Supplementary material for "Nonparametric LAD cointegrating regression"

Details of (23), (25), (27), (33), and (34) are given here.

(23), (25), (27):

We can establish (23), (25), and 27) by combining the standard arguments in the literature of nonparametric quantile regression. First put

$$a_i = \frac{1}{2}\Big(\frac{X_i - x_0}{h}\Big)^2 h^2 g''(\bar{X}_i) \text{ and } b_i = \tau_n^{-1}\eta_i^T\theta.$$

and notice that a_i and b_i tends to 0 uniformly in i since we can assume $|X_i - x_0| \leq Ch$.

(23): v_i^* is defined in (9) as $v_i^* = v_i + a_i$. Then

$$|v_i^*| \leq C|b_i| \Rightarrow -C|b_i| - a_i \leq v_i \leq C|b_i| - a_i.$$

Recall that $a_i/\tau_n^{-1} = O(1)$ uniformly in i from Assumption H. Hence we obtain (23) from from Assumptions V and U2.

(25): When $a_i \geq 0$ and $b_i \geq 0$, $B_{2i}(\theta)$ is not 0 only when $-a_i \leq v_i \leq -a_i + b_i$. Then we have

$$B_{2i}(\theta) = -2(v_i + a_i - b_i)$$

and

$$-2\int_{-a_i}^{-a_i+b_i}(v_i + a_i - b_i)f_{v_i}(v_i|\mathcal{E})dv_i = b_i^2 f_{v_i}(0|\mathcal{E}) + o_p(b_i^2)$$

uniformly in i from Assumption V and U2. We can deal with the other cases in the same way.

(27): When $a_i > 0$, we have

$$\text{sign}(v_i^*) - \text{sign}(v_i) = 2I(-a_i < v_i < 0)$$

and

$$2(F_{v_i}(0|\mathcal{E}) - F_{v_i}(-a_i|\mathcal{E})) = 2a_i f_{v_i}(0|\mathcal{E}) + o_p(a_i)$$

uniformly in i from Assumptions V and U2. We can deal with the other case in the same way.

(33), (34):

It is not easy to establish (33) and (34).

(33): Recall that

$$v_i^{**} = v_i + \delta_i^{**} \quad \text{and} \quad \delta_i^{**} = O(h).$$

When $\delta_i^{**} > 0$, we have

$$\mathrm{sign}(v_i^{**}) - \mathrm{sign}(v_i) = 2I(-\delta_i^{**} < v_i < 0)$$

and

$$\begin{aligned} & 2h^{-2}(F_{v_i}(0|\mathcal{E}) - F_{v_i}(-\delta_i^{**}|\mathcal{E})) \\ &= 2h^{-2}\delta_i^{**} f_{v_i}(0|\mathcal{E}) - h^{-2}(\delta_i^{**})^2 f'_{v_i}(0|\mathcal{E}) + O(|f'_{v_i}(0|\mathcal{E}) - f'_{v_i}(\bar{\delta}_i^{**}|\mathcal{E})|), \end{aligned}$$

where $\bar{\delta}_i^{**}$ is between 0 and δ_i^{**}.

Assumption V and (35) imply that

$$\begin{aligned} & |f'_{v_i}(0|\mathcal{E}) - f'_{v_i}(\bar{\delta}_i^{**}|\mathcal{E})| \\ &\le C(\sup_{|m_u - m| \le Ch} |f'_{u_i}(m_u|\mathcal{E}^i_{i-m_0}) - f'_{u_i}(m|\mathcal{E}^i_{i-m_0})| + o(1)) \end{aligned}$$

uniformly in i. Since

$$\lim_{h\to 0} \mathrm{E}\{\sup_{|m_u - m| \le Ch} |f'_{u_i}(m_u|\mathcal{E}^i_{i-m_0}) - f'_{u_i}(m|\mathcal{E}^i_{i-m_0})|\} = 0,$$

we get from [22] (for example, see Proposition 1 of this paper) that

$$\tau_n^{-2}\sum_{i=1}^n K_i |f'_{v_i}(0|\mathcal{E}) - f'_{v_i}(\bar{\delta}_i^{**}|\mathcal{E})| = o_p(1).$$

(34): First write

$$\begin{aligned} f_{v_i}(0|\mathcal{E}) &= f_{u_i}(m_u|\mathcal{E}^i_{i-m_0})\Big(\frac{\partial v}{\partial u}(X_i, m_u)\Big)^{-1} \\ &= f_{u_i}(m_u|\mathcal{E}^i_{i-m_0})(w(x_0) + (X_i - x_0)w'(x_0) + o(|X_i - x_0|)), \end{aligned}$$

where $w(x)$ is clearly defined in the above equation. Using the above notation, we have

$$
\begin{aligned}
&\frac{2h^{-2}}{\tau_n^2}\sum_{i=1}^{n} K_i\delta_i^{**} f_{v_i}(0|\mathcal{E}) \\
= \;& \frac{2h^{-2}}{\tau_n^2}\sum_{i=1}^{n} K_i\Big\{\frac{X_i-x_0}{h}hg'(x_0)+\frac{1}{2}\Big(\frac{X_i-x_0}{h}\Big)^2 h^2 g''(\bar{X}_i)\Big\} \\
& \times f_{u_i}(m_u|\mathcal{E}_{i-m_0}^i)(w(x_0)+(X_i-x_0)w'(x_0)+o(|X_i-x_0|)) \\
= \;& 2\tau_n^{-2}\sum_{i=1}^{n} K_i f_{u_i}(m_u|\mathcal{E}_{i-m_0}^i)\Big\{\Big(\frac{X_i-x_0}{h}\Big)^2 g'(x_0)w'(x_0) \qquad (37)\\
& +\frac{1}{2}\Big(\frac{X_i-x_0}{h}\Big)^2 g''(x_0)w(x_0)+o(h^2)\Big\} \\
& +\frac{2h^{-1}}{\tau_n^2}\sum_{i=1}^{n} K_i\frac{X_i-x_0}{h} f_{u_i}(m_u|\mathcal{E}_{i-m_0}^i)g'(x_0)w(x_0)+o_p(1).
\end{aligned}
$$

We can handle the first term of (37) by using Proposition 1.

Finally we consider the second term of (37). Write

$$
\begin{aligned}
&\frac{h^{-1}}{\tau_n^2}\sum_{i=1}^{n} K_i\frac{X_i-x_0}{h} f_{u_i}(m_u|\mathcal{E}_{i-m_0}^i) \\
= \;& \frac{\tau_n^{-1}}{(nh^6)^{1/4}}\sum_{i=1}^{n} K_i\frac{X_i-x_0}{h}\{(f_{u_i}(m_u|\mathcal{E}_{i-m_0}^i)-f_u(m_u))+f_u(m_u)\}. \qquad (38)
\end{aligned}
$$

We can use Theorem 2.1 of [23] to show $\tau_n^{-1}\sum_{i=1}^n\{(X_i-x_0)/h\}K_i = O_p(1)$. We can deal with the first term inside the braces of (38) by using a result similar to the second element of Proposition 1.