1. Introduction

In this article we propose a singular nonlinear partial differential equation (PDE) which is derived from the Hamilton-Jacobi-Bellman (HJB) equation for the value function in the optimal investment problem. We recall that optimal behavior within continuous time economics environments has been an intensive area of research and that many models have already been introduced within the stochastic control framework. The analysis is then often reduced to the treatment of the HJB equation for the value function. However, the HJB equation is typically fully nonlinear and hard to solve; it may not be an exaggeration to say that all that we can do is merely guess a shape of solution and manage to arrange the parameters. See for instance [1].

We here propose a different approach and derive a singular quasilinear PDE from the HJB equation. Although essential difficulties are equivalent to those expressed by the HJB equation, the derived PDE is rather simple looking when viewed from the theory of nonlinear PDE. Moreover, the unknown quantity is related to the Arrow-Pratt coefficient of relative risk aversion [2] with respect to the optimal value function. We show the existence of monotone traveling wave solutions and the nonexistence of non-monotone such solutions, which are suitable from the standpoint of financial economics.

Keywords optimal economic behavior, Arrow-Pratt coefficient of relative risk aversion, risk preference, singular nonlinear partial differential equation, traveling wave solutions

Research Activity Group Mathematical Finance

Traveling wave solutions to the nonlinear evolution equation for the risk preference

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Abstract

A singular nonlinear partial differential equation (PDE) is introduced, which can be interpreted as the evolution of the risk preference in the optimal investment problem under the random risk process. The unknown quantity is related to the Arrow-Pratt coefficient of relative risk aversion with respect to the optimal value function. We show the existence of monotone traveling wave solutions and the nonexistence of non-monotone such solutions, which are suitable from the standpoint of financial economics.

The main purpose of this article is to prove the existence of monotone traveling wave solution to this PDE. The solutions can be interpreted positively from the viewpoint of financial economics. In addition, we show the nonexistence of non-monotone traveling wave solutions, by which we refer to those whose derivative changes sign several times. This observation is also welcome as a financial concept.

We here perform an analytical study. A numerical investigation, in particular for the monotone traveling wave solution, is attempted in [5]. See also [6-9].

The organization of the paper is as follows. In Section 2 we recall the model and introduce our PDE. Sections 3 and 4 are devoted to proving the existence of monotone traveling wave solutions and the nonexistence of non-monotone traveling wave solutions, respectively. We conclude with discussions in Section 5.

2. Model

Here we briefly review our model. Suppose that the wealth \(X_t\) at time \(t \geq 0\) of the company is subject to a fluctuating process, and the company wants to invest in one risky stock. We assume that the price \(P_t\) of the stock available for investment is governed by the stochastic differential equation of Black-Scholes-Merton type [10, 11] \(dP_t = P_t(\mu dt + \sigma dW_t^{(1)})\), where \(\mu\) and \(\sigma\) are constants and \(\{W_t^{(1)}\}_{t \geq 0}\) is a standard Brownian motion. The fluctuating process, which directly affects the wealth of the company, is denoted by \(Y_t\), and is assumed to evolve as \(dY_t = \alpha dt + \beta dW_t^{(2)}\), where \(\alpha\) and \(\beta\) (\(\beta > 0\)) are constants and \(\{W_t^{(2)}\}_{t \geq 0}\) is another standard Brownian motion. It is allowed that these two Brownian motions be correlated with the correlation coefficient \(\rho\) (\(0 \leq |\rho| < 1\)).

The investment policy \(f = \{f_t\}_{0 \leq t \leq T}\) of the company is a suitable admissible adapted control process. Here \(T\)
stands for the maturity date. The stochastic process of the
wealth \( X_t \) of the company is then assumed to be
expressed as
\[
\frac{dX_t}{X_t} = f_t \frac{dP_t}{P_t} + dY_t
\]
\[
= (f_t \mu + \alpha) dt + f_t \sigma dW_t^{(1)} + \beta dW_t^{(2)},
\]
\( X_0 = x \in \mathbb{R}. \)

Suppose that the company aims to maximize the util-
ity \( u(x) \) from his terminal wealth. The utility func-
tion \( u(x) \) is customarily assumed to satisfy \( u' > 0 \) and
\( u'' < 0 \). Let
\[
V(x, t) := \sup_y E[u(X_T) | X_t = x].
\]
(1)

Now the Hamilton-Jacobi-Bellman equation for the
value function (1) becomes
\[
\sup_{\mathcal{A}} \mathcal{A} V(x, t) = 0, \quad V(x, T) = u(x),
\]
(2)
where the generator \( \mathcal{A} \) is given by
\[
(\mathcal{A} g)(x, t) := \frac{\partial g}{\partial t} + (f_t \mu + \alpha) x \frac{\partial g}{\partial x}
\]
\[
+ \frac{1}{2} \left( f_t^2 \sigma^2 + \beta^2 + 2 \beta \sigma \rho f \right) x^2 \frac{\partial^2 g}{\partial x^2}.
\]

Suppose that (2) has a classical solution \( V \) with
\( \partial V/\partial x \geq 0, \partial^2 V/\partial x^2 < 0 \). We then discover that the
optimal policy \( \{f_t^*\}_{0 \leq t \leq T} \) is
\[
f_t^* = -\frac{\mu}{\sigma^2} \frac{\partial V}{\partial x} - \frac{\beta \rho f}{\sigma}.
\]
(3)

Placing (3) back into (2) we obtain
\[
\begin{cases}
0 = \frac{\partial V}{\partial t} + \left( \alpha - \frac{\beta \rho f}{\sigma} \right) x \frac{\partial V}{\partial x} - \frac{u^2}{2 \sigma^2} \frac{\partial^2 V}{\partial x^2} \\
+ \frac{1}{2} \beta^2 (1 - \rho^2) x^2 \frac{\partial^2 V}{\partial x^2} & \text{for } 0 < t < T,
\end{cases}
\]
(4)

\[
V(T, x) = u(x).
\]

Let \( \tau := 2(1 - \rho^2)^{-1} \beta^{-2}(T - t) \) and put \( V(x, t) = V(x, \tau) \) by abuse of notation, we find that
\[
\begin{cases}
\frac{\partial V}{\partial \tau} = x^2 \frac{\partial^2 V}{\partial x^2} - a^2 \left( \frac{\partial V}{\partial x} \right)^2 - bx \frac{\partial V}{\partial x},
\end{cases}
\]
(5)
where we have set
\[
a^2 := \frac{\mu^2}{(1 - \rho^2) \beta^2 \sigma^2}, \quad b := \frac{2(\rho \mu \beta - \alpha \sigma)}{(1 - \rho^2) \beta^2 \sigma}.
\]
Eq. (5) is fully nonlinear parabolic type \[8\].

Now we define
\[
r(x, \tau) := -\frac{x \frac{\partial^2 V}{\partial x^2}}{-x \frac{\partial V}{\partial x}} = -x \frac{\partial}{\partial x} \log \left| \frac{\partial V}{\partial x}(x, \tau) \right|,
\]
(6)
which extends the Arrow-Pratt coefficient of relative risk
aversion for the utility function. Here we note that \( r \)
is introduced with respect to the optimal value function.
A similar transformation is considered in [12], where the
transformation \(-V_x/V_{xx}\) is employed.

Following [13], we make a change of variables \( x = e^y \)
\( (y = \log x) \) and put \( r(y, \tau) = r(x, \tau) \); we infer that
\[
\frac{\partial r}{\partial \tau} = \left( \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial y} \right) (r - \frac{a^2}{r}) - (2r + b) \frac{\partial r}{\partial y}
\]
for \(-\infty < y < \infty, \quad \tau > 0.\)
(7)

In the following two sections, we prove the existence of
monotone traveling wave solutions and the nonexistence of
non-monotone solutions to (7).

3. Monotone traveling wave solution

For a standard risk averse investor, the coefficient of
relative risk aversion is expected to be non-increasing
[14]. In addition, it is easy to see that every constant
function verifies (7). We thus wish to seek a traveling
wave solution \( r = r(y - v \tau) \) with the property
\[
\begin{cases}
\left. \begin{align}
\frac{\partial r}{\partial \tau} &< 0 & \text{for } -\infty < y < \infty, \\
r(y) &\to r_+ & \text{as } y \to -\infty, \\
r(y) &\to r_- & \text{as } y \to \infty,
\end{align} \right\} \quad \text{for } r(y) \neq 0
\end{cases}
\]
(8)
where \( r_- > r_+ \geq 0 \) are prescribed constants and the
wave speed \( v \in \mathbb{R} \) should be determined later on.

Putting \( r(y, \tau) = r(y - v \tau) \) into (7), we derive the
ordinary differential equation
\[
-v r' = \left( r - \frac{a^2}{r} \right)' + \left( r - \frac{a^2}{r} \right)' - (2r + br)',
\]
(9)
where \( r = r(y) \) and \( \tau' = d\tau/dy \). Integrating once, we obtain
\[
\left( r - \frac{a^2}{r} \right) + \frac{a^2}{r} - r^2 - br + vr = C.
\]
(10)
Here \( C \) denotes a constant, and from the boundary con-
dition (8) we deduce that
\[
\begin{cases}
C = r_- r_+ - \frac{r_- - r_+}{r_- r_+} a^2, \\
v = r_- + r_+ - \frac{a^2}{r_- r_+} + b - 1.
\end{cases}
\]
(11)
Eq. (10) can be written in the separable form
\[
\frac{r^2 + a^2}{r^3 + (b - v - 1)r^2 + Cr + a^2} dr = dy.
\]
(12)
We define \( f(r) := r^3 + (b - 1 - br)^2 + Cr + a^2 \), which is the
factor of the denominator in (12). The condition (11)
implies that \( f(r_-) = f(r_+) = 0 \). Since \( f(0) = a^2 > 0 \),
the solution \( r \) for (10) which fulfills (8) is constructed
implicitly through the integration of (12) on the interval
\( r \in (r_+, r_-) \) and \( \infty > y > -\infty \), provided the prescribed constants \( r_- > r_+ (> 0) \) are realized as positive real numbers.

We examine such criterion. Taking account of \( f'(r) = 3r^2 - 2(v+1-b)r + C \), we learn that they are

(i) \((v+1-b)^2 - 3C > 0,\)

(ii) \( v + 1 - b > 0,\)

(iii) \( f \left( \frac{v+1-b+\sqrt{(v+1-b)^2 - 3C}}{3} \right) < 0.\)

In view of (11), condition (i) is reduced to

\[ r_-^2 - r_r + r_+^4 + a^2 \left( \frac{r_-^2 - r_r + a^2}{r_r} + \frac{a^2}{r_+^2 + r_+^2} \right) > 0, \]

which is true for \( r_- > r_+ > 0 \). Condition (ii) results in \( r_- + r_+ - a^2 r_+ r_-^{-1} > 0 \), which should be imposed beforehand. Finally, condition (iii) becomes, after a tedious calculation,

\[ 8a^4 < 2(r_- + r_+)(r_+^2 + r_- r_+ + r_+^2) a^2 + \frac{(r_- + r_+)^2}{r_+^2 r_-^2 a^2} (r_+^2 + r_-^2) a^4 + \frac{2(r_- + r_+)}{r_+^2 r_-^2} (r_- - r_+)^2 a^6 + \frac{(r_- - r_+)^2}{r_+^2 r_-^2} a^8 + r_+^2 (r_- - r_+)^2. \]

By virtue that \( r_-^2 r_+^2 - r_-^2 r_+^2 r_+^2 (r_+^2 + r_+^2) \geq 8, \) this requirement is always satisfied.

To summarize, we have completed the proof of the next theorem.

**Theorem 1** For any \( r_- > r_+ > 0 \) satisfying \( r_- r_+ (r_- + r_+) > a^2 \), there exists a traveling wave solution \( r = r(y) \) to (7) with \( v = r_+ + r_- - r_+^2 r_-^{-1} a^2 + b - 1 \) such that

\[ r'(y) < 0 \text{ for } -\infty < y < \infty, \]

and \( r(y) \to r_{\pm} \text{ as } y \to \pm \infty, \) respectively.

4. Nonexistence of non-monotone traveling wave solutions

In this section we make an elementary observation that there exists no non-monotone traveling wave solutions to (7). Here, by non-monotone solution, we mean a traveling wave solution whose derivative changes sign several times. As examples, the solution \( r = r(y) \) to (9) with \( r'(y) > 0 \) on \( -\infty < y < l_0 \) and \( r'(y) < 0 \) on \( l_0 < y < \infty \) for some \( l_0 \in R \) is referred to as a “one-pulse” solution; \( r = r(y) \) with \( r'(y) < 0 \) on \( -\infty < y < l_0, \)

\[ l_{2i-1} < y < l_{2i} / (i = 1, 2, \ldots, m) \text{ and } r'(y) > 0 \text{ on } l_{2i} < y < l_{2i+1} / (i = 0, 1, \ldots, m - 1), \]

and

\[ l_{2m} < y < \infty \text{ for some } -\infty < l_0 < l_1 < \cdots < l_{2i-1} < l_{2i} < \cdots < l_{2m} < \infty \text{ is referred to as an “}(m+1)\text{-} \text{ bump}\text{” solution. We remark that the similar nonexistence result holds true for solutions whose derivative changes sign an even number of times.} \]

The proof proceeds as follows. Suppose the solution \( r = r(y) \) to (9) changes the sign of its derivative and let

\[ r'(l_0) = 0 \text{ for some } l_0 \in R. \]

We know that the ODE (9) is equivalent to the first order system

\[ \frac{d}{dy} \left( \frac{r'}{r} \right) + \left( \frac{2(r + b - v)}{1 + a^2 r' / r^2} - 1 \right) \left( \frac{1 + a^2 r' / r^2}{r' / r} \right) = 0. \]

Since this system is regular at \((r(l_0), r'(l_0)) = (r(l_0), 0)\) and \( r(y) \equiv r(l_0) \) solves the system, we conclude that the solution \( r = r(y) \) should be the constant function, thanks to the uniqueness theorem of ODE. This is a contradiction and we obtain the next theorem.

**Theorem 2** There exists no traveling wave solution \( r = r(y - vy) \) to (7) such that \( r' \) changes sign.

5. Discussions

We have introduced a singular quasilinear parabolic equation for the risk preference. The unknown function is related to the coefficient of relative risk aversion with respect to the value function in the optimal investment problem. We established the existence of monotone traveling wave solutions and the nonexistence of non-monotone traveling wave solutions. Since the coefficient of relative risk aversion is claimed to be nonincreasing, our existence theorem of monotone solutions, as well as the nonexistence theorem of non-monotone solutions, is welcome in the standpoint of financial economics.

The nonexistence theorem of non-monotone solutions perfectly corresponds with the economic theory where clearly the coefficient of risk aversion is always nonnegative. Despite resulting in nonnegative wave solutions, the existence of monotone solutions, however, casts doubt on what happens in the markets. From the traveling wave solution, as the maturity gets closer, the solution will decrease. This means a company (or an individual) is less risk averse (recall that such a solution is determined as the coefficient of relative risk aversion). We can infer that the company is less risk averse in short-term investment and more risk averse in long-term investment. This is, however, counterintuitive from the general case, where it should be the opposite. In brief, long-term investors tend to be less risk averse than short-term investors as annualized volatilities of returns on some assets are lower in the longer term.

Nevertheless, we may interpret this counterintuitive property as a special case. For example, when an economy has been stable for a long period or is in the recovering process from its trough, it seems that an individual or a company will be much cautious about its investment strategy in longer term. The company, thus, is less risk averse for short-term investment (when it predicts that markets are stable) and more risk averse for long term investment (when it forecasts that markets will be more volatile).

Also, as to the derived equation (7) itself, there certainly exist many remaining open questions. For instance, a general existence theorem is an interesting problem, which is worth further research.
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