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INTRODUCTION

• Optimal behaviors in economic environment have been studied intensively since Merton (1969, 1971)
• Standard method of investigation is based on stochastic control framework and Hamilton-Jacobi-Bellman (HJB) equation, resulting in nonlinear equations which are typically hard to solve
• Economists are interested in consumption level and growth. Investment planners are interested in optimal policy for investment.
• I try to focus on both issues based on basic models from Chapter 6 of Munk (2010)
• The optimal consumption and investment strategy can be solved for exact solutions
• To study more in-depths of consumption, I use “Keynes-Ramsey rule” and “Expected growth rate of consumption” as means to analyze the consumption
LITERATURE REVIEWS

• Obstfeld (1994) develops a continuous-time stochastic model with the use of HJB equation in which international risk-sharing can yield substantial welfare gains through its effect on expected consumption growth
• His model makes this theoretical point cleanly, but its empirical applicability is limited by several factors
• Sennewald and Walde (2006) also use the HJB equation to derive both a Keynes-Ramsey rule and a closed-form solution for an optimal consumption-investment problem with labor income. Uncertainty stems from a Poisson process.
• Munk (2010) considers a simple case of dynamic asset allocation with constant investment opportunities
• For the CRRA utility maximization problem in a market with constant interest rate, expected rate of returns, and volatility, he derives the optimal investment strategy and optimal consumption rate
THE MODEL (1)

• Consider a simple case in which the short-term interest rate \( r \), the expected rates of return \( \mu \), and the volatility \( \sigma \) of the risky assets are assumed to be constant through time.

• Also considering the case where the state variable is one-dimensional, the dynamics of the \( d \) risky asset prices can be given by

\[
dP_t = \text{diag}(P_t) \left[ \mu_t \, dt + \sigma_t \, d\,z_t \right]
\]

\[
= \text{diag}(P_t) \left[ \left( r_t 1 + \sigma_t \lambda_t \right) dt + \sigma_t \, d\,z_t \right]
\]

• Where the market price of risk is given by

\[
\lambda_t = \left( \sigma_t \right)^{-1} \left( \mu_t - r_t 1 \right)
\]

• Wealth process can be written as

\[
dW_t = W_t \left[ r_t + \pi'_t \sigma_t \lambda_t \right] dt - c_t \, dt + W_t \pi'_t \sigma_t \, d\,z_t
\]
The indirect utility function (or the value function) is a function of only current wealth and time

\[ J(W,t) = \sup_{(c_s, \pi_s)_{s\in[t,T]}} E_{W,t} \left[ \int_t^T e^{-\delta(s-t)} u(c_s) ds + e^{-\delta(T-t)} \bar{u}(W_T) \right] \]

Time preference rate

- For CRRA utility, we have

\[ u(c) = \varepsilon_1 \frac{c^{1-\gamma}}{1-\gamma} \quad \text{and} \quad \bar{u}(W) = \varepsilon_2 \frac{W^{1-\gamma}}{1-\gamma} \]

- Then, from Theorem 6.2 in Munk (2010), we obtain

\[ J(W,t) = g(t) \frac{W^{1-\gamma}}{1-\gamma} \quad \text{and} \quad g(t) = \frac{1}{A} \left( \varepsilon_1^{1/\gamma} + \left[ \varepsilon_2^{1/\gamma} A - \varepsilon_1^{1/\gamma} \right] e^{-A(T-t)} \right) \]

with

\[ A = \frac{\delta + r(\gamma - 1)}{\gamma} + \frac{1}{2} \frac{\gamma - 1}{\gamma^2} \| \lambda \|^2 \]

- The optimal investment strategy and consumption rate are given by

\[ \Pi(W,t) = \frac{1}{\gamma} \left( \sigma' \right)^{-1} \lambda = \frac{1}{\gamma} \left( \sigma \sigma' \right)^{-1} \left( \mu - r1 \right) \quad \text{and} \quad c(W,t) = \varepsilon_1^{1/\gamma} \frac{W}{g(t)} \]
KEYNES-RAMSEY RULE (1)

- One of the best-known and most widely used in Economics
- In deterministic case, it says that consumption increases when the interest rate is higher than the time preference
- So how about Keynes-Ramsey rule in stochastic case with \( d \) risky assets?
- From the model, we can write the HJB equation as follows

\[
\delta J(W, t) = u(c) - cJ_W + WJ_W \sigma'_t \sigma \lambda_t + \frac{1}{2} J_W W^2 \sigma' \sigma' \pi + J_t + rWJ_W
\]

- Using the envelope theorem

\[
\delta J_W = u'(c)c'(W) - (cJ_{WW} + J_W c'(W))cJ_W + \left( r + \sigma'_t \sigma \lambda_t \right)(WJ_W + J_W)
\]

\[
+ \frac{1}{2} \sigma' \sigma' \pi \left( W^2 J_{WWW} + 2J_{WW}W \right) + J_{tW}
\]

where \( u'(c) = J_W \)
KEYNES-RAMSEY RULE (2)

- Therefore, it can be written that
  \[
  \left[ \delta - \left( r + \pi_t', \sigma_t, \lambda_t \right) \right] J_W - J_{ww} W \pi' \sigma' \pi \\
  = J_{tw} + J_{ww} \left[ W \left( r + \pi_t', \sigma_t, \lambda_t \right) - c \right] + \frac{1}{2} W^2 J_{www} \pi' \sigma' \pi
  \]

- Now, using Itô's lemma to \( J_W \), we obtain
  \[
  dJ_W = \left( J_{tw} + J_{ww} \left[ W \left( r + \pi_t', \sigma_t, \lambda_t \right) - c \right] + \frac{1}{2} W^2 J_{www} \pi' \sigma' \pi \right) dt + J_{ww} W \pi_t' \sigma_t d z_t
  \]

- Replacing the drift term by the same expression above gives
  \[
  dJ_W = \left\{ \left[ \delta - \left( r + \pi_t', \sigma_t, \lambda_t \right) \right] J_W - J_{ww} W \pi' \sigma' \pi \right\} dt + J_{ww} W \pi_t' \sigma_t d z_t
  \]

- Finally, we can replace \( J_W \) by marginal utility from the FOC, \( u'(c) = J_W \)
  \[
  du'(c) = \left\{ \left[ \delta - \left( r + \pi_t', \sigma_t, \lambda_t \right) \right] u'(c) - u''(c) c_w W \pi' \sigma' \pi \right\} dt + u''(c) c_w W \pi_t' \sigma_t d z_t
  \]
KEYNES-RAMSEY RULE (3)

• Define \( f(u'(c)) = c \) and apply Ito’s lemma to \( f(u'(c)) \)

\[
df(u'(c)) = \left\{ \begin{array}{l}
  f_t + \left( \delta - \left[ r + \pi_t \sigma_t \frac{\gamma'}{\gamma} \right] \right) u'(c) - u''(c)c_w W \sigma_t \sigma' \sigma \pi \frac{c}{u'(c)} f''(u'(c)) \\
  + \frac{1}{2} \left( u''(c)c_w W \right)^2 \sigma_t \sigma' \sigma \pi \\
  + f'(u'(c)) u''(c)c_w W \sigma_t \sigma' \sigma_t d\zeta_t
\end{array} \right\} dt
\]

where \( f_t = c_t \) and \( f(u'(c)) = \frac{1}{u''(c)} \)

• Multiplying both sides with \(- \frac{u''(c)}{u'(c)}\), we obtain the Keynes-Ramsey rule

\[
- \frac{u''(c)}{u'(c)} dc = \left\{ r + \pi'_t \sigma_t \frac{\gamma'}{\gamma} \lambda_t - \delta + \frac{u''(c)}{u'(c)} c_w W \sigma_t \sigma' \sigma \pi + \frac{1}{2} \frac{u''(c)}{u'(c)} \left( c_w W \right)^2 \sigma_t \sigma' \sigma \pi - \frac{u''(c)}{u'(c)} c_t \right\} dt
\]

\[
- \frac{u''(c)}{u'(c)} c_w W \sigma_t \sigma_t d\zeta_t
\]

• For CRRA case \(- \frac{u''(c)}{u'(c)} = \frac{\gamma}{c}\) and \(- \frac{u'''(c)}{u'(c)} = \frac{\gamma(\gamma + 1)}{c^2}\)
KEYNES-RAMSEY RULE (4)

- The rule describes the evolution of consumption under optimal behavior for a household that faces uncertainty resulting from Wiener process.
- For CRRA utility function, the rule becomes

$$\frac{dc}{c} = \frac{1}{\gamma} \left\{ r + \pi_t' \frac{\lambda_t}{\sigma_t} - \delta - \frac{\gamma}{c} c_w W \pi' \sigma \pi + \frac{1}{2} \gamma (\gamma + 1) \left( \frac{c_w W}{c} \right)^2 \pi' \sigma' \pi + \gamma \frac{c_t}{c} \right\} dt$$

$$+ \frac{c_w W}{c} \pi' \sigma \, d z_t$$

- As we know consumption rate and its derivatives from Theorem 6.2, therefore we obtain

$$\frac{dc}{c} = \left\{ \left( \frac{\gamma + 1}{2 \gamma^2} \right) \| \lambda \|^2 + \frac{r - \delta}{\gamma} \right\} \left[ g(t) \left[ \epsilon_2^{1/\gamma} A - \epsilon_1^{1/\gamma} \right] e^{-A(t-t)} \right] dt + \frac{1}{\gamma} \lambda' \, d z_t$$

- In the case of infinite horizon, there will be no time horizon term and the analysis of Keynes-Ramsey rule will be easier.
KEYNES-RAMSEY RULE (5)

• From the previous equation, we can write the expected growth rate of consumption

$$\frac{E(\text{dc})}{dt \cdot c} = \left( \frac{\gamma + 1}{2\gamma^2} \right) \| \Delta \|^2 + \frac{r - \delta}{\gamma} - g(t) \left[ \varepsilon_2^{1/\gamma} A - \varepsilon_1^{1/\gamma} \right] e^{-A(T-t)}$$

• The expected growth rate is difficult to interpret due to the time horizon related term

• An empirical or numerical example will provide more details and can lead to better interpretation of the consumption issue
SUMMARY AND NEXT STEP

• I review past research related to asset allocation, consumption growth, and investment strategy
• From Munk (2010), the basic model of dynamic asset allocation is studied and leads to the exact solution of optimal consumption and investment strategy
• To study more in-depths of consumption, “Keynes-Ramsey rule” and “Expected growth rate of consumption” are used as means to analyze the consumption growth
• The next step is to do more possible analysis on consumption
• The more sophisticated model with bonds as one of risky assets
• A numerical example on international asset allocation
REFERENCES


THANK YOU