximated dynamics of the forward swap rate in forward LIBOR model can be seen to be as following proposition.

**Proposition 1.** For $t \leq T_{a}$, under forward measure $Q_{T}^{\alpha}$, the approximated dynamic process of the forward swap rate in the lognormal forward LIBOR model is as follows:

$$ \frac{dS_{a,a+\epsilon}(t)}{S_{a,a+\epsilon}(t)} = \tilde{m}_{a,a+\epsilon}(t)dt + v_{a,a+\epsilon}(t)d\tilde{W}_{T}^{\alpha}(t) $$

(7)

where $\tilde{m}_{a,a+\epsilon}(t) = \sum_{k=a+1}^{a+\epsilon} \tilde{W}_{a,a+\epsilon}^k G_k(t)$ is the drift term of the swap rate and the relative volatility of the swap rate is

$$ v_{a,a+\epsilon}(t) = \left( \sum_{k=a+1}^{a+\epsilon} \tilde{W}_{a,a+\epsilon}^k \sigma_k(t) \right), $$

and the weight is

$$ \tilde{W}_{a,a+\epsilon}^k = \frac{P(0,T_{a+\epsilon}) - P(0,T_{a})}{P(0,T_{a}) - P(0,T_{a-1})}. $$

Furthermore, analogous to the dynamics in (2), the forward LIBOR rates under forward measure $Q_{T_{a+1}}^{\alpha}$ are as follows:

$$ \begin{align*}
\{ dF_k(t) &= F_k(t)\sigma_k(t) \left[ \sum_{j=k+1}^{k+\epsilon} \frac{\tau_j \sigma_j(t)F_j(t)}{1+\tau_j F_j(t)} dt + d\tilde{W}_{T}^{\alpha}(t) \right], a+1 < k \\
n dF_{a+1}(t) &= F_{a+1}(t)\sigma_{a+1}(t)d\tilde{W}_{T}^{\alpha}(t) 
\end{align*} $$

where $t \leq T_{a}$ and $\tilde{W}_{T}^{\alpha}(t)$ denotes an n-dimensional Brownian Motion defined on the measure space $(\Omega,F,Q_{T_{a+1}}^{\alpha})$. Using above equation, we can derive the following proposition.

**Proposition 2.** For $t \leq T_{a-1}$, under forward measure $Q_{T}^{\alpha}$, the approximated dynamic
process of the forward swap rate in the lognormal forward LIBOR model is as follows:

\[
\frac{dS_{\alpha-1,\alpha-1+c}(t)}{S_{\alpha-1,\alpha-1+c}(t)} = \tilde{n}_{\alpha-1,\alpha-1+c}(t)dt + \nu_{\alpha-1,\alpha-1+c}(t)d\tilde{W}(t)
\]  

(8)

where \( \tilde{n}_{\alpha-1,\alpha-1+c}(t) = \sum_{k=1}^{a-1} \tilde{w}^i_{\alpha-1,\alpha-1+c} H_i(t) \) is the drift term of the swap rate and

\[
H_i(t) = \sigma_i(t) \sum_{j=a+1} \frac{\tau_i \sigma_j(t) F_j(0)}{1 + \tau_i F_j(0)}
\]

2.2 CMS products

A constant-maturity swap (CMS) is like IRS contract that the payments also are exchanged between two differently indexed legs. Formally, at payment date \( T_\alpha \), the floating leg pays the \( c \) periods spot swap rate \( S_{a,a,c}(T_\alpha) \tau_\alpha \) to the fixed leg. There are two differences between IRS and CMS contracts. One is that the reference rate of the floating leg is the \( c \) periods swap rate in the CMS contract but it is the LIBOR rate in the IRS contract. The other is that the payment date is the reset date of the corresponding swap rate in the CMS contracts but the IRS contracts pay later in the next reset date.

Constant maturity swap spread (CMSS) options are considered on the spread option under the two different swap rates in this paper. More precisely, given a fixed maturity \( T_\alpha \), two positive real numbers \( a_1 \) and \( a_2 \), and a strike price \( K \), the payoff of the CMSS option between two reference rates is defined as

\[
V_i(T_\alpha) = \tau_\alpha \max(a_1 \delta S_{a,a+c_1}(T_\alpha) - a_2 \delta S_{a,a+c_2}(T_\alpha) - \delta K, 0),
\]

(9)

where \( \delta = 1 \) for a call and \( \delta = -1 \) for a put.

CMS ratchet options are considered on the options with two swap rates between the two different maturity dates in this paper. More precisely, given a fixed maturity \( T_\alpha \) and a previous maturity \( T_{\alpha-1} \), two positive real number \( a_1 \) and \( a_2 \), and a strike price
\( K \), the payoff of the ratchet option on two different maturity swap rates with \( c \) periods is denoted as

\[
V(z(T_a)) = \tau_\alpha \max(a_\delta S_{a-\delta, a-\delta}(T_a) - a_\delta S_{a-\delta+1, a-\delta+1}(T_{a-1}) - \delta K, 0), \quad (10)
\]

where \( \delta = 1 \) for a call and \( \delta = -1 \) for a put.

It reveals that the two processes in equations (7) and (8) possess the lognormal distribution and it is advantageous to price CMS products. The valuations of two CMS products will be derived in next section. One is the CMS spread option; the other is the CMS ratchet option.
3. Valuation of Constant Maturity Swap Products

According to the existence of a unique equivalent martingale measure, there exists a unique arbitrage-free price for the two products at time 0 as follows:

$$V_i(0) = P(0, T_a) \mathbb{E}^{T_a} \left[ \frac{V_i(T_a)}{P(T_a, T_a)} \right] \quad i = 1, 2.$$  \hspace{1cm} (11)

Hence the approximated value is discounted by the payoffs in equation (11) under forward martingale measure. In spite of the covariance matrix of forward rates is a multi-factor framework, but empirical results reveals that it is often rank one (cf Brace et al, 1998). Under this circumstance, the volatilities $\sigma(t)$'s of the forward rates could be reduced to one dimension. In order to value two products for simplicity, we derive the lemma as follows.

**Lemma 1.** Assume the dynamic processes of two reference rates are

$$\frac{dY_i(t)}{Y_i(t)} = \mu_i(t)dt + \sigma_i(t)dW(t)$$

$$\frac{dY_2(t)}{Y_2(t)} = \mu_2(t)dt + \sigma_2(t)dW(t)$$

where $\mu_i(t)$ and $\sigma_i(t)$, $i = 1, 2$, are deterministic functions, $W(t)$ is one-dimension Brownian motion under measure space $(\Omega, (\mathcal{F})_{\geq t}, P)$. Let $T_1 > 0$, $T_2 > 0$, $a_i > 0$, $a_2 > 0$, $K \geq 0$ and $g(x) = \beta_1 e^{\gamma_1 x} - \beta_2 e^{\gamma_2 x} - K$ with $x = \frac{\ln(\beta_1 \sqrt{T_1} - a_1 Y_i(0) \exp \left( \int_0^{T_1} (\mu_i(t) - 0.5 \sigma_i^2(t))dt \right))}{\gamma_2 \sqrt{T_2} - \beta_2 \sqrt{T_i}}$, $\gamma_i = \sqrt{\frac{\int_0^{T_i} \sigma_i^2(t)dt}{T_i}}$, $\beta_i = a_i Y_i(0) \exp \left( \int_0^{T_i} (\mu_i(t) - 0.5 \sigma_i^2(t))dt \right)$, for $i = 1, 2$

then the spread option with two reference rates can be divided into four case as follows:

**Case 1:** $g(x)$ has two real roots $x_1$ and $x_2$ with $x_1 < x^* < x_2$. 

\[
\mathbb{E}_p \left( \delta a Y_1(T_1) - \delta a Y_2(T_2) - \delta K \right)^* = \delta a Y_1(0) e^{\int_0^t \mu_1(t) dt} \left( N(\text{sign}(g(x^*))) \delta d_{2,1} \right) \\
- \text{sign}(g(x^*)) \delta N(d_{1,1}) \\
- \delta a Y_2(0) e^{\int_0^t \mu_2(t) dt} \left( N(\text{sign}(g(x^*))) \delta d_{2,2} \right) \\
- \text{sign}(g(x^*)) \delta N(d_{1,2}) \\
- \delta K \left( N(\text{sign}(g(x^*)) \delta x_1) - \text{sign}(g(x^*)) \delta N(x_1) \right)
\]

**Case 2:** \( g(x) \) has unique real root with \( x^* < x_2 \).

\[
\mathbb{E}_p \left( \delta a Y_1(T_1) - \delta a Y_2(T_2) - \delta K \right)^* = \delta a Y_1(0) e^{\int_0^t \mu_1(t) dt} \left( N(\text{sign}(g(x^*))) \delta d_{2,1} \right) - \\
\delta a Y_2(0) e^{\int_0^t \mu_2(t) dt} \left( N(\text{sign}(g(x^*))) \delta d_{2,2} \right) - \delta K N(\text{sign}(g(x^*)) \delta x_2)
\]

**Case 3:** \( g(x) \) has unique real root with \( x_1 < x^* \).

\[
\mathbb{E}_p \left( \delta a Y_1(T_1) - \delta a Y_2(T_2) - \delta K \right)^* = \delta a Y_1(0) e^{\int_0^t \mu_1(t) dt} \left( -\text{sign}(g(x^*)) \delta d_{1,1} \right) \\
- \delta a Y_2(0) e^{\int_0^t \mu_2(t) dt} \left( -\text{sign}(g(x^*)) \delta d_{1,2} \right) - \delta K N(-\text{sign}(g(x^*)) \delta x_1)
\]

**Case 4:** \( g(x) \) has no roots. (It is in the money if the option is call option and out the money if the option is put option)

\[
\mathbb{E}_p \left( \delta a Y_1(T_1) - \delta a Y_2(T_2) - \delta K \right)^* = \text{sign}(g(x^*)) \left( a Y_1(0) e^{\int_0^t \mu_1(t) dt} - \\
a Y_2(0) e^{\int_0^t \mu_2(t) dt} - K \right) I_{[\text{sign}(g(x^*))]} \\
\]

where \( d_{1,1} = x_1 - \gamma_1 \sqrt{T_1}, d_{1,2} = x_1 - \gamma_2 \sqrt{T_2}, d_{2,1} = x_2 - \gamma_1 \sqrt{T_1}, d_{2,2} = x_2 - \gamma_2 \sqrt{T_2} \).

### 3.1 Valuation of CMS spread options

From the dynamics of \( c_1 \)-period forward swap rate and \( c_2 \)-period swap rate, we adopt the dynamics in Proposition 1 by putting \( c_1 \) and \( c_2 \) into \( c \) and the instantaneous correlation of two forward swap rates \( S_{\alpha,\alpha+c_1} \) and \( S_{\alpha,\alpha+c_2} \) in (9) is denoted as
Then, based on the payoff in equation (9), we derive the pricing formula of the general CMS spread option by Lemma 3.1 as shown in the following theorem.

**Theorem 1.** For the dynamics of two forward swap rates with \( c_1 \) and \( c_2 \) period in Proposition 1, the pricing formula of a general CMS spread option with equation (9) is

\[
V_1(0) = P(0,T_a)E^{T_a} \left[ V_1(T_a) \right] = \tau_a P(0,T_a) M_g(x_1, x_2, x^*)
\]

where

\[
M_g(x_1, x_2, x^*) = \delta a_1 S_{a, a+1} \left( 0 \right) e^{\tilde{w}_{a, a+1} \left( 0 \right) dt} \left( N(\text{sign}(g(x^*)) \delta d_{1,1}) - \text{sign}(g(x^*)) \delta N(d_{1,1}) \right)
\]

\[
- \delta a_2 S_{a, a+2} \left( 0 \right) e^{\tilde{w}_{a, a+2} \left( 0 \right) dt} \left( N(\text{sign}(g(x^*)) \delta d_{1,2}) - \text{sign}(g(x^*)) \delta N(d_{1,2}) \right),
\]

\[
- \delta K \left( N(\text{sign}(g(x^*)) \delta x_2) - \text{sign}(g(x^*)) \delta N(x_2) \right)
\]

\[
g(x) = \beta_1 e^{\gamma_1 T_a} - \beta_2 e^{\gamma_2 T_a} - K, \quad x^* = \frac{\ln(\beta_1 \gamma_1 \sqrt{T_a}) - \ln(\beta_2 \gamma_2 \sqrt{T_a})}{\gamma_1 \sqrt{T_a} - \gamma_2 \sqrt{T_a}}
\]

\[
\gamma_i = \left( \int_0^{T_a} V_{a, a+e_j} \left( 0 \right) dt \right)^{2/3}, \quad \beta_i = a_i S_{a, a+e_i} \left( 0 \right) \exp \left( \int_0^{T_a} \left( \tilde{m}_{a, a+1} \left( t \right) - 0.5 \tilde{v}_{a, a+1} \left( t \right)^2 \right) dt \right), \text{ for } i = 1, 2.
\]

\[
d_{1,1} = x_1 - \gamma_1 \sqrt{T_a}, \quad d_{1,2} = x_1 - \gamma_2 \sqrt{T_a}, \quad d_{2,1} = x_2 - \gamma_1 \sqrt{T_a}, \quad \text{and} \quad d_{2,2} = x_2 - \gamma_2 \sqrt{T_a}.
\]

\( x_1 \) and \( x_2 \) are two real roots of \( g \) with \( x_1 < x^* < x_2 \) and \( x_1 = -\infty \) if \( x_1 \) does not exist and \( x_2 = \infty \) if \( x_2 \) does not exist.

### 3.2 Valuation on CMS ratchet options

According to the two dynamics of \( c \)-period forward swap rate which is reset at \( T_a \) and \( T_{a-1} \) respectively, we adopt the approximated dynamics of swap rate in Proposition 1 and Proposition 2 and the instantaneous correlation of two forward swap rates \( S_{a, a+c} \) and \( S_{a-1, a-1+c} \) in (10) is denoted as
Then, based on the payoff in equation (10), the pricing formula of the general ratchet option is derived using Lemma 3.1 as shown in the following theorem.

**Theorem 2.** For the approximated dynamics of the forward swap rate and the ahead forward swap rate at time $T_a$ and $T_{a-1}$ with $c$ years in Proposition 1 and 2, the pricing formula of a CMS ratchet option with equation (10) is

$$V_2(0) = \tau_a P(0, T_a) E_{T_a} [V_2(T_a)] = \tau_a P(0, T_a) M_g(x_1, x_2, x^*)$$

where

$$M_g(x_1, x_2, x^*) = \delta a_i S_{a, \alpha, \tau_0}(0)e^{\int_{T_{a-1}}^{T_a} (N(\text{sign}(g(x^*)))\delta d_{2,i}) - \text{sign}(g(x^*))\delta N(d_{1,i})}$$

$$- \delta a_i S_{a-1, \alpha, \tau_0}(0)e^{\int_{T_{a-1}}^{T_a} (N(\text{sign}(g(x^*)))\delta d_{2,i}) - \text{sign}(g(x^*))\delta N(d_{1,i})}$$

$$- \delta K \left( N(\text{sign}(g(x^*)))\delta x_2 - \text{sign}(g(x^*))\delta N(x_i) \right)$$

$$g(x) = \beta_1 e^{\gamma_1 \sqrt{T_a}} - \beta_2 e^{\gamma_2 \sqrt{T_{a-1}}} - K, \quad x^* = \frac{\ln(\gamma_1 \sqrt{T_a}) - \ln(\gamma_2 \sqrt{T_{a-1}})}{\gamma_2 \sqrt{T_{a-1}} - \gamma_1 \sqrt{T_a}},$$

$$\gamma_i = \left( \int_0^{T_{a-1}} \frac{v_{a,i+1, \alpha, \tau_0}}{T_{a-1}} \right) \text{ for } i = 1, 2$$

$$\beta_1 = a_i S_{a, \alpha, \tau_0}(0) \exp \left( \int_0^{T_{a-1}} (\tilde{a}_{a, \alpha, \tau_0}(t) - 0.5v_{a, \alpha, \tau_0}^2(t))dt \right)$$

$$\beta_2 = a_i S_{a-1, \alpha, \tau_0}(0) \exp \left( \int_0^{T_{a-1}} (\tilde{a}_{a-1, \alpha, \tau_0}(t) - 0.5v_{a-1, \alpha, \tau_0}^2(t))dt \right)$$

$$d_{1,i} = x_1 - \gamma_1 \sqrt{T_a}, \quad d_{1,i} = x_1 - \gamma_2 \sqrt{T_{a-1}},$$

$$d_{2,i} = x_2 - \gamma_1 \sqrt{T_a}, \quad \text{and } d_{2,2} = x_2 - \gamma_2 \sqrt{T_{a-1}}, \quad x_i \text{ and } x_2 \text{ are two real roots of } g \text{ with}$$

$$x_1 < x^* < x_2 \text{ and } x_1 = -\infty \text{ if } x_1 \text{ does not exist and } x_2 = \infty \text{ if } x_2 \text{ does not exist.}$$
4. Calibration Procedure and Numerical analyses

Ultimately, in order to carry out the setting of one-factor Libor market model, we implement the principal component analysis to obtain a suitable covariance matrix of the forward Libor rates, and examine the numerical effects of pricing the two products for the model parameters in Section 3 between Monte Carlo method and approximated formulas under the framework of LIBOR market model.

4.1 parameters setting and calibration procedure

In our model, an endogenous correlation structure is set to fit swaptions if the yield curves and volatilities of LIBOR forward rates are specific. Therefore, we calibrate market data in two stages. The first stage is to bootstrap the curve of forward rates and volatilities from IRS and Caps, and the second is to calibrate from the swaption market data.

We consider a full-rank time homogeneous instantaneous correlation framework (cf. Rabonato, 2004) as follows:

\[ \rho_{ij} = \phi_0 + (1 - \phi_0) \exp(-\phi_1 |i - j|), \]  
(14)

where \( \rho_{ij} \) is the correlation coefficients between of \( i \)-th and \( j \)-th forward LIBOR rates. There are some advantages in such setting. First, it carries out nonnegative definite correlation matrix. Second, relatively small movements in the \( \rho_{ij} \) cause relatively small changes in \( \phi_0 \) and \( \phi_1 \).

In this procedure of fitting the swaptions, we employ the Rebonato swaption formula (cf. Rebonato, 2004) to match market swaption volatility under LIBOR market model. Finally, the value of correlation between two forward rates tend to diminish as the distance of their maturity increases. Finally, we compute the covariance matrix of
forward rates and find the eigenvector corresponding to the largest eigenvalue.

We adopt the market data to calibrate forward LIBOR rates and volatilities of forward LIBOR rates from Bloomberg data from 14th, August, 2008 as in Tables 1 and 2.

Table 1 The calibrated forward LIBOR rates from Bloomberg

<table>
<thead>
<tr>
<th>Year</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2.8069%</td>
<td>3.3764%</td>
<td>3.2119%</td>
<td>3.3822%</td>
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<tr>
<td>2</td>
<td>3.3859%</td>
<td>3.4457%</td>
<td>3.5040%</td>
<td>3.5634%</td>
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<td>3</td>
<td>4.1515%</td>
<td>4.3256%</td>
<td>4.4958%</td>
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<td>4.8469%</td>
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<tr>
<td>5</td>
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<td>4.7648%</td>
<td>4.8390%</td>
</tr>
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</tr>
<tr>
<td>7</td>
<td>5.0066%</td>
<td>5.0642%</td>
<td>5.1197%</td>
<td>5.1774%</td>
</tr>
<tr>
<td>8</td>
<td>5.0951%</td>
<td>5.1423%</td>
<td>5.1883%</td>
<td>5.2357%</td>
</tr>
<tr>
<td>9</td>
<td>5.1628%</td>
<td>5.2021%</td>
<td>5.2396%</td>
<td>5.2794%</td>
</tr>
<tr>
<td>10</td>
<td>5.2174%</td>
<td>5.2507%</td>
<td>5.2824%</td>
<td>5.3163%</td>
</tr>
</tbody>
</table>

Table 2 The calibrated volatilities of forward LIBOR rates from Bloomberg

<table>
<thead>
<tr>
<th>Year</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
</tr>
</thead>
<tbody>
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<td>27.72%</td>
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<tr>
<td>2</td>
<td>35.37%</td>
<td>35.39%</td>
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</tr>
<tr>
<td>3</td>
<td>28.49%</td>
<td>27.50%</td>
<td>26.47%</td>
<td>25.47%</td>
</tr>
<tr>
<td>4</td>
<td>25.22%</td>
<td>24.22%</td>
<td>23.29%</td>
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</tr>
<tr>
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<td>22.58%</td>
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</tr>
<tr>
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</tr>
<tr>
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</tr>
<tr>
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<td>13.38%</td>
<td>13.26%</td>
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</tr>
<tr>
<td>9</td>
<td>18.20%</td>
<td>18.09%</td>
<td>17.99%</td>
<td>17.92%</td>
</tr>
<tr>
<td>10</td>
<td>20.32%</td>
<td>20.25%</td>
<td>20.19%</td>
<td>20.14%</td>
</tr>
</tbody>
</table>
4.2 The closed-form formula vs Monte Carlo simulation

This subsection provides practical examples to present the accuracy of the approximated pricing formulas and compares the relative errors with Monte Carlo simulation. The used market data are in Appendix C. In this numerical study, the notional principal is assumed to be $1 and the simulations are 100,000 paths.

In Table 4, we calculate the value of the CMS spread call option with 2Y-CMS and 5Y-CMS by Monte Carlo simulation and the closed-form formula. The relative error is the ratio between the absolute error and the correct value. The relative errors are very small for the CMS spread options with many strikes. In Table 5, we calculate the value of the CMS ratchet option by Monte Carlo simulation and the closed-form formula. Similarly, the relative errors are also very small for the CMS spread options with many strikes. This is to say, the approximated formulas of CMS spread options are sufficient accurate by comparing with Monte Carlo simulations and are worth recommending for practical implementing.

<table>
<thead>
<tr>
<th>Strike</th>
<th>Closed Form</th>
<th>Monte Carlo</th>
<th>relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 bp</td>
<td>12.45731285</td>
<td>12.4436535</td>
<td>0.11%</td>
</tr>
<tr>
<td>10 bp</td>
<td>10.0105815</td>
<td>9.98250304</td>
<td>0.28%</td>
</tr>
<tr>
<td>20 bp</td>
<td>7.564367352</td>
<td>7.53664619</td>
<td>0.37%</td>
</tr>
<tr>
<td>30 bp</td>
<td>5.153084696</td>
<td>5.12505085</td>
<td>0.55%</td>
</tr>
<tr>
<td>40 bp</td>
<td>2.831834454</td>
<td>2.81075685</td>
<td>0.75%</td>
</tr>
<tr>
<td>50 bp</td>
<td>0.780413148</td>
<td>0.78953136</td>
<td>-1.15%</td>
</tr>
</tbody>
</table>

The times of Monte Carlo simulation are 100,000 to compute the value of the CMS spread option with a spread of 2 year and 5 year swap rates. The relative error is the ratio between the absolute error and the correct value where the value of the CMS spread option by Monte Carlo simulation is the correct value.
Table 5 The CMS ratchet option by the closed-form formula vs Monte Carlo simulation

<table>
<thead>
<tr>
<th>Strike</th>
<th>Closed Form</th>
<th>Monte Carlo</th>
<th>relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 bp</td>
<td>6.84226557</td>
<td>6.87351366</td>
<td>-0.45%</td>
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<tr>
<td>10bp</td>
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<td>-0.87%</td>
</tr>
<tr>
<td>20bp</td>
<td>4.492045871</td>
<td>4.534604046</td>
<td>-0.94%</td>
</tr>
<tr>
<td>30bp</td>
<td>3.658125721</td>
<td>3.621659865</td>
<td>1.01%</td>
</tr>
<tr>
<td>40bp</td>
<td>2.894345728</td>
<td>2.85797662</td>
<td>1.29%</td>
</tr>
<tr>
<td>50bp</td>
<td>2.261687529</td>
<td>2.231637842</td>
<td>1.35%</td>
</tr>
</tbody>
</table>

The times of Monte Carlo simulation are 100,000 to compute the mean value of the CMS ratchet options. The relative error is the ratio between the absolute error and the correct value where the value of the CMS spread option by Monte Carlo simulation is the correct value. The computational time of Monte Carlo simulation is about 600 seconds, and the computational time of the closed-form formula is about one second.

5. Hedge

For hedging purpose, delta hedge is the popular sensitive analysis. In tradition, the delta for CMS product on Libor market model is practiced by computing buckets to market rate. However, it is time-consuming and unstable due to thorough proceeding Monte Carlo simulation. In this paper, we provide an efficient manner for delta hedging CMS product.

**Lemma 1.** Under the condition in the lemma(3.1), let

\[ F(Y_1(0), Y_2(0)) = \mathbb{E}_p \left( \delta a_1 Y_1(T_1) - \delta a_2 Y_2(T_2) - \delta K \right) \]

be the spread option, then the delta of spread option with two reference rates can be divided into four case as follows:

**Case 1:** \( g(x) \) has two real roots \( x_1 \) and \( x_2 \) with \( x_1 < x^* < x_2 \).

\[ \Delta_1^+ = \frac{\partial F}{\partial Y_1(0)} = \delta a_1 e^{\mu_1 N(d_{11})} \left( N(\text{sign}(g(x^*)) N(d_{11}) - \text{sign}(g(x^*)) \delta N(d_{11}) \right) \]

\[ \Delta_1^- = -\frac{\partial F}{\partial Y_2(0)} = -\delta a_2 e^{\mu_2 N(d_{22})} \left( N(\text{sign}(g(x^*)) N(d_{22}) - \text{sign}(g(x^*)) \delta N(d_{22}) \right) \]
Case 2: $g(x)$ has unique real root with $x^* < x_2$.

$$\Delta_i^* = \delta a_i \exp\left(\int_0^{t_i} \mu_i(t)dt\right) N(\delta \text{sign}(g(x^*))d_{i,1})$$

$$\Delta_i^* = -\delta a_i \exp\left(\int_0^{t_i} \mu_i(t)dt\right) N(\delta \text{sign}(g(x^*))d_{i,2})$$

Case 3: $g(x)$ has unique real root with $x_i < x^*$.

$$\Delta_i^* = \delta a_i \exp\left(\int_0^{t_i} \mu_i(t)dt\right) N(-\text{sign}(g(x^*))d_{i,1})$$

$$\Delta_i^* = -\delta a_i \exp\left(\int_0^{t_i} \mu_i(t)dt\right) e^{\int_0^{\frac{T}{T}} \mu_2(t)dt} N(-\text{sign}(g(x^*))d_{i,2})$$

Case 4: $g(x)$ has no roots. (It is in the money if the option is call option and out the money if the option is put option)

$$E_p \left( \delta a_i Y_i(T_1) - \delta a_i Y_2(T_2) - \delta K \right) = \text{sign}(g(x^*)) \left( a_i Y_i(0)e^{\int_0^{t_i} \mu_i(t)dt} - a_i Y_2(0)e^{\int_0^{\frac{T}{T}} \mu_2(t)dt} - K \right) I_{[\delta \text{sign}(g(x^*))]}$$

where $d_{i,1} = x_i - \gamma_1 \sqrt{T_1}, d_{i,2} = x_i - \gamma_2 \sqrt{T_2}, d_{2,1} = x_2 - \gamma_1 \sqrt{T_1}, d_{2,2} = x_2 - \gamma_2 \sqrt{T_2}$.

Analogous to the

6. Conclusions

The valuations of constant maturity swap (CMS) spread derivatives almost use Monte Carlo method in a lognormal forward-LIBOR model in previous researches. In this paper, based on a lognormal LIBOR framework, we develop the approximated dynamics of the swap rate and provide the approximated closed-form formulas of CMS spread options and CMS ratchet options. More generally, some CMS products can be derive with the approximated dynamics of the swap rate.

According to the approximated closed-form formulas, the CMS products have many
applications. CMS products can be used as ancillary instruments for interest rates swaps to enhance profit from a change in the spread between two interest rate swaps or to lock in current spread and manage interest-rate risk. The approximation formulas of CMS spread options are sufficient accurate by comparing with Monte Carlo simulations and are worth recommending for practical implementing.
Reference


Appendix A: Proof of Lemma 3.1.

We first consider expectation of theorem 1

$\mathbb{E}_p (\delta a_1 Y_1(T) - \delta a_2 Y_2(T) - \delta K)^+.$

Using the properties of Geometric Brownian Motion, it can be represented to

$$\mathbb{E}_p \left( \delta a_1 Y_1(0) \exp \left( \int_0^T (\mu_1(t) - 0.5 \sigma_1^2(t))dt + \int_0^T \sigma_1(t) dW(t) \right) ight. - \delta a_2 Y_2(0) \exp \left( \int_0^T (\mu_2(t) - 0.5 \sigma_2^2(t))dt + \int_0^T \sigma_2(t) dW(t) \right) - \delta K \bigg)^+$$

$$= \int_{-\infty}^{\infty} \left( \delta \beta e^{\delta T} \sqrt[2n]{x_{11}} e^{\beta T} - \delta \beta e^{\delta T} \sqrt[2n]{x_{12}} e^{\beta T} - \delta K \right)^+ n(x) dx$$

$$= \int_{-\infty}^{\infty} \left( \delta g(x) \right)^+ n(x) dx.$$

Thus, we are interesting to find region in the range of $\delta g(x)$ which is nonnegative.

Next, we search the roots of $\delta g(x)$. First, differential $g(x)$ with respect to $x$, we have

$$\frac{dg(x)}{dx} = \beta_1 \gamma_1 \sqrt{T_1} e^{\eta_1 \sqrt{T_1}} - \beta_2 \gamma_2 \sqrt{T_2} e^{\eta_2 \sqrt{T_2}},$$

and find that $g(x)$ has unique critical point

$$x^* = \frac{\ln(\beta_1 \gamma_1 \sqrt{T_1}) - \ln(\beta_2 \gamma_2 \sqrt{T_2})}{\gamma_2 \sqrt{T_2} - \gamma_1 \sqrt{T_1}}.$$

Let $\eta = \beta_1 \gamma_1 \sqrt{T_1} e^{\eta_1 \sqrt{T_1}} = \beta_2 \gamma_2 \sqrt{T_2} e^{\eta_2 \sqrt{T_2}} > 0$, and compute the second derivative at $x^*$, then we have

$$\frac{d^2 g(x^*)}{dx^2} = \beta_1 \gamma_1 e^{\eta_1 \sqrt{T_1}} \gamma_1 \sqrt{T_1} - \beta_2 \gamma_2 e^{\eta_2 \sqrt{T_2}} \gamma_2 \sqrt{T_2} = \eta (\gamma_1 \sqrt{T_1} - \gamma_2 \sqrt{T_2}).$$

Then $\frac{d^2 g(x^*)}{dx^2} > 0$ iff $\gamma_1 \sqrt{T_1} > \gamma_2 \sqrt{T_2}$. Hence $g(x)$ has unique min value or max value.

So, there are most two roots of $\delta g(x)$, we compute the expectation of theorem 1 in accordance with roots of $g$.

Define $g_c(x) = \max(g(x),0)$, $g_p(x) = \max(-g(x),0)$, $n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $n_+(x) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}$.

Case 1: $g(x)$ has two real roots $x_1, x_2$ with $x_1 < x^* < x_2$. 

If \( g(x^*) < 0 \), then \( g(x) \) has nonnegative value on \((\infty, x_1] \cup [x_2, \infty)\) and \(g(x)\) has non positive value on \([x_1, x_2]\)

\[
\int_{x_1}^{x_2} g_x(x) u(x) \, dx = \int_{-\infty}^{x_1} g(x) u(x) \, dx + \int_{x_2}^{\infty} g(x) u(x) \, dx
\]

\[
\int_{x_1}^{x_2} g(x) u(x) \, dx = \int_{-\infty}^{x_1} \beta_1 e^{\gamma_1} n_{\gamma_1}^2 (x) \, dx - \int_{-\infty}^{x_1} \beta_2 e^{\gamma_2} n_{\gamma_2}^2 (x) \, dx - K \int_{-\infty}^{x_1} u(x) \, dx
\]

\[
= \beta_1 e^{\gamma_1} N(d_{1,1}) - \beta_2 e^{\gamma_2} N(d_{1,2}) - K N(x_1)
\]

where \( d_{1,1} = x_1 - \gamma_1 \sqrt{T_1}, d_{1,2} = x_1 - \gamma_2 \sqrt{T_2} \)

\[
\int_{x_1}^{x_2} g(x) u(x) \, dx = \int_{-\infty}^{x_1} g(x) u(x) \, dx - \int_{x_2}^{x_1} g(x) u(x) \, dx
\]

\[
= \beta_1 e^{\gamma_1} - \beta_2 e^{\gamma_2} - K
\]

\[
- \left[ \beta_1 e^{\gamma_1} N(d_{2,1}) - \beta_2 e^{\gamma_2} N(d_{2,2}) - K N(x_2) \right]
\]

where \( d_{2,1} = x_2 - \gamma_1 \sqrt{T_1}, d_{2,2} = x_2 - \gamma_2 \sqrt{T_2} \)

\[
\int_{x_1}^{x_2} g_x(x) u(x) \, dx = \int_{-\infty}^{x_1} g_x(x) u(x) \, dx - \int_{x_2}^{x_1} g(x) u(x) \, dx
\]

\[
= \beta_1 e^{\gamma_1} \left( 1 + N(d_{1,1}) - N(d_{2,1}) \right) - \beta_2 e^{\gamma_2} \left( 1 + N(d_{1,2}) - N(d_{2,2}) \right)
\]

\[
- K \left( 1 + N(x_1) - N(x_2) \right)
\]

\[
= \beta_1 e^{\gamma_1} \left( N(d_{1,1}) + N(-d_{1,1}) \right) - \beta_2 e^{\gamma_2} \left( N(d_{1,2}) + N(-d_{1,2}) \right)
\]

\[
- K \left( N(x_1) + N(-x_2) \right)
\]

\[
\int_{x_1}^{x_2} g_p(x) u(x) \, dx = \int_{-\infty}^{x_1} g_p(x) u(x) \, dx - \int_{x_2}^{x_1} g(x) u(x) \, dx
\]

\[
= -\int_{x_1}^{-x_2} g(x) u(x) \, dx
\]

\[
= \beta_1 e^{\gamma_1} \left( N(d_{1,1}) - N(d_{2,1}) \right) - \beta_2 e^{\gamma_2} \left( N(d_{1,2}) - N(d_{2,2}) \right)
\]

\[
- K \left( N(x_1) - N(x_2) \right)
\]
\begin{align*}
\mathbb{E}_p\left( a Y(T_i) - a_2 Y(T_2) - K \right)^+ &= a Y(0) e^{\int_0^{T_i} \mu_1(t) dt} \left( N(-d_{2,2}) + N(d_{1,1}) \right) \\
&\quad - a_2 Y(0) e^{\int_0^{T_2} \mu_2(t) dt} \left( N(-d_{2,2}) + N(d_{1,1}) \right) - K \left( N(-x_2) + N(x_1) \right) \\
\mathbb{E}_p \left( K - a Y(T_i) + a_2 Y(T_2) \right)^+ &= -a Y(0) e^{\int_0^{T_i} \mu_1(t) dt} \left( N(d_{2,2}) - N(d_{1,1}) \right) \\
&\quad + a_2 Y(0) e^{\int_0^{T_2} \mu_2(t) dt} \left( N(d_{2,2}) - N(d_{1,1}) \right) + K \left( N(x_2) - N(x_1) \right) \\
\mathbb{E}_p \left( a \delta Y(T_i) - a_2 \delta Y(T_2) - \delta K \right)^+ &= a Y(0) e^{\int_0^{T_i} \mu_1(t) dt} \left( N(-\delta d_{2,2}) + \delta N(d_{1,1}) \right) \\
&\quad - a_2 Y(0) e^{\int_0^{T_2} \mu_2(t) dt} \left( N(-\delta d_{2,2}) + \delta N(d_{1,1}) \right) - \delta K \left( N(-x_2) + \delta N(x_1) \right)
\end{align*}

If \( g(x^+) > 0 \), then \( g(x) \) has non-negative value on \([x_1, x_2]\) and \( g(x) \) has non-positive value on \((-\infty, x_1] \cup [x_2, \infty)\)

\[
\int_{-\infty}^{x_1} g_p(x) n(x) dx = \int_{x_1}^{x_2} g(x) n(x) dx \\
= \beta e^{\frac{1}{\gamma_T} \left( N(d_{2,2}) - N(d_{1,1}) \right) - \beta_2 e^{\frac{1}{\gamma_T} \left( N(d_{2,2}) - N(d_{1,1}) \right)} \\
- K \left( N(x_2) - N(x_1) \right)
\]

\[
\int_{-\infty}^{x_2} g_p(x) n(x) dx = -\int_{-\infty}^{x_1} g(x) n(x) dx - \int_{x_2}^{\infty} g(x) n(x) dx \\
= \beta_2 e^{\frac{1}{\gamma_T} \left( -N(-d_{2,2}) - N(d_{1,1}) \right) - \beta e^{\frac{1}{\gamma_T} \left( -N(-d_{2,2}) - N(d_{1,1}) \right)} \\
- K \left( -N(-x_2) - N(x_1) \right)
\]

\begin{align*}
\mathbb{E}_p\left( a Y(T_i) - a_2 Y(T_2) - K \right)^+ &= a Y(0) e^{\int_0^{T_i} \mu_1(t) dt} \left( N(-d_{2,2}) + N(d_{1,1}) \right) \\
&\quad - a_2 Y(0) e^{\int_0^{T_2} \mu_2(t) dt} \left( N(-d_{2,2}) + N(d_{1,1}) \right) - K \left( N(-x_2) + N(x_1) \right) \\
\mathbb{E}_p \left( K - a Y(T_i) + a_2 Y(T_2) \right)^+ &= a Y(0) e^{\int_0^{T_i} \mu_1(t) dt} \left( -N(-d_{2,2}) - N(d_{1,1}) \right) \\
&\quad - a_2 Y(0) e^{\int_0^{T_2} \mu_2(t) dt} \left( -N(-d_{2,2}) - N(d_{1,1}) \right) + K \left( N(x_2) - N(x_1) \right) \\
\mathbb{E}_p \left( \delta a Y(T_i) - \delta a_2 Y(T_2) - \delta K \right)^+ &= \delta a Y(0) e^{\int_0^{T_i} \mu_1(t) dt} \left( N(\delta d_{2,2}) - \delta N(d_{1,1}) \right) \\
&\quad - \delta a_2 Y(0) e^{\int_0^{T_2} \mu_2(t) dt} \left( N(\delta d_{2,2}) - \delta N(d_{1,1}) \right) - \delta K \left( N(x_2) - \delta N(x_1) \right)
\end{align*}

We combine (A1) and (A2) to acquire

\[
\mathbb{E}_p \left( \delta a Y(T_i) - \delta a_2 Y(T_2) - \delta K \right)^+
\]
\[
= \delta a_1 Y(0) e^{\int_0^{\mu_{1}(t)} dt} \left( N(\text{sign}(g(x^*))) \delta d_{1,1} - \text{sign}(g(x^*)) \delta N(d_{1,1}) \right) \\
- \delta a_2 Y(0) e^{\int_0^{\mu_{2}(t)} dt} \left( N(\text{sign}(g(x^*))) \delta d_{2,2} - \text{sign}(g(x^*)) \delta N(d_{2,2}) \right) \\
- \delta K \left( N(\text{sign}(g(x^*))) \delta x_2 - \text{sign}(g(x^*)) \delta N(x_2) \right)
\]

Case 2: \( g(x) \) has unique real root \( x_2 \) more than \( x^* \).

If \( g(x^*) < 0 \), then \( g(x) \) has nonnegative value on \([x_2, \infty)\) and
\( g(x) \) has non positive value on \((-\infty, x_2]\).

\[
\int_{-\infty}^{x_2} g_{x}(x)n(x)dx = \int_{x_2}^{\infty} g(x)n(x)dx \\
= \beta e^{\frac{1}{2\gamma T_i}} \left( -1 - N(d_{1,1}) \right) - \beta_2 e^{\frac{1}{2\gamma T_i}} \left( 1 - N(d_{2,2}) \right) - K \left( 1 - N(x_2) \right) \\
= \beta e^{\frac{1}{2\gamma T_i}} N(-d_{1,1}) - \beta_2 e^{\frac{1}{2\gamma T_i}} N(-d_{2,2}) - KN(-x_2)
\]

\[
\int_{-\infty}^{x_2} g_{p}(x)n(x)dx = -\int_{-\infty}^{x_2} g(x)n(x)dx \\
= -\left[ \beta e^{\frac{1}{2\gamma T_i}} N(d_{1,1}) - \beta_2 e^{\frac{1}{2\gamma T_i}} N(d_{2,2}) - KN(x_2) \right] \\
= KN(x_2) + \beta_2 e^{\frac{1}{2\gamma T_i}} N(d_{2,2}) - \beta e^{\frac{1}{2\gamma T_i}} N(d_{1,1})
\]

\[
E_p \left( a_1 Y(T_1) - a_2 Y_2(T_2) - K \right) = a_1 Y(0) e^{\int_0^{\mu_{1}(t)} dt} N(-d_{2,1}) - \\
a_2 Y_2(0) e^{\int_0^{\mu_{2}(t)} dt} N(-d_{2,2}) - KN(-x_2)
\]

\[
E_p \left( K - a_1 Y(T_1) + a_2 Y_2(T_2) \right) = KN(x_2) + a_2 Y_2(0) e^{\int_0^{\mu_{2}(t)} dt} N(d_{2,2}) - \\
a_1 Y(0) e^{\int_0^{\mu_{1}(t)} dt} N(d_{1,1})
\]

\[
E_p \left( \delta a_1 Y(T_1) - \delta a_2 Y_2(T_2) - \delta K \right) = \delta a_1 Y(0) e^{\int_0^{\mu_{1}(t)} dt} N(-\delta d_{2,1}) - \\
\delta a_2 Y_2(0) e^{\int_0^{\mu_{2}(t)} dt} N(-\delta d_{2,2}) - \delta KN(-\delta x_2) \tag{A3}
\]

If \( g(x^*) > 0 \), then \( g(x) \) has nonnegative value on \((-\infty, x_2]\) and
\( g(x) \) has non positive value on \([x_2, \infty)\)

\[
\int_{-\infty}^{x_2} g_{x}(x)n(x)dx = \int_{x_2}^{\infty} g(x)n(x)dx \\
= \beta e^{\frac{1}{2\gamma T_i}} N(d_{2,1}) - \beta_2 e^{\frac{1}{2\gamma T_i}} N(d_{2,2}) - KN(x_2)
\]
\[
\int_{-\infty}^{\infty} p(x)u^0(x)dx = -\int_{-\infty}^{\infty} g(x)u(x)dx \\
= \beta_1 e^{x^2 T_1} N(d_{1,1}) - \beta_2 e^{x^2 T_2} N(d_{1,2}) - KN(x_i) \\
= -\beta_1 e^{x^2 T_1} N(-d_{1,1}) + \beta_2 e^{x^2 T_2} N(-d_{1,2}) + KN(-x_i)
\]

\[
E_p \left( a_1 Y_1(T_1) - a_2 Y_2(T_2) - K \right) = a_1 Y_1(0)e^{\int_{0}^{t} \mu_1(s)dt} N(d_{2,1}) \\
- a_2 Y_2(0)e^{\int_{0}^{t} \mu_2(s)dt} N(d_{2,2}) - KN(x_i)
\]

\[
E_p \left( K - a_1 Y_1(T_1) + a_2 Y_2(T_2) \right) = KN(-x_i) + a_2 Y_2(0)e^{\int_{0}^{t} \mu_2(s)dt} N(-d_{2,2}) \\
- a_1 Y_1(0)e^{\int_{0}^{t} \mu_1(s)dt} N(-d_{2,1})
\]

\[
E_p \left( \delta a_1 Y_1(T_1) - \delta a_2 Y_2(T_2) - \delta K \right) = \delta a_1 Y_1(0)e^{\int_{0}^{t} \mu_1(s)dt} N(\delta d_{2,1}) \\
- \delta a_2 Y_2(0)e^{\int_{0}^{t} \mu_2(s)dt} N(\delta d_{2,2}) - \delta KN(\delta x_i) \tag{A4}
\]

We combine (A3) and (A4) to acquire

\[
E_p \left( \delta a_1 Y_1(T_1) - \delta a_2 Y_2(T_2) - \delta K \right) = \delta a_1 Y_1(0)e^{\int_{0}^{t} \mu_1(s)dt} N(\text{sign}(g(x^*))\delta d_{2,1}) - \\
\delta a_2 Y_2(0)e^{\int_{0}^{t} \mu_2(s)dt} N(\text{sign}(g(x^*))\delta d_{2,2}) - \delta KN(\text{sign}(g(x^*))\delta x_i)
\]

Case 3: \( g(x) \) has unique real root \( x_i \) less than \( x^* \).

If \( g(x^*) < 0 \), then \( g(x) \) has nonnegative value on \((-\infty,x_i] \) and

\( g(x) \) has non positive value on \([x_i,\infty) \)

\[
\int_{-\infty}^{\infty} g_x(x)u(x)dx = \int_{-\infty}^{x_i} g(x)u(x)dx \\
= \beta_1 e^{x^2 T_1} N(d_{1,1}) - \beta_2 e^{x^2 T_2} N(d_{1,2}) - KN(x_i) \\
= \beta_1 e^{x^2 T_1} \left( N(d_{1,1}) - 1 \right) - \beta_2 e^{x^2 T_2} \left( N(d_{1,2}) - 1 \right) - K \left( N(x_i) - 1 \right)
\]

\[
E_p \left( a_1 Y_1(T_1) - a_2 Y_2(T_2) - K \right) = a_1 Y_1(0)e^{\int_{0}^{t} \mu_1(s)dt} N(d_{1,1}) - \\
a_2 Y_2(0)e^{\int_{0}^{t} \mu_2(s)dt} N(d_{1,2}) - KN(x_i)
\]

\[
E_p \left( K - a_1 Y_1(T_1) + a_2 Y_2(T_2) \right) = -a_1 Y_1(0)e^{\int_{0}^{t} \mu_1(s)dt} N(-d_{1,1}) + \\
a_2 Y_2(0)e^{\int_{0}^{t} \mu_2(s)dt} N(-d_{1,2}) + KN(-x_i)
\]

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\[
\mathbb{E}_p \left( \delta a Y_1(T_1) - \delta a Y_2(T_2) - \delta K \right) = \delta a Y_1(0) e^{\int_0^{d_1(t)} dt} N(\delta d_1) - \\
\delta a Y_2(0) e^{\int_0^{d_2(t)} dt} N(\delta d_2) - \delta KN(\delta x_r) 
\]  
(A5)

If \( g(x^* ) > 0 \), then \( g(x) \) has nonnegative value on \([x_1, \infty)\) and

\[ g(x) \text{ has non positive value on } (-\infty, x_1] \]

\[
\int_{-\infty}^{x} g_x(x)n(x)dx = \int_{x_1}^{\infty} g(x)n(x)dx 
= \beta e^{\frac{1}{2} x^2 \gamma_{11}} N(-d_{11}) - \beta e^{\frac{1}{2} x^2 \gamma_{12}} N(-d_{12}) - KN(-x_1) 
\]

\[
\int_{-\infty}^{x} g_p(x)n(x)dx = -\int_{-\infty}^{x} g(x)n(x)dx 
= -\left[ \beta e^{\frac{1}{2} x^2 \gamma_{11}} N(d_{11}) - \beta e^{\frac{1}{2} x^2 \gamma_{12}} N(d_{12}) - KN(x_1) \right] 
= KN(x_1) + \beta e^{\frac{1}{2} x^2 \gamma_{11}} N(d_{11}) - \beta e^{\frac{1}{2} x^2 \gamma_{12}} N(d_{12}) 
\]

\[
\mathbb{E}_p \left( aY_1(T_1) - aY_2(T_2) - K \right) = aY_1(0) e^{\int_0^{d_1(t)} dt} N(-d_{11}) 
- aY_2(0) e^{\int_0^{d_2(t)} dt} N(-d_{12}) - \delta KN(-x_1) 
\]

\[
\mathbb{E}_p \left( K - aY_1(T_1) + aY_2(T_2) \right) = KN(x_1) + aY_1(0) e^{\int_0^{d_1(t)} dt} N(d_{11}) 
- aY_2(0) e^{\int_0^{d_2(t)} dt} N(d_{12}) 
\]

\[
\mathbb{E}_p \left( \delta a Y_1(T_1) - \delta a Y_2(T_2) - \delta K \right) = \delta a Y_1(0) e^{\int_0^{d_1(t)} dt} N(\delta d_{11}) - \\
- \delta a Y_2(0) e^{\int_0^{d_2(t)} dt} N(\delta d_{12}) - \delta KN(\delta x_r) 
\]  
(A6)

We combine (A5) and (A6) to acquire

\[
\mathbb{E}_p \left( \delta a Y_1(T_1) - \delta a Y_2(T_2) - \delta K \right) = \delta a Y_1(0) e^{\int_0^{d_1(t)} dt} N(-\text{sign}(g(x^*))) \delta d_{11}) 
- \delta a Y_2(0) e^{\int_0^{d_2(t)} dt} N(-\text{sign}(g(x^*))) \delta d_{12}) - \delta KN(-\text{sign}(g(x^*))) \delta x_r) 
\]

Case 4: \( g(x) \) has no real root

If \( g(x^* ) < 0 \) then \( g(x) < 0 \) for all \( x \)

\[
\int_{-\infty}^{x} g_x(x)n(x)dx = 0 
\]

\[
\int_{-\infty}^{x} g_p(x)n(x)dx = \int_{-\infty}^{x} -g(x)n(x)dx = K - \beta e^{\frac{1}{2} x^2 \gamma_{11}} + \beta e^{\frac{1}{2} x^2 \gamma_{12}} 
\]
\[ E_p \left( a, Y_1(T_1) - a, Y_2(T_2) - K \right)^* = 0 \]
\[ E_p \left( K - a, Y_1(T_1) + a, Y_2(T_2) \right)^* = K - a, Y_1(0) e^{\int_0^{T_1} \mu_1(t) dt} + a, Y_2(0) e^{\int_0^{T_2} \mu_2(t) dt} \]
\[ E_p \left( \delta a, Y_1(T_1) - \delta a, Y_2(T_2) - \delta K \right)^* = - \left( a, Y_1(0) e^{\int_0^{T_1} \mu_1(t) dt} - a, Y_2(0) e^{\int_0^{T_2} \mu_2(t) dt} - K \right) I_{\{d = 1\}} \] \quad (A7)

If \( g(x^*) > 0 \), then \( g(x) > 0 \) for all \( x \)

\[
\int_{-\infty}^{\infty} g_1(x)\eta(x)dx = \int_{-\infty}^{\infty} g(x)\eta(x)dx = \beta_1 e^{\frac{1}{2}x^2 T_1} - \beta_2 e^{\frac{1}{2}x^2 T_2} - K
\]
\[
\int_{-\infty}^{\infty} g_p(x)\eta(x)dx = 0
\]

\[ E_p \left( a, Y_1(T_1) - a, Y_2(T_2) - K \right)^* = a, Y_1(0) e^{\int_0^{T_1} \mu_1(t) dt} - a, Y_2(0) e^{\int_0^{T_2} \mu_2(t) dt} - K \]
\[ E_p \left( K - a, Y_1(T_1) + a, Y_2(T_2) \right)^* = 0 \]

\[ E_p \left( \delta a, Y_1(T_1) - \delta a, Y_2(T_2) - \delta K \right)^* = \left( a, Y_1(0) e^{\int_0^{T_1} \mu_1(t) dt} - a, Y_2(0) e^{\int_0^{T_2} \mu_2(t) dt} - K \right) I_{\{d = 1\}} \] \quad (A8)

We combine (A7) and (A8) to acquire

\[ E_p \left( \delta a, Y_1(T_1) - \delta a, Y_2(T_2) - \delta K \right)^* = \text{sign}(g(x^*)) \left( a, Y_1(0) e^{\int_0^{T_1} \mu_1(t) dt} - a, Y_2(0) e^{\int_0^{T_2} \mu_2(t) dt} - K \right) I_{\{d = \text{sign}(g(x^*))\}} \]