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Abstract

We propose a Bayesian procedure to estimate heteroscedastic variances of the regression error term ω , when the form of heteroscedasticity is *unknown*. The prior information on ω is elicited from the well-known Eicker–White Heteroscedasticity Consistent Variance–Covariance Matrix Estimator. Markov Chain Monte Carlo algorithm is used to simulate posterior pdf's of the unknown elements of ω . In addition to the numerical examples, we present an empirical investigation of the stock prices of Japanese pharmaceutical and biomedical companies to demonstrate usefulness of the proposed method.

key words

Eicker–White HCCM, orthogonal regressors, informative prior pdf's, MCMC, stock return variance

JEL Classification

C11, C13

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1 Introduction

We propose a Bayesian procedure to estimate heteroscedastic variances of the regression error term, when the form of heteroscedasticity is unknown. As pointed out by Amemiya (1985, p.199), the crucial ω vector¹ cannot be consistently estimated because as the number of parameters increases, the sample size also increases at the same rate, leading to the lack of identifiability of ω . In asymptotics framework, Eicker (1963) and White (1980) independently developed a well-known consistent variance-covariance matrix estimator (“HCCM” hereafter) for the OLS regression coefficient estimator. The methodology we propose in this paper is a Bayesian that uses information obtained from the HCCM, in terms of a candidate density of a Metropolis-Hastings (“M-H” hereafter) algorithm in Markov Chain Monte Carlo simulation. The lack of identifiability of ω poses no problem. For one thing, as in Amemiya (1985) we use an orthogonal regression that circumvents possible underidentifiability of ω , and we shall explain this method in detail later². Second, we impose a prior on ω so that the vector becomes identifiable in a Bayesian context.

The trend in the HCCM literature seems to be how to improve the finite sample performance of tests of the linear restriction(s) on the coefficient vector, *e.g.*, Long and Ervin(2000) and Godfrey (2006), among others. We note that our focus in this paper is in the direct estimation of the elements ω vector. There are papers that deal with statistical inferences of regression coefficients, when the scedastic function of the error term is unconstrained. Robinson (1987), for example, assumes it to be a function of regressors, and

¹The ω vector has in its elements, all the *normalized* diagonal elements of variance-covariance matrix of the regression error term. The normalization rule for the matrix is given just below equation (1).

²Amemiya *op cit* proposes and uses an orthogonal regression to obtain a better performing GLS.

derives an GLS estimator that is more efficient than the existing ones. Our Bayesian estimation of heteroscedasticity should sharpen posterior density of regression coefficient vector β and/or lead to a better predictive density. It may also lead to more efficient estimator of β in terms of asymptotic theory framework as well.

We need to discuss the direct estimation of the ω vector. In financial returns, ω is nothing but the volatility. In order to access an option pricing, what we need to do first is to come up with a reasonable estimate of volatility. Our estimation of ω needs no parametric model for the volatility process such as the GARCH model, since we use information obtained from the HCCM estimation, in our MCMC simulation. If we wish to estimate a volatility process in time series data nonparametrically, what we usually do is to calculate a historical volatility series. But this is just a descriptive statistic without a theoretical background. Moreover, when it comes to cross section data, historical volatility calculation breaks down for obvious reasons. Our Bayesian method, on the other hand, should provide a good deal of theoretical support for cross sectional data.

Our strategy to estimate the ω vector is Bayesian. After assuming a usual prior density for the parameters in the regression model, we are able to write down a joint posterior density. The usual parameters such as regression coefficients may be easily simulated using the Gibbs sampler scheme. It is in the simulation of the elements of ω that we use the HCCM. We use results from HCCM to form the candidate density in the M-H algorithm.

The rest of this paper is organized as follows. In section 2, we set our regression model. Prior pdf's are assumed here, and the joint posterior pdf is derived. Section 3 starts out with our Bayesian MCMC calculation by a

Gibbs sampler. We propose to use the Eicker–White result to simulate $\boldsymbol{\omega}$ by a M-H scheme. Our numerical illustration and an empirical investigation of the stock prices of Japanese pharmaceutical and biomedical companies, are given in section 4. Section 5 concludes.

2 The Model and the Joint Posterior pdf

Suppose a heteroscedastic regression model,

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad (1)$$

where $\mathbf{X} \sim n \times K$ is a purely exogenous variable matrix, $\boldsymbol{\varepsilon} \sim \mathbf{N}(\mathbf{0}, \sigma^2 \boldsymbol{\Omega})$, $\boldsymbol{\Omega} = \text{diag}(\boldsymbol{\omega}) = \text{diag}(\omega_1, \dots, \omega_n)$, $\boldsymbol{\omega} \subset \mathfrak{R}_+^n$, $\text{tr}(\boldsymbol{\Omega}) = \sum_{i=1}^n \omega_i = n$, $\sigma^2 > 0$, and n is the sample size³. Let the column vector $\boldsymbol{\theta}$ be all the parameters in the model, *i.e.*, $\boldsymbol{\theta}' = (\boldsymbol{\beta}' \ \sigma^2 \ \boldsymbol{\omega}') \sim 1 \times (K + 1 + n)$. The likelihood function, then, becomes

$$p(\mathbf{y}|\boldsymbol{\theta}, \mathbf{X}) \propto |\boldsymbol{\Omega}|^{-1/2} \sigma^{-n} \exp\left(-\frac{1}{2\sigma^2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})' \boldsymbol{\Omega}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})\right). \quad (2)$$

Assuming that the components of $\boldsymbol{\theta}$ are independent *a priori*, and specifically the prior of $\boldsymbol{\beta}$ is diffuse, we obtain the following prior pdf under the $\text{tr}(\boldsymbol{\Omega}) = n$ assumption:

$$p(\boldsymbol{\theta}) \propto |\sigma^2 \ \boldsymbol{\Omega}|^{-1/2} \mathbf{1}_{[\underline{\omega}, \bar{\omega}]}(\boldsymbol{\omega}), \quad (3)$$

where “ $\mathbf{1}_{[\underline{\omega}, \bar{\omega}]}(\boldsymbol{\omega})$ ” designates an indicator function such that it takes on the value one if an element of the $\boldsymbol{\omega}$ vector is contained in the range $[\underline{\omega}, \bar{\omega}]$, and zero otherwise.

We shall discuss how we obtained the upper bound $\bar{\omega}$ and the lower bound $\underline{\omega}$. Our scaling assumption of $\boldsymbol{\omega}$ is given by $\sum_{i=1}^n \omega_i = n$. This means

³“ $\text{tr}(\boldsymbol{\Omega}) = n$ ” is a technical assumption that is often employed for identifiability purpose in regression models with heteroscedastic error term. See, *e.g.*, Greene (2008, p.158).

that the sample mean of $\boldsymbol{\omega}$ is one, *i.e.*, $\bar{\omega} = 1$. Consider a rather extreme case that ten percent of ω_i 's have ten times the magnitude of $\bar{\omega} = 1$, *i.e.*, 10% of elements of $\boldsymbol{\omega}$ is $\omega_i = 10$. This leaves the rest of ninety percent of ω_j 's, zeros only, *i.e.*, $\omega_j = 0$ for $j \neq i$, where the i belong to the first ten percent of the entire n . Hence, along with the assumption $\omega_i > 0$ for $i = 1, \dots, n$, the fact that $\bar{\omega} = 1$ encourages us to set a certain range for each ω_i that depends on the sample size, n . In the rest of this paper, our sample size in simulation and also empirical investigation, is fifty. We may, therefore, safely set $(\underline{\omega}, \bar{\omega}) = (0.025, 12.5)$, say. What does this imply? Consider an example of the first four observations taking up the bulk of the entire volatility. That is $\sum_{i=1}^4 \omega_i = 46$ while $\sum_{i=5}^{50} \omega_i = 4$. If this is the case, our $\underline{\omega}$ and $\bar{\omega}$ may be given by

$$\frac{4}{46} = 0.087 > 0.025 = \underline{\omega} \quad \text{and} \quad \frac{46}{4} = 11.5 < 12.5 = \bar{\omega}.$$

The above implies that even in this extreme case, our lower bound, $\underline{\omega}$, is far smaller than the average of the latter forty six ω_i 's, while our upper bound, $\bar{\omega}$, far exceeds the average of the first four ω_i 's. We emphasize that this data dependent prior on the bounds of ω_i for $i = 1, \dots, n$, are relatively easy to set, since the only information needed to form it is n . We also emphasize that our experiments show insensitivity of MCMC draws to the bounds, in the sense that the results are robust to the values of approximately ten percent changes in the values of $(\underline{\omega}, \bar{\omega})$. Our prior pdf for $\boldsymbol{\theta}$ in this paper is given by

$$p(\boldsymbol{\theta}) \propto \sigma^{-1} |\boldsymbol{\Omega}|^{-1/2} \mathbf{1}_{[0.025, 12.5]}(\boldsymbol{\omega}), \quad (4)$$

and this replaces equation (3). Note that this specification of $(\underline{\omega}, \bar{\omega})$ amounts to

$$(\underline{\omega}, \bar{\omega}) = (0.025/50, 12.5/50) = (0.0005, 0.25)$$

that is $\underline{\omega} = 0.05$ percent of n , while $\bar{\omega} = 25$ percent of n . This is the rule of thumb that we recommend. When implementing this “rule of thumb,” around ten percent of each bound, should be explored.

Our joint posterior pdf is proportional to the product of equation (2) times (4):

$$p(\boldsymbol{\theta}|\mathbf{y}, \mathbf{X}) \propto |\boldsymbol{\Omega}|^{-1} \sigma^{-n-1} \exp\left(-\frac{1}{2\sigma^2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\boldsymbol{\Omega}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})\right) \mathbf{1}_{[0.025, 12.5]}(\boldsymbol{\omega}) \quad (5)$$

3 Posterior Simulation: MCMC

We shall simulate $\boldsymbol{\beta}$ and σ using a Gibbs sampler, and then combine a Metropolis-Hastings (“MH” hereafter) algorithm to simulate the entire $\boldsymbol{\theta}$ ⁴.

3.1 Simulating $\boldsymbol{\beta}$

Let $\boldsymbol{\theta}_{-\boldsymbol{\beta}}$ represent a vector $\boldsymbol{\theta}$ without $\boldsymbol{\beta}$. Then the Gibbs sampler for $\boldsymbol{\beta}$ starts out with

$$\boldsymbol{\beta}^{(1)}|\boldsymbol{\theta}_{-\boldsymbol{\beta}}^{(0)} \sim p(\boldsymbol{\beta}|\boldsymbol{\theta}_{-\boldsymbol{\beta}}^{(0)}, \mathbf{y}, \mathbf{X}),$$

where the superscript r in $\boldsymbol{\beta}^{(r)}$ denotes the r th simulated $\boldsymbol{\beta}$ (thus $r = 0$ is the initial value) for $r = 1, \dots, R$, and R is the number of replications in the MCMC. The conditional probability in the right hand side represents a fully conditional posterior pdf of $\boldsymbol{\beta}$, where a notation “ $A \sim p(\cdot)$ ” is used to indicate that a value A is generated from $p(\cdot)$ pdf. We use the OLS estimated residual vector $\mathbf{e} = \mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}$ to obtain $s^2 = \mathbf{e}'\mathbf{e}/(n - K)$, and this is used as the initial value of σ^2 . Next, note that it is pretty reasonable to set the initial values of $\boldsymbol{\omega}$, $\boldsymbol{\omega}^{(0)}$, as $\boldsymbol{\omega}^{(0)} = \boldsymbol{\iota}_n$, where $\boldsymbol{\iota}_n \sim n \times 1$ is a vector of all one’s. This comes from the scaling assumption of $\boldsymbol{\omega}$, *i.e.*, $\bar{\omega} = 1$. As a

⁴Following several sections, in particular the Gibbs sampler subsection, may seem to be a textbook content. But the notations introduced in these subsections are needed in the subsection where we outline simulation of $\boldsymbol{\omega}$ (Section 3.3).

result, our initial value vector becomes $\boldsymbol{\theta}_{-\beta}^{(0)} = (s^2 \boldsymbol{\iota}_n)'$. Notice that the full conditional distribution of $\boldsymbol{\beta}$ given $\boldsymbol{\theta}_{-\beta}$ is

$$\boldsymbol{\beta} | (\boldsymbol{\theta}_{-\beta}, \mathbf{y}, \mathbf{X}) \sim \text{N} \left(\hat{\boldsymbol{\beta}}, \sigma^{-2} (\mathbf{X}' \boldsymbol{\Omega}^{-1} \mathbf{X})^{-1} \right).$$

Then, using $\boldsymbol{\theta}_{-\beta}^{(0)}$, we may generate $\boldsymbol{\beta}^{(1)}$ from a multivariate normal conditional density:

$$\boldsymbol{\beta}^{(1)} \sim \text{N} \left(\hat{\boldsymbol{\beta}}, s^2 (\mathbf{X}' \mathbf{X})^{-1} \right),$$

where it turns out that the diagonal elements of the initial value of $\boldsymbol{\Omega}$ is simply $\text{diag}(\boldsymbol{\iota}_n) = \mathbf{I}_n$. In $r \geq 2$ iterations, above $\boldsymbol{\beta}^{(1)}$ generation scheme is replaced by

$$\boldsymbol{\beta}^{(r)} \sim \text{N} \left(\boldsymbol{\beta}^{(r-1)}, \sigma^{(r-1)^2} (\mathbf{X}' \boldsymbol{\Omega}^{(r-1)^{-1}} \mathbf{X})^{-1} \right), \quad (6)$$

however.

3.2 Simulating σ

We, next, turn to generation of σ . The conditional posterior pdf of σ becomes

$$p(\sigma | \boldsymbol{\theta}_{-\sigma^2}, \mathbf{y}, \mathbf{X}) \propto \sigma^{-n-1} \exp \left(-\frac{\boldsymbol{\varepsilon}' \boldsymbol{\Omega}^{-1} \boldsymbol{\varepsilon}}{2} / \sigma^2 \right), \quad (7)$$

where $\boldsymbol{\varepsilon} = \mathbf{y} - \mathbf{X}\boldsymbol{\beta}$. This is the kernel of an inverse gamma pdf, $\mathcal{IG} \left(\frac{n}{2}, \frac{\boldsymbol{\varepsilon}' \boldsymbol{\Omega}^{-1} \boldsymbol{\varepsilon}}{2} \right)$ ⁵.

The initial value of $\boldsymbol{\theta}_{-\sigma^2}$ becomes $\boldsymbol{\theta}_{-\sigma^2}^{(0)} = (\boldsymbol{\beta}^{(1)'}, \boldsymbol{\iota}_n)'$, hence $\sigma^{(1)}$ is generated as follows:

$$\sigma^{(1)} | \boldsymbol{\theta}_{-\sigma^2}^{(0)} \sim \mathcal{IG} \left(\frac{n}{2}, \frac{\boldsymbol{\varepsilon}^{(1)'}, \boldsymbol{\varepsilon}^{(1)}}{2} \right),$$

⁵We denote $f(x|\alpha, \beta) \propto x^{-(\alpha+1)} \exp(-\beta/x) \mathbf{1}_{[0,+\infty)}(x)$ distribution as $\mathcal{IG}(\alpha, \beta)$, inverse gamma with parameters α and β . Such x has $\text{E}(x) = \beta/(\alpha - 1)$ for $\alpha > 1$, and $\text{Var}(x) = \beta^2/((\alpha - 1)^2(\alpha - 2))$ for $\alpha > 2$. See, *e.g.*, Robert and Casella (2004, p.582). Our fully conditional σ pdf is given in equation (7), however, has σ^2 in the denominator in the exponential function. To match the inverse gamma expression to the above, make a transformation of random variables from x to σ and we obtain $f(\sigma|\alpha, \beta) \propto \sigma^{-2(\alpha+1)} \exp(-\beta/\sigma^2) \mathbf{1}_{[0,+\infty)}(\sigma)$. $\mathcal{IG}(\cdot)$ expression in the text, thus, follows.

where $\boldsymbol{\varepsilon}^{(1)} = \mathbf{y} - \mathbf{X}\boldsymbol{\beta}^{(1)}$. For $r \geq 2$ simulations, the above is replaced by

$$\sigma^{(r)} | \boldsymbol{\theta}_{-\sigma^2}^{(r-1)} \sim \mathcal{IG} \left(\frac{n}{2}, \frac{\boldsymbol{\varepsilon}^{(r)'} \boldsymbol{\Omega}^{(r-1)-1} \boldsymbol{\varepsilon}^{(r)}}{2} \right). \quad (8)$$

3.3 Simulating $\boldsymbol{\omega}$ Using HCCM Information

3.3.1 Conditional pdf of ω_i

Let us now derive conditional distribution, $\omega_i | \boldsymbol{\theta}_{-\omega_i}, \mathbf{y}, \mathbf{X}$. It is easy to see that the conditional distribution of $\boldsymbol{\omega}$ is in the form,

$$p(\boldsymbol{\omega} | \boldsymbol{\theta}_{-\boldsymbol{\omega}}, \mathbf{y}, \mathbf{X}) \propto |\boldsymbol{\Omega}|^{-1} \exp \left(-\frac{1}{2\sigma^2} \boldsymbol{\varepsilon}' \boldsymbol{\Omega}^{-1} \boldsymbol{\varepsilon} \right) \mathbf{1}_{[0.025, 12.5]}(\boldsymbol{\omega}).$$

Inside the exponential function in the above, $\boldsymbol{\varepsilon}' \boldsymbol{\Omega}^{-1} \boldsymbol{\varepsilon}$ becomes $\boldsymbol{\varepsilon}' \boldsymbol{\Omega}^{-1} \boldsymbol{\varepsilon} = \sum_{i=1}^n \left(\frac{\varepsilon_i}{\sqrt{\omega_i}} \right)^2 = \boldsymbol{\varepsilon}'^* \boldsymbol{\varepsilon}^*$, where $\boldsymbol{\varepsilon}^* = (\varepsilon_1^*, \dots, \varepsilon_n^*)' \sim n \times 1$ and $\varepsilon_i^* = \varepsilon_i / \sqrt{\omega_i}$, since $\boldsymbol{\Omega} = \text{diag}(\boldsymbol{\omega})$. Note also $|\boldsymbol{\Omega}|^{-1} = \prod_{i=1}^n \omega_i^{-1}$, we then have for a particular ω_i ,

$$p(\omega_i | \boldsymbol{\theta}_{-\omega_i}, \mathbf{y}, \mathbf{X}) \propto \omega_i^{-1} \exp \left(-\frac{\varepsilon_i^2}{2\sigma^2 \omega_i} \right) \mathbf{1}_{[0.025, 12.5]}(\omega_i),$$

where $\varepsilon_i = y_i - \mathbf{x}_i' \boldsymbol{\beta}$ and \mathbf{x}_i is the i th row of \mathbf{X} . The above is not a tractable pdf for two reasons, however. First, if we did not have a bounds constraint, $[\underline{\omega}, \bar{\omega}]$, this function cannot be integrated, *i.e.*, $\int_0^\infty \omega_i^{-1} \exp \left(-\frac{\varepsilon_i^2}{2\sigma^2 \omega_i} \right) d\omega_i$ is not finite. Secondly, even if we let the first parameter of the inverse gamma pdf to be some positive value $n_o > 1^6$, it would be awfully difficult to simulate a truncated inverse gamma random variable that has parameters, $\mathcal{IG}(n_o, \frac{\varepsilon_i^2}{2\sigma^2})$, in a Gibbs scheme, *i.e.*, always accepting whatever is generated.

3.3.2 HCCM Information to be used in M-H

Let us, for a moment, depart from the problem of generating ω_i , and see what the sampling theory has to tell us about the heteroscedastic $\boldsymbol{\varepsilon}$. An

⁶We note that setting such $n_o > 0$, when it is not, is quite arbitrary. To deal with this possibly unidentified conditional posterior pdf, we propose to set bounds on the $\boldsymbol{\omega}$ vector, and employ an M-H scheme, as discussed in section 3.3.3, below. Instead of this avenue of approach, we might assume a prior on the first parameter in the inverse gamma pdf, to turn our model to a hierarchical Bayes model.

OLS estimated residual vector, $\mathbf{e} = (e_1, \dots, e_n)'$, is already defined for equation (1). Eicker (1963) and White (1980)'s result may be used to form a sandwich variance-covariance matrix estimator of $\hat{\boldsymbol{\beta}}$, HCCM, that is consistent: $\left(\frac{1}{n}\mathbf{X}'\mathbf{X}\right)^{-1}\hat{\mathbf{H}}\left(\frac{1}{n}\mathbf{X}'\mathbf{X}\right)^{-1}$, where $\hat{\mathbf{H}} = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i'e_i^2$ and $\mathbf{X}' = (\mathbf{x}_1, \dots, \mathbf{x}_n) \sim K \times n$. Under the usual assumptions such as a full rank matrix $\lim \frac{1}{n}\mathbf{X}'\mathbf{X} \equiv \mathbf{Q} < \infty$, it is well known that

$$\hat{\mathbf{H}} \xrightarrow{p} \mathbf{H} = \lim \frac{\sigma^2}{n}\mathbf{X}'\boldsymbol{\Omega}\mathbf{X} = \lim \frac{\sigma^2}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i'\omega_i \quad (9)$$

holds. We may, hence, postulate a multivariate regression model of the following:

$$\hat{\mathbf{H}} = \mathbf{H} + \mathbf{V}, \quad (10)$$

where $\mathbf{V} \sim K \times K$ is an error term matrix that is constrained to make both \mathbf{H} and $\hat{\mathbf{H}} > \mathbf{0}$.⁷ After some manipulations (see the appendix to this paper), the above becomes

$$\hat{\mathbf{h}} = \boldsymbol{\mathcal{X}}_n\sigma^2\boldsymbol{\omega} + \mathbf{v}, \quad (11)$$

where $\hat{\mathbf{h}} = \text{vech}(\hat{\mathbf{H}}) \sim K' \times 1$, $\boldsymbol{\mathcal{X}}_n = [\text{vech}(\mathbf{x}_1\mathbf{x}_1') \dots \text{vech}(\mathbf{x}_n\mathbf{x}_n')] \sim K' \times n$, $\boldsymbol{\omega} = (\omega_1 \dots \omega_n)' \sim n \times 1$, and $\boldsymbol{\omega} \subset \mathfrak{R}_+^n$, $\mathbf{v} = \text{vech}(\mathbf{V}) \sim K' \times 1$, and $K' \equiv \frac{1}{2}K(K+1)$. This is a regression of K' dimension $\hat{\mathbf{h}}$ variable on $K' \times n$ explanatory variable matrix $\boldsymbol{\mathcal{X}}_n$. Hence the degrees of freedom, here, is $\text{df} = K' - n$, and under the usual situations such “df” could even be negative, *i.e.*, the $\hat{\mathbf{h}}$ regression is unestimable. We have now found that the $\boldsymbol{\omega}$ parameters in equation (11) are identifiable only when the $\boldsymbol{\beta}$ vector in equation (1) is identifiable. To restate,

the $\boldsymbol{\omega}$ vector is identifiable if the inequalities $K' > n$ and $n > K$ simultaneously hold.

⁷We use the notation “ $\mathbf{A} > \mathbf{0}$ ” to denote that the matrix \mathbf{A} is positive definite.

In a typical case like $n = 50$ and $K = 3$, ω is unidentifiable since $K' = 6$, however, take a case like $n = 20$ and $K = 10$ then $K' = 55$ which makes ω to be identifiable. But this is, indeed, a rare case. ω may become identifiable if the column dimension of \mathbf{X} increases, then, the “sample size” in equation (11), *i.e.*, K' , should also increase. What we need to do is to increase K' without altering \mathbf{e} , estimated regression residual vector from equation (1), that constitutes a part of dependent variable in equation (11).

Amemiya (1983) proposed “partially generalized least squares” estimation method to improve upon efficiency over OLSE. We may use his idea to circumvent the problem that we stated in the previous paragraph. Let the regressor matrix in equation (1) to be

$$\mathbf{Z} = (\mathbf{X} \ \mathbf{W}) \sim n \times (K + K_W), \quad (12)$$

where we require that \mathbf{W} is orthogonal to *all* the variables in equation (1), *i.e.*, $\mathbf{W}'(\mathbf{y} \ \mathbf{X}) = \mathbf{0} \sim K_W \times (K + 1)$ ⁸. It is easy to see that \mathbf{b} and \mathbf{e} from equation (1) remain the same even after we regress \mathbf{y} on \mathbf{Z} instead of \mathbf{X} alone. We now note that the the identifiability condition given in the previous paragraph becomes $K'' > n$ and $n > K$, where $K'' \equiv (K + K_W)(K + K_W + 1)/2$. We could now increase K_W to the extent that $K'' > n$ condition is satisfied⁹. In summary, we now have the control over the number of additional orthogonal variables, K_W , so that the above two conditions are met, while the \mathbf{e} vector is unchanged.

⁸It is well known that the OLSE of β for the transformed regression, $\mathbf{Z}'\mathbf{y} = \mathbf{Z}'\mathbf{X}\beta + \mathbf{Z}'\mathbf{e}$, is identical to the OLSE for the original equation. Incidentally, Amemiya *op.cit.* showed that the GLS estimator for the transformed model is more efficient than the OLSE.

⁹For instance, take the previous numerical case of $n = 50$ and $K = 3$. If we set $7 < K_W < 47$ then the conditions are satisfied

3.3.3 HCCM Information Accomodated in Candidate Density

We have thus shown that using \mathbf{Z} instead of \mathbf{X} as the regressor matrix in equation (1), the $\boldsymbol{\omega}$ vector is always identifiable. Let the OLSE of $\sigma^2\boldsymbol{\omega}$ be $\sigma^2\hat{\boldsymbol{\omega}}$. “ σ^2 ” has no hat on it since in our Bayesian MCMC, σ^2 is already generated in the previous step by the Gibbs sampler scheme. We turn to an M-H algorithm, to generate ω_i . For this purpose, we use a truncated normal as our candidate generating density:

$$\omega_i \sim N(\hat{\omega}_i, \tilde{c} \times \hat{\omega}_i^2) \mathbf{1}_{\omega_i \in (.25, 12.5)}, \quad (13)$$

for $i = 1, \dots, n$ where \tilde{c} is a constant that has been determined in the following way. In the footnote that discusses inverse gamma distribution (footnote 7), we see that $\text{Var}(x) = \text{E}(x)^2(\alpha - 2)^{-1}$ for $x \sim \mathcal{IG}(\alpha, \beta)$. Hence, if we were to let α a little over “2” so that the variance would not vanish, then almost any value of \tilde{c} suffices. The value ought to be such that the average acceptance rate in the independence chain M-H to be 20% to 50%. The value we came up with is $\tilde{c} = 15$ after some experiments, and the acceptance rate for such candidate generating density function, in our experiment, is 33%, which is reasonable¹⁰. For $r \geq 2$ sequence, equation (13) has been employed as the candidate generating density.

3.4 Hybrid of Gibbs Sampler and M-H

Let us describe our MCMC sequence in its entirety. For the parameter vector $\boldsymbol{\theta} = (\boldsymbol{\beta}' \ \sigma \ \boldsymbol{\omega}')$, we set the initial value $\boldsymbol{\theta}^{(0)} = (\hat{\boldsymbol{\beta}}' \ s^2 \ \boldsymbol{\iota}'_n)'$ to generate the first round $\boldsymbol{\theta}^{(1)}$. Note here that we first draw $\boldsymbol{\beta}^{(1)}$, hence $\hat{\boldsymbol{\beta}}$ is not really needed in $\boldsymbol{\theta}^{(0)}$. What we actually need is $\boldsymbol{\theta}_{-\boldsymbol{\beta}}^{(0)}$. The sequence of our MCMC continues

¹⁰In M-H, an acceptance rate between 20% and 50% is regarded as reasonable, *e.g.*, see Koop (2003, p.98). In fact in a series of experiments, we found that the average acceptance rate of 21% when $\tilde{c} = 10$, 49% when $\tilde{c} = 25$, and 58% when $\tilde{c} = 35$. We may add that despite these differences in the average acceptance rates, however, the results are fairly robust.

in the way specified by equations (6), (8) and (13) for $r \geq 2$, combining the Gibbs sampler and M-H. The details of monitoring and diagnostic checking concerning this MCMC sequence, is given in the next section.

4 Numerical and Empirical Investigation

We first generate a data set to make sure that our Bayesian procedure works. After that, we move on to use a Japanese stock price data set to show that our Bayesian method can be used to estimate stock return volatilities without assuming *any* structure on them *a priori*.

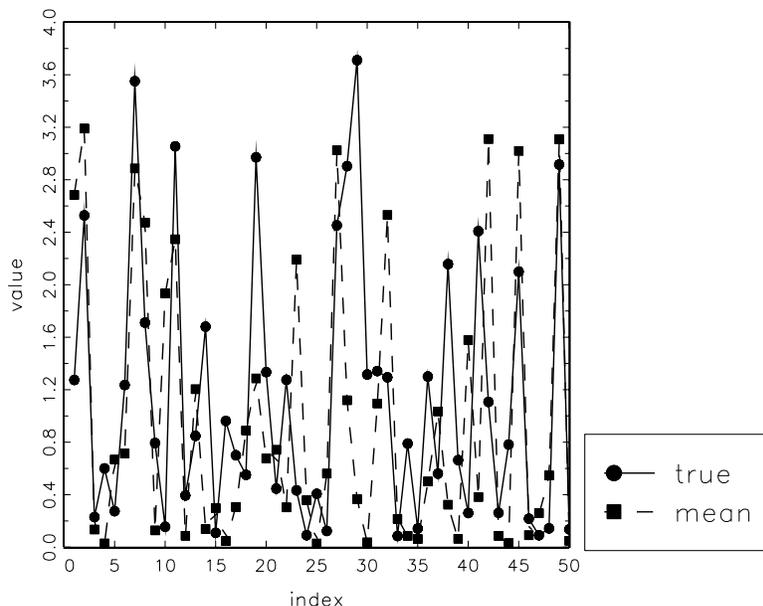
4.1 Numerical Illustration

For the generation purpose, we let the parameters of our model be $\beta = (1, 3, -2)'$; $\sigma^2 = 10$; sample size $n = 50$; $\mathbf{X} = (\boldsymbol{\iota}_n, \mathbf{x}_2, \mathbf{x}_3) \sim n \times 3$, where $\boldsymbol{\iota}_n$ is an $n \times 1$ vector of all one's, $\mathbf{x}_2 \sim N(3, I_n)$, and $\mathbf{x}_3 \sim N(6, 2^2 I_n)$. As to $\boldsymbol{\omega}$, we need to specify total of 50 elements. We generated ω_i 's using an uniform random number generator. Assigned ω_i 's will be given in a chart that compares them to the Bayesian posterior mean values, later (see Figure 1 below). Using these prespecified set of parameter values, we generated data on \mathbf{y} . This is used to generate \mathbf{W} matrix to form $\mathbf{Z} = (\mathbf{X} \ \mathbf{W})$ ¹¹. By regressing \mathbf{y} on \mathbf{Z} , we obtain one set of $\mathbf{e} \sim n \times 1$, and hence a $\hat{\mathbf{h}} \sim K'' \times 1$ vector. We now have one sample of \mathbf{y} , \mathbf{X} and $\hat{\mathbf{h}}$.

In the MCMC computation, we drew total of 30,000 $\boldsymbol{\theta}$'s. After some experiments, using such diagnostics devices as Geweke's (1992), we decided to throw away the first 10,000 simulated values as burn-in's, and took every fifth drawn values after that to lessen the possible serial correlation inherent in MCMC. The average acceptance rate in the M-H step was 33%. We

¹¹In our *GAUSS* programming, we used the "NULL(\cdot)" command to obtain $\mathbf{W} \sim n \times K_W$ that is orthogonal to \mathbf{X} . NULL command uses the QR decomposition to obtain the orthogonal complement of \mathbf{X} .

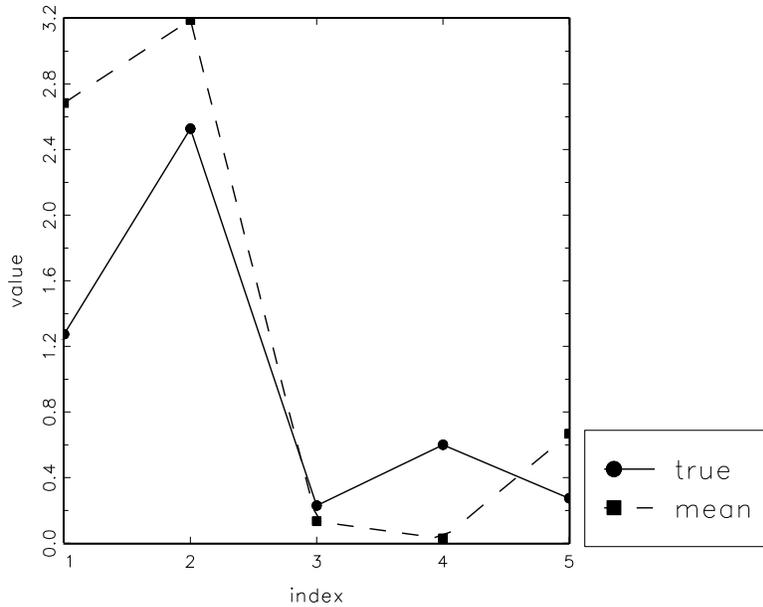
Figure 1: Posterior Means of the 50 ω_i 's



now have a set of $20,000/5 = 4,000$ simulated values for each of σ^2 and ω_i 's ($i = 1, \dots, 50$) available. Since giving 50 posterior pdf's would not contribute to the understanding of our procedure, we decided to present posterior means of the fifty ω_i 's, and this is given in Figure 1.

Fifty assigned ω_i values are given using a solid line, and posterior means using a dotted line. We see that there are some discrepancies between the true (or given) value and the posterior mean value, however, mostly they are close. Their closeness may be observed more clearly for the first five ω_i 's of Figure 1, in Figure 2. In particular, posterior means of ω_1 to ω_3 , *i.e.*, $E(\omega_1|\text{data})$ to $E(\omega_3|\text{data})$, are seen to follow randomly assigned values very closely. To further this argument, we used some pseudo R-squared measures *e.g.*, given in Wooldridge (2002, pp.465-466), as measures of closeness to the assigned (or true) ω . Let one of such pseudo R-squared be $\tilde{R}_{[E(\omega|\text{data})]}^2$. In all cases that we computed, we found $\tilde{R}_{E(\omega|\text{data})}^2$ to be significantly larger than $\tilde{R}_{\text{homogeneous}}^2$, where $\tilde{R}_{\text{homogeneous}}^2$ is the pseudo R-squared

Figure 2: Posterior Means of the First Five ω_i 's



measure for the homogeneous case, *i.e.*, $\omega =$ “a constant vector.” This shows our $E(\omega_i|\text{data})$'s closeness to the assigned ω_i .

We depicted the marginal posterior pdf's of the first two $\sigma^2\omega_i$'s in Figure 3. We see that the two marginal posterior pdf's seem to be equipped with typical characteristics of a variance pdf, *i.e.*, a gamma like shape with a hump close to the origin and skewed to the right. We emphasize that the shapes have been obtained without any prior shape information on them. This fact may indicate an appropriateness of our MCMC simulator.

We next examine the posterior standard deviations computed from the marginal densities of ω_i 's. The 50 posterior standard deviations are given in Figure 4. We observe that they tend to fluctuate a lot. For instance, posterior standard deviation of ω_1 and ω_2 are slightly less than “1.8” while that of ω_4 is close to zero. An interesting observation may be made, when we compare Figure 4 to Figure 1 or to Figure 2. There is a tendency that

Figure 3: Marginal Posterior's of $\sigma^2\omega_1$ and $\sigma^2\omega_2$

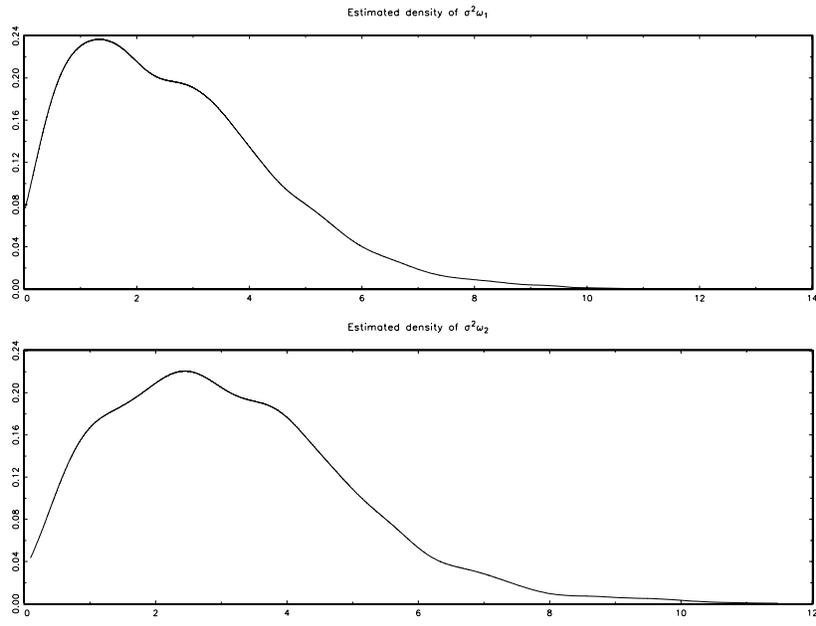
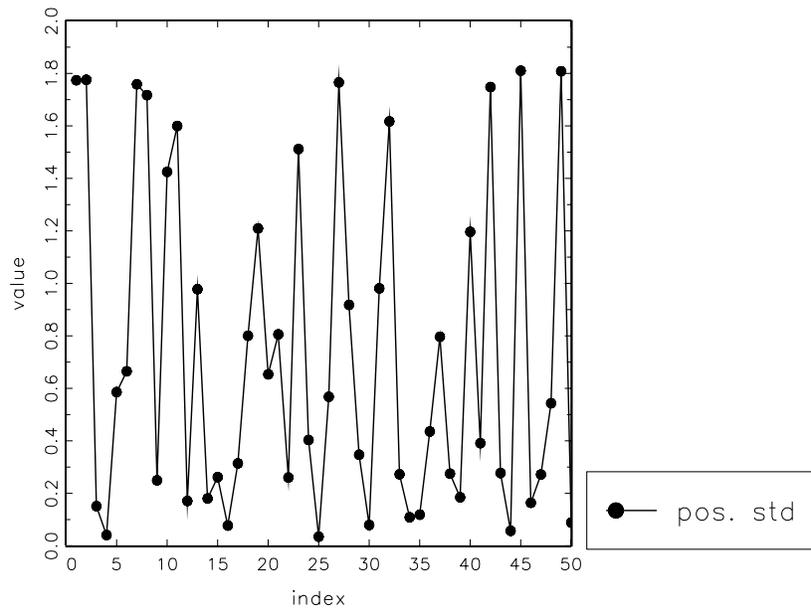


Figure 4: Posterior Standard Deviations of the 50 ω_i 's



posterior means and standard deviations, move in the same direction. This may be interpreted to come from the fact that marginal posterior pdf of ω_i 's are in gamma like shape¹².

4.2 Empirical Investigation: Japanese Stock Returns

4.2.1 Theoretical Model

In this section, we use a panel data set to show that our procedure works well in practice. Saying “panel data set” may be far stretched, however, since we do not implement, for example, random effects model. We carry out a two-step time series to cross section regressions, in a way similar to the Fama-Macbeth procedure (see *e.g.*, Cochrane (2001 p.244)). We used data on the daily stock prices of fifty Japanese pharmaceutical/ biomedical venture capital companies. To obtain an excess return series, we used 10-year Japanese Government Bond (JGB) rate for the risk free rate. For the market return, we used TOPIX. The sample period is from May 6, 2005 to April 28, 2006, hence the sample size is 245 in total. We obtained stock return data and the JGB data from Yahoo Finance and Nikko Financial Intelligence web site, respectively.

We begin with a multifactor multivariate return generating equation:

$$\mathbf{R} = (\iota_T \mathbf{F}) \begin{pmatrix} \boldsymbol{\alpha} \\ \mathbf{B} \end{pmatrix} + \boldsymbol{\varepsilon}, \quad (14)$$

where $\mathbf{R} = (\mathbf{R}_1 \cdots \mathbf{R}_N) \sim T \times N$ is a T period excess returns for N firms,
 $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) \sim 1 \times N$ vector of constants,
 $\mathbf{B} = (\boldsymbol{\beta}_1 \cdots \boldsymbol{\beta}_N) \sim F \times N$ is a matrix of beta's,
 $\mathbf{F} = (\mathbf{f}_1 \cdots \mathbf{f}_F) \sim T \times F$ is a matrix of F factors,
 $\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}_1 \cdots \boldsymbol{\varepsilon}_N) \sim T \times N$ matrix of error terms,
 N is the number of stocks, and T is the time series sample size. When $F = 1$

the above reduces to a single factor return generating equation:

$$\mathbf{R} = (\iota_T \mathbf{f}) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \boldsymbol{\varepsilon}, \quad (15)$$

¹²Suppose a gamma distributed random variable γ has $E(\gamma) = \theta$, then its standard deviation becomes $SD(\gamma) = \theta/\sqrt{\eta}$, where η is the scale parameter.

where $\mathbf{F} = \mathbf{f}$ is a $T \times 1$ vector of one factor,
 $\mathbf{B} = \boldsymbol{\beta}$ is a $1 \times N$ vector of beta's.

Equation (15) is just a set of N time series regressions. We obtain an OLSE of $\boldsymbol{\beta}$, $\hat{\boldsymbol{\beta}} \sim 1 \times N$ from equation (15). Define sample mean of \mathbf{R} to be an N dimensional vector $\bar{\mathbf{R}}$, we then obtain a cross sectional regression model

$$\bar{\mathbf{R}} = \hat{\boldsymbol{\beta}}' \lambda + \mathbf{u}, \quad (16)$$

where $\bar{\mathbf{R}} = \frac{1}{T} \mathbf{R}' \boldsymbol{\nu}_T = (\bar{R}_1 \cdots \bar{R}_N)' \sim N \times 1$ vector of average excess returns,
 $\lambda \sim 1 \times 1$ scalar is a risk premium associated with the factor \mathbf{f} ,
 $\mathbf{u} \sim N \times 1$ is a vector of pricing errors.

Equation (16) is the one factor type CAPM *without* an intercept term given in Cochrane *op.cit.*, p.235, among others¹³.

In this section so far, we have used a set of notations that are common in empirical finance, and in this paper it corresponds to equation (1). We, thus, need to clarify the notational correspondences between the ones used so far in the current section, and in the previous subsections. The correspondences are given in below.

$\bar{\mathbf{R}} \sim N \times 1$	(corresponds to \mathbf{y} in (1),
$\hat{\boldsymbol{\beta}}' \sim N \times 1$	(corresponds to \mathbf{X} in (1)),
$\mathbf{u} \sim N \times 1$	(corresponds to $\boldsymbol{\varepsilon}$ in (1)),
$\lambda \sim \text{scalar}$	(corresponds to $\boldsymbol{\beta}$ in (1)),
$F = 1$ is the number of factors	(corresponds to K in (1)),
N the number of stocks	(corresponds to n in (1)).

4.2.2 Bayesian Test of Heteroscedasticity

Since we do not know whether our data set exhibits a heteroscedasticity or not, we carried out a Bayesian test of heteroscedasticity *a la* Lancaster(1994, p.149). In equation (1) or equivalently in the present case equation (16),

¹³This specification is found *e.g.*, in Cochrane *op.cit.* equation 12.10. We have regressed with an intercept term, and the OLSE of it is 0.003 (0.048) and the slope estimate is 0.104 (0.062), where the figures inside the parentheses are estimated standard errors. Without an intercept term, the slope estimate is 0.108 (0.024), and there is very little difference whether we include an intercept term or not.

under the null hypothesis of homogeneity, H_0 , we have $\varepsilon \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_N)$. Hence, letting $\varepsilon_i = \bar{R}_i - \hat{\beta}_i \lambda$, we have $E(\varepsilon_i^2 - \sigma^2)^2 = E(\varepsilon_i^4 - \sigma^2 \varepsilon_i^2 + \sigma^4) = 2\sigma^4$, under H_0 , since our normality assumption on ε_i calls for $E(\varepsilon_i^4) = 3\sigma^4$. Lancaster *op.cit.*, thus, proposes $T(\lambda, \sigma^2) = \left(\frac{\sum_{i=1}^N (\hat{\varepsilon}_i^2 - \sigma^2)^2}{N} \right) - 2\sigma^4$, where $\hat{\varepsilon}_i = \bar{R}_i - \hat{\beta}_i \hat{\lambda}$ and $\hat{\lambda}$ is an MCMC generated value, to be used in a Highest Posterior Density Interval (HPDI) test. This test would reject H_0 if the mass of the T statistic distribution does not contain “zero,” *i.e.*, homoscedasticity¹⁴.

It is easy to draw λ and σ from their respective marginal posterior pdf, since each pdf boils down to an univariate normal and an \mathcal{IG} , respectively. We will not discuss the details of the MCMC procedure involved in simulating T under H_0 . Simulated T statistic is shown in Fig. 4. Our calculations show that for the distribution of T , $P(T < 0) = 0.016$, which is smaller than the usual significance level even if we multiply it by 2, *i.e.*, $0.016 \times 2 = 3.2\% < 5\%$, say. We would safely reject H_0 . We also carried out several well-known sampling theory heteroscedasticity tests. They all reject H_0 .

4.2.3 Empirical Estimation of ω_i 's

The MCMC simulation procedure, here, is exactly the same as the one discussed in the numerical illustration of section 4.1. Posterior means and standard deviations are given, respectively, in Figures 5 and 6 in below. Since we have confirmed that our method works in section 4.1, we need to

¹⁴Lancaster *op. cit.* does not call T , a test statistic, however. He rightfully regards T as a “checking” device. For one thing, the empirical distribution obtained from simulated T is *not* the marginal posterior pdf under H_0 . In other words, we (or Lancaster *op. cit.* too) did not compute the posterior pdf of the T statistic. Secondly, in a sampling theory framework, obviously $\text{plim}T = 0$ under H_0 as $N \rightarrow \infty$. This implies that whether $T > 0$ or $T < 0$ is not known under the alternative hypothesis as $N \rightarrow \infty$. It seems like we ought to take a $100(1 - \alpha)\%$ HPDI of the T statistic, and reject H_0 if it does not contain zero, where α is a significance level. This is in essence, what we did.

Figure 5: Marginal Posterior pdf of the T Statistic

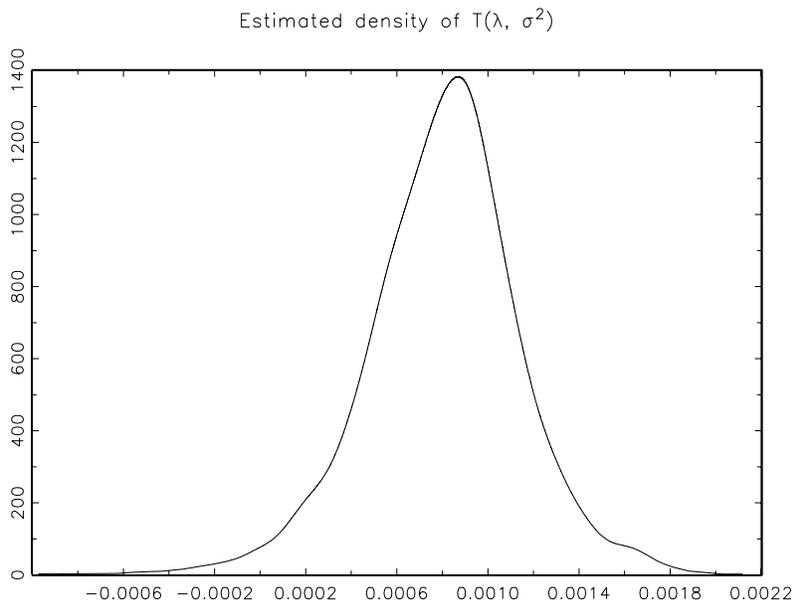


Figure 6: Marginal Posterior Means of the $\sigma^2\omega_i$'s

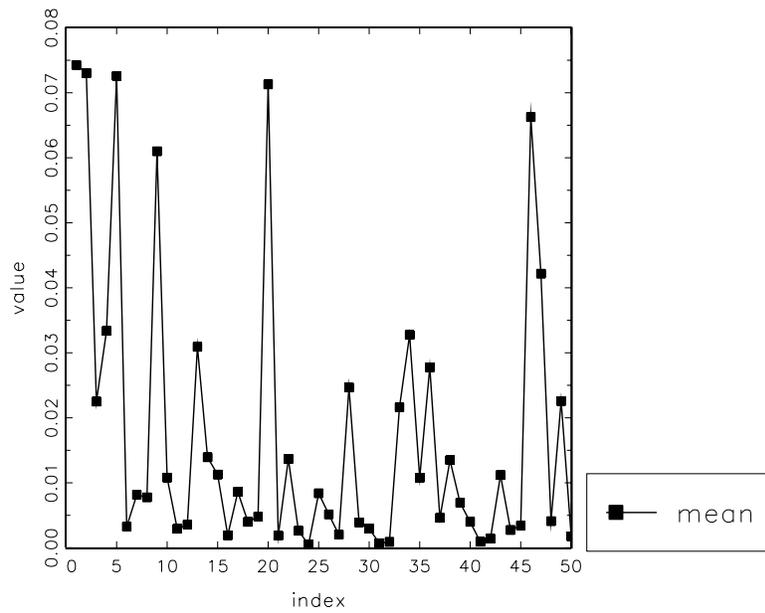
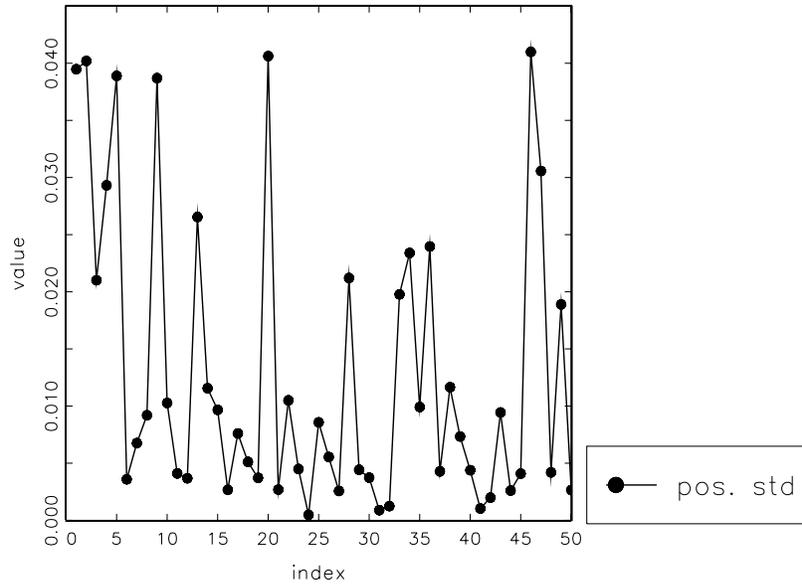


Figure 7: Marginal Posterior Standard Deviations of the $\sigma^2\omega_i$'s



make sure that our results may be reasonable as an empirical investigation.

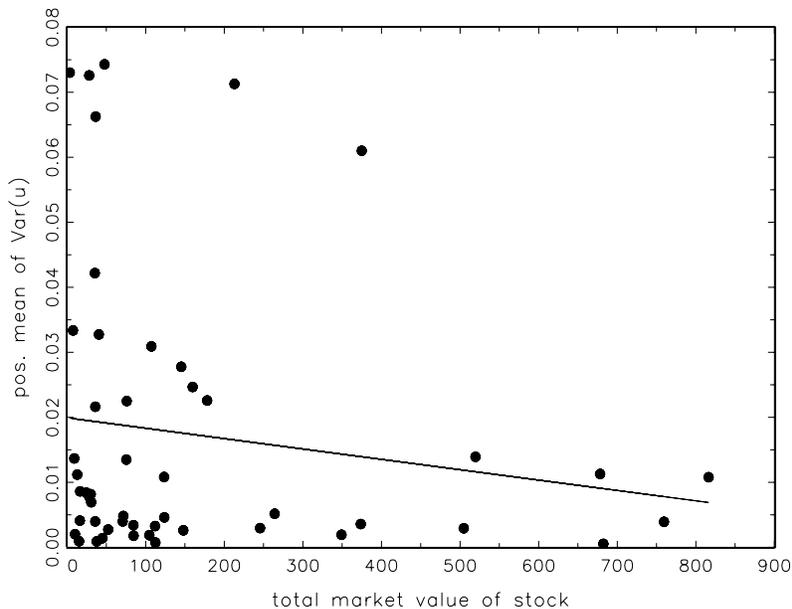
In the U.S. and world wide, starting with a seminal paper by Banz (1981) many observed “size effect” that is the smaller the company is the higher its return¹⁵. This phenomenon could be naively associated to the mean-variance efficiency to yield a thesis that says smaller companies are expected to be more riskier, *i.e.*, the smaller the size, the larger the mean and volatility of returns. In this section, we intend to investigate whether larger (smaller) size companies have smaller (larger) ω_i 's.

For the fifty stock data, we now collected the market value of its equity (MVE) data from Yahoo Finance (in one billion yen)¹⁶. We then drew a graph with the MVE on the horizontal axis, and the $\sigma^2\omega_i$'s on the vertical

¹⁵For the size effect in Japan, see *e.g.*, Chan and Chen (1991), among others.

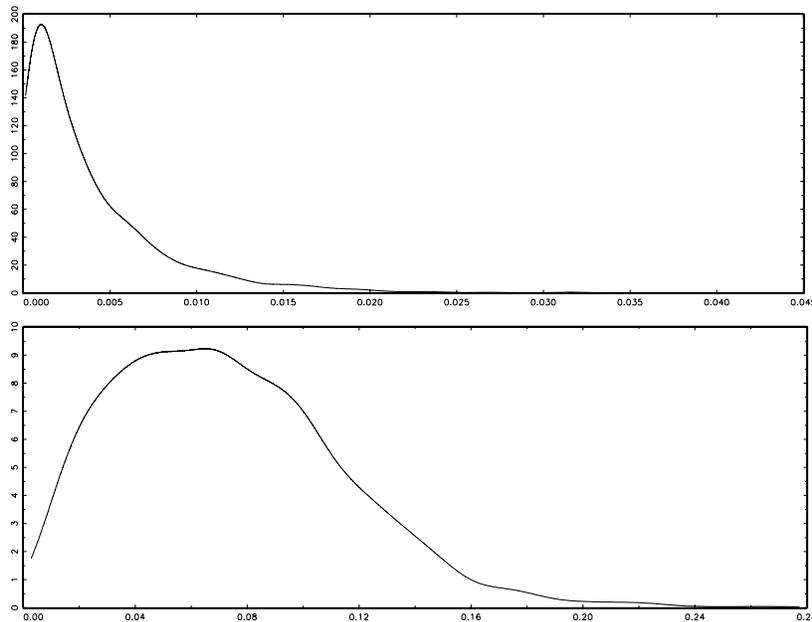
¹⁶Berk (1997) among others, however, examined the so-called *size effect* and proposed that a size of a company should not be measured by MVE but some other variables such as sales.

Figure 8: Volatility versus Size: $\sigma^2\omega_i$'s and MVE



axis. This is shown in Figure 7. The downward sloping solid line in Figure 7, is the OLS estimated line. This figure clearly shows the larger the MVE, the smaller the volatility as measured by $\sigma^2\omega_i$. We further selected two stocks (1) that has large MVE and small $\sigma^2\omega_i$, and (2) that has small MVE and large $\sigma^2\omega_i$, to see what the marginal posterior pdf's of $\sigma^2\omega_i$ of these companies look like. Specifically, we chose Taisho Pharmaceutical Co., Ltd. for (1), and Site Support Institute Co., Ltd. for (2). They are given in Figures 8 below. Notice that the two charts have different horizontal axis scale. The smaller MVE stock has very large volatility (see the lower chart) compared to the that of the larger MVE stock (see the upper chart). The two pdf's have quite reasonable shapes. We conclude that our Bayesian estimation of volatility supports the view that the smaller the size of the stock, the larger is the volatility.

Figure 9: Marginal Posterior pdf of $\sigma^2\omega_i$ for Taisho Phamaceutical Co. and Site Support Institute



5 Concluding Remarks

In this paper, we proposed a fully Bayesian method to estimate regression error term scedastic function ω *without* assuming any structure on it. One unique feature of our methodology is in using the Eicker-White result in a candidate generating function in our M-H step of MCMC. We have supplied a numerical example, and also an empirical example to show that our method works reasonably. Finally, as demonstrated in our empirical research, obtaining \mathbf{W} such that $\mathbf{W}'(\mathbf{y} \ \mathbf{X}) = \mathbf{0}$ is no problem since \mathbf{W} is not found as an empirical data but rather computer generated data.

One possible arbitrariness might remain in this paper. That is, a reader might feel uneasy about our setting bounds for ω prior pdf. As we suggested in footnote 8, we may adopt a hierarchical Bayes model.

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Appendix : Derivation of equation (11)

\mathbf{H} is defined by $\mathbf{H} = \mathbf{X}'\mathbf{\Omega}\mathbf{X}$. Our purpose in this appendix is to derive equation (9) in the text, *i.e.*, $\mathbf{h} \equiv \text{vech}(\mathbf{H}) = \mathbf{X}_n\boldsymbol{\omega}$. To this effect, we note the following series of lemmas.

Lemma 1 . $\text{vec}(\mathbf{H}) = (\mathbf{X}' \otimes \mathbf{X}')\text{vec}(\mathbf{\Omega}) \sim K^2 \times 1$.

proof : Just use the usual $\text{vec}(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A})\text{vec}(\mathbf{B})$ formula. See, *e.g.*, Lütkepohl (1996, p.97).

Lemma 2 . $\mathbf{X}' \otimes \mathbf{X}' = [(\mathbf{x}_1 \otimes \mathbf{x}_1) \dots (\mathbf{x}_1 \otimes \mathbf{x}_n)] \dots [(\mathbf{x}_n \otimes \mathbf{x}_1) \dots (\mathbf{x}_n \otimes \mathbf{x}_n)] \sim K^2 \times n^2$.

proof : Note $\mathbf{X}' = (\mathbf{x}_1 \dots \mathbf{x}_n) \sim K \times n$, and the result follows.

Lemma 3 . $\text{vec}(\mathbf{H}) = [(\mathbf{x}_1 \otimes \mathbf{x}_1) \dots (\mathbf{x}_n \otimes \mathbf{x}_n)]\boldsymbol{\omega}$, where $\boldsymbol{\omega} = (\omega_1 \dots \omega_n)' \sim n \times 1$.

proof : Note

$$\text{vec}(\mathbf{\Omega}) = \text{vec}(\text{diag}(\boldsymbol{\omega})) = (\omega_1 \ 0 \dots 0 \ : \ 0 \ \omega_2 \ 0 \dots 0 \ : \ \dots \ : \ \dots \ \omega_n)' \sim n^2 \times 1,$$

where $\boldsymbol{\omega}' = (\omega_1, \dots, \omega_n)$, “diag($\boldsymbol{\omega}$)” denotes a diagonal matrix with the vector $\boldsymbol{\omega}$ in its diagonal. Note that $\text{vec}(\mathbf{\Omega})$ is a vector of size $n^2 \times 1$, and picks up the columns $(\mathbf{x}_t \otimes \mathbf{x}_t)$ only for $t = 1, \dots, n$ from $\mathbf{X}' \otimes \mathbf{X}'$.

Lemmas 1 and 2, in conjunction with the above proves the lemma.

Lemma 4 . $\mathbf{x}_t \otimes \mathbf{x}_t = \text{vec}(\mathbf{x}_t\mathbf{x}_t') \sim K^2 \times 1$.

i.e., $\mathbf{a}_{j(-1,2)}$ deletes the first two elements from \mathbf{a}_j . Then,

$$\mathbf{L}_{K'} \text{vec}(\mathbf{A}) = \mathbf{L}_{K'} \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_K \end{pmatrix} = \mathbf{L}_{K'} \begin{pmatrix} \mathbf{a}_1 \\ \dots\dots\dots \\ 0 \\ \mathbf{a}_{2(-1)} \\ \dots\dots\dots \\ 0 \\ 0 \\ \mathbf{a}_{3(-1,2)} \\ \dots\dots\dots \\ \vdots \\ a_{KK} \end{pmatrix} = \text{vech}(\mathbf{A}).$$

Lemma 6 . $\text{vech}(\mathbf{x}_t \mathbf{x}'_t) = \mathbf{L}_{K'} \text{vec}(\mathbf{x}_t \mathbf{x}'_t)$.

proof : Omitted.

We are now, in a position to derive the $\text{vech}(\mathbf{H}) = \mathcal{X}_n \boldsymbol{\omega}$ relationship.

Use Lemmas 2 to 4 on the Lemma 1 relationship to obtain

$$\text{vec}(\mathbf{H}) = [\text{vec}(\mathbf{x}_1 \mathbf{x}'_1) \dots \text{vec}(\mathbf{x}_n \mathbf{x}'_n)] \boldsymbol{\omega}.$$

Further, use Lemmas 5 and 6 to obtain the following.

$$\begin{aligned} \text{vech}(\mathbf{H}) &= \mathbf{L}_{K'} \text{vec}(\mathbf{H}) \boldsymbol{\omega} \\ &= [\mathbf{L}_{K'} \text{vec}(\mathbf{x}_1 \mathbf{x}'_1) \dots \mathbf{L}_{K'} \text{vec}(\mathbf{x}_n \mathbf{x}'_n)] \boldsymbol{\omega} \\ &= \mathcal{X}_n \boldsymbol{\omega}. \end{aligned}$$

where $\mathcal{X}_n = [\text{vech}(\mathbf{x}_1 \mathbf{x}'_1) \dots \text{vech}(\mathbf{x}_n \mathbf{x}'_n)] \sim K' \times n$.