Search, money and capital in an overlapping generations model

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Abstract

In this note, we incorporate capital accumulation into Zhu ("An overlapping generations model with search", Journal of Economic Theory 2008). I show that the Friedman rule may not be optimal. Intergenerational monetary transfer between young and old is too high under the Friedman rule. The deviation from the rule lowers the utility at the decentralized market, but the welfare loss is dominated by the welfare gain through the reduction of the intergenerational transfer. We also show that monetary policy is not neutral and inflation lowers the real interest rate.

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1 Introduction

Search-theoretic models of monetary exchange have been intensively investigated in modern macroeconomics. Lagos and Wright (2005) (henceforth LR) consider an infinite horizons model with buyer and seller and show that the Friedman rule is optimal. Aruoba and Wright (2005) incorporate capital accumulation into LR and show that the rule is still optimal and that capital and labor are dichotomized.

In this note, we consider capital accumulation in a monetary OLG model with search. The set-up is very close to Zhu(2008). The only difference is that our model has two assets with (possibly) different rates of returns, money and capital. We first show that under some parametric restrictions, positive nominal interest rates raise steady state welfare of agents. The optimal monetary policy deviates from the Friedman rule. We next show that inflation enhances capital accumulation.

Optimality of the Friedman rule in OLG models have been investigated by many authors. For example, Gahvari (2008) shows that if capital tax is available, FR is not the only optimal policy. However, these models are lack of microfoundation. Here we consider the similar problem in a micro-founded model.

In our model, money is not neutral in the sense that inflation enhances more capital accumulation. Therefore our model observes Tobin effect. Rapach (2005) studies the relationship between inflation and real interest rate for 14 countries. He shows that monetary neutrality rejected for all countries and that inflation lowers real interest rates for all countries. This observation is clearly inconsistent with the infinite horizons model of Aruoba and Wright (2005) who shows the superneutrality of money.
Many authors including Zhu (2008) argue that the monetary neutrality in the monetary search model crucially depends on the quasi-linearity of the utility function. However, the utility function is still quasi-linear in our model and we still observe the capital accumulation by the monetary expansion.

In the following, Section 2 describes the model. Section 3 characterizes the competitive equilibrium. Section 4 shows the non-optimality of the Friedman rule. Section 5 shows the non-neutrality of money. Section 6 concludes the paper. Proofs are in the Appendix.

2 Environment

This section sets up the model which is based on Zhu(2008).

2.1 Set-up

Time is discrete and continues forever. There is a [0,1] continuum of agents who live two periods. The economy consists of two markets, centralized market and decentralized market. There are general good, special good, money and capital. Production of general good requires both labor and capital.

Money is perfectly divisible and agents can carry non-negative amount of money. Only money can be used in a decentralized market. In this sense money is essential. Each date has two stages, day and night and individual lives for three stages.

Day market is centralized, but the night market is decentralized and buyer needs
to have money to get the special good in the night market. At the 1st stage (day): agents supply unit labor \( n_t = 1 \), receive wage income and hold money and capital. In a later section, we consider a case with endogenous labor supply.

At the second stage that is open at night, agents have equal probability \( \pi \in (0, 0.5] \) to be a buyer and a seller. There is no possibility of double coincident of wants. Buyer’s utility is \( u(q) \) and seller’s disutility is \( c(q) \). Without loss of generality, we assume \( u(q) = q \).

At the third stage, agents consume but do not supply labor. The utility function at the this stage is linear and is given by \( u_3(c) = \beta c \), with \( \beta > 0 \). The constant \( \beta \) can be considered as the discount factor.

There is a large number of identical competitive firms. Production function \( F(k_t, n_t) = Ak_t^{\alpha}n_t^{1-\alpha} \) is constant returns to scale, where \( n_t = 1 \) is labor, \( k_t \) is capital at time \( t \) and \( \alpha \in (0, 1) \) is capital share. Let \( f(k_t) = Ak_t^{\alpha} \). Capital is fully depreciated. Factor markets are competitive. The wage is \( W_t = A(1 - \alpha)k_t^{\alpha} \equiv W(k_t) \) and the capital rental rate is \( R_t = A\alpha k_t^{\alpha-1} \equiv R(k_t) \). Note that the function \( \omega(k) = W(k) - k = A(1 - \alpha)k^\alpha - k \) is maximized when \( k_\omega = [\alpha(1 - \alpha)A]^{1/(1-\alpha)} \) and the maximum is \( \omega(k_\omega) = \rho_{\text{max}} = (1/\alpha - 1)[\alpha(1 - \alpha)A]^{1/(1-\alpha)} \).

The amount of money supply at time \( t \) is \( M_t > 0 \). The growth rate of money is \( \tau = M_{t+1}/M_t - 1 \), which is controlled by the government and is set to be constant. If we let \( p_t \) denote the price of the general good in terms of money, the real value of money held by the old is \( \rho_t = M_t/p_t \).
2.2 Preference

The agents who are born at time $t$ have the following preferences:

$$U = \pi(q_b - \beta \rho_{t+1} l_b) + \pi(\beta \rho_{t+1} l_s - c(q_s)) + \beta(\rho_{t+1} x + R_{t+1} a_{t+1}).$$

Here $q_b$ is the amount of special good that the agent obtains when he becomes buyer, $q_s$ is the amount of special good that the agent produces when he becomes seller, $l_b(l_s)$ is the monetary transfer that buyer pays (seller receives) and $x$ denotes the individual states that show the ratio of money holdings to the total money holdings $M_{t+1}$. Therefore the individual money holding when he is old (stage 3) is equal to $xM_{t+1}$. The budget constraint when he is young is

$$W_t = (1 + \tau)\rho_t x - \tau \rho_t + a_{t+1}. \quad (1)$$

This equality implies that agents save their wage income in the form of money and capital. New money $\tau \rho_t$ is injected in the form of lump sum transfers. At the equilibrium, savings $a_t$ coincides with the capital $k_t$.

Following Zhu (2008), we restrict our attention on the stationary equilibrium with $x = 1$, $\rho_t = \rho$, $l_s = l_b = l$, $a_t = k_t = k$ and $q_b = q_s = q$. If we let $s(q) = q - c(q)$ denote a surplus at the decentralized market, the utility at the stationary equilibrium is $U = \pi s(q) + \rho + \beta R(k)k$. Substitution of the young agent’s budget constraint
$W(k) = \rho + k$ into the utility function yields

$$U = \pi s(q) + \beta(f(k) - k).$$

(2)

The real money holdings $\rho$ is interpreted as transfer from young to old. Eq. (2) implies that the consumption of the old $c$, the capital $k$ and the amount of available resources $f(k) = F(k, 1)$ satisfies the resource constraint $c + k = F(k, 1)$.

2.3 Pairwise meeting

The payoff of a person who enters old age in state $z$, $\beta(\rho z + Rk)$ is linear. Hence Nash bargaining problem between seller with state $(z_s, k_s)$ and buyer with state $(z_b, k_b)$ is independent of their capital holdings:

$$\max_{q \geq 0, \ 0 \leq t \leq z_b} (u(q) - \beta pl)^{\theta} (-c(q) + \beta pl)^{1-\theta}. \quad (3)$$

Here $\theta$ is the bargaining power of the buyer. We assume the following on the cost function.

**Assumption 1:** The function $c$ satisfies the following.

1) The function $c$ is positive, strictly increasing and strictly convex.

2) The surplus $q - c(q)$ is maximized if $q = \bar{q} > 0$.

For the efficient value of the production $\bar{q}$, let $\bar{\rho} = \{\theta c(\bar{q}) + (1 - \theta)\bar{q}\}/\beta$. We denote the solution to the first order conditions $\theta[\beta m - c(q)] = (1 - \theta)c'(q) [q - \beta m]$ as $q(m)$. 

6
The function $q(m)$ is the inverse function of

$$m(q) = \frac{1}{\beta} \frac{\theta c(q) + (1 - \theta)qc'(q)}{\theta + (1 - \theta)c'(q)}.$$  

**Lemma 1** The function $q(m)$ is increasing, $q(0) = 0$, $q'(0) = \infty$, $q(\bar{q}) = \bar{q}$ and $q'(\bar{q}) < \beta$.

**Proof.** See Appendix. ■

We put the following assumption on the function $q$:

**Assumption 2:** The function $q(m)$ is strictly concave.

The following example satisfies Assumption 2.

**Example 1** For $\theta = 1/2$, $\beta = 1$ and $c(q) = q^2$, one gets $q(m) = 3^{-1}\{m + \sqrt{m^2 + 3m}\}$.

It is concave, since $3q'(m) = 1 + [1 - 9/4(m + 3/2)^{-2}]^{-1}$ is obviously decreasing.

The surplus $s(q)$ satisfies $s'(q) > (<)0$ if $q < (>)\bar{q}$ and is maximized when $q = \bar{q}$. Assumption 2 and Lemma 1 imply that the function $S(\rho) = s(q(\rho))$ is concave, increasing if $\rho \leq \bar{\rho}$, maximized when $\rho = \bar{\rho}$ and $S'(0) = +\infty$.

The optimal $(q,l)$ in this bargaining problem (3), $(q^*,l^*)$ is

$$(q^*,l^*) = (q(\rho z_b), \rho z_b) \quad \text{if } \rho z_b \leq \bar{\rho}.$$  

$$= (\bar{q}, \bar{\rho}) \quad \text{if } \rho z_b > \bar{\rho}.$$  

If the buyer’s money holdings is large, then efficient trade is possible. In what follows, we say the real money holding is *sufficient* if it is more than $\bar{\rho}.$
3 Competitive equilibrium

This section characterizes the competitive equilibrium. We solve the model backward. First we fix the value of money $\rho x$ and the capital $k$ and derives the utility of agent with state $x$. Here we investigate a case with $\rho < \bar{\rho}$ and the one with $\rho \geq \bar{\rho}$ separately.

3.1 Case with insufficient money holdings

Here we consider a case in which the total money holding is less than the efficient value ($\rho < \bar{\rho}$). First assume $\rho x \leq \bar{\rho}$. This inequality must in a degenerate equilibrium with $x = 1$. If agent with state $x$ becomes a buyer, one gets $(q^*, \rho l^*) = (q(\rho x), \rho x)$, since his money holding $\rho x$ is insufficient. On the other hand, if he becomes a seller and meets buyer with $x = 1$, the optimal trade is $(q^*, \rho l^*) = (q(\rho), \rho)$. since the buyer’s money holdings $\rho$ are also insufficient. If we denote the expected utility with fixed $k$ and $x$ as $U(x, k)$, one has

$$U(x, k) = \pi(q(\rho x) - \beta \rho x) + \pi(-c(q(\rho)) + \beta \rho) + \beta(\rho x + Rk).$$

Next suppose $\rho x > \bar{\rho}$. If agent with state $x$ becomes a buyer, one gets $(q^*, \rho l^*) = (\bar{q}, \bar{\rho})$, since his money holding $\rho x$ is more than the efficient value. On the other hand, if he meets with seller who meets buyer with $x = 1$, one gets $(q^*, \rho l^*) = (q(\rho), \rho)$ since his money holding $\rho$ is insufficient. The utility is given by

$$U(x, k) = \pi(q(\bar{\rho}) - \beta \bar{\rho}) + \pi(-c(q(\rho)) + \beta \rho) + \beta(\rho x + Rk).$$  (4)
Note that the degenerate competitive equilibrium with \( x = 1 \) cannot satisfy \( \rho x > \bar{\rho} \).

This means that if such an equilibrium exists, then the utility above must be less than the one with \( x < 1 \). Later we look for the money growth rate \( \tau \) so that agents optimally choose \( x = 1 \). If such \( \tau \) exists, then we can say that the competitive equilibrium with money holding \( \rho < \bar{\rho} \) exists.

### 3.2 Case with sufficient money holdings

Now assume that agents have sufficient amount of money (\( \rho > \bar{\rho} \)). First suppose \( \rho x \geq \bar{x} \). Note that the stationary equilibrium with \( x = 1 \) satisfies the condition. When the agent with state \( x \) becomes buyer, one gets \((q^*, \rho l^*) = (\bar{q}, \bar{\rho})\). If becomes seller who meets buyer with \( x = 1 \), one gets \((q^*, \rho l^*) = (\bar{q}, \bar{\rho})\). The utility \( U(x, k) \) is:

\[
U(x, k) = \pi s(q(\bar{\rho})) + \beta(\rho x + Rk). \tag{5}
\]

Remember that \( \bar{q} = 1/2 \).

Next suppose \( \rho x < \bar{x} \). This cannot be the degenerate equilibrium with \( x = 1 \). If agent with state \( x \) becomes buyer, one has \((q^*, l^*) = (q(\rho x), x)\). If he becomes seller who meets buyer with \( x = 1 \), \((q^*, l^*) = (\bar{q}, \bar{\rho}/\rho)\). The utility is written as

\[
U(x, k) = \pi(q(\rho x) - \beta px) + \pi(-c(\bar{\rho}) + \beta \bar{\rho}) + \beta(\rho x + Rk). \tag{6}
\]
3.3 Equilibrium allocation

This section characterizes the competitive equilibrium allocation. First suppose $\rho \leq \bar{\rho}$. If the agent becomes buyer, the amount of transaction at the decentralized market is no more than the efficient money holdings $\bar{\rho}$. The expected utility is

$$U = \pi [q(\min\{\rho x, \bar{\rho}\}) - \beta \min\{\rho x, \bar{\rho}\}] + \pi \{\beta \rho - c(q(\rho))\} + \beta (\rho x + R(k)k).$$

Here the nominal interest rates is $i = R(k)(1 + \tau) - 1$. Under the Friedman rule, $i = 0$. Substituting the budget constraint $W = (1 + \tau)\rho x - \tau \rho + k$ into the second period consumption yields $Rk + \rho x = Rw - i\rho x + R\tau \rho$. One has

$$U(x) = \pi [q(\min\{\rho x, \bar{\rho}\}) - \beta \min\{\rho x, \bar{\rho}\}] - \beta i\rho x + C_0.$$ 

where $C_0 = \pi \{\rho - c(q(\rho))\} + \beta (Rw + R\tau \rho)$ is constant. If the nominal interest rates are negative, $1 > R(1 + \tau)$ and then $\lim_{x \to -\infty} U(x) = +\infty$. The optimal $x$ does not exist. Hence $i \geq 0$ at the competitive equilibrium.

The utility function $U$ is kinked at $x = \bar{\rho}/\rho$:

$$U(x) = \begin{cases} \pi \{q(\rho x) - \beta \rho x\} - \beta i\rho x + C_0 & \text{if } x \leq \bar{\rho}/\rho \\ -\beta i\rho x + \{\pi q(\bar{\rho}) - \beta \pi \bar{\rho} + C_0\} & \text{otherwise.} \end{cases}$$

The function $U$ is decreasing function if $x > \bar{\rho}/\rho$. Moreover, if $x = \bar{\rho}/\rho$, we have $U'(\bar{\rho}/\rho - 0) = \pi \rho \{q'(\bar{\rho}) - \beta\} - \beta i\rho x < 0$ since $q'(\bar{\rho}) < \beta$. Hence the optimal choice of $x$
is interior and satisfies $\pi q'(\rho x) = \beta (\pi + i)$. When the agent optimally chooses $x = 1$, the first order conditions are:

$$\beta i = \pi (q'(\rho) - \beta).$$

(7)

The utility at the stationary equilibrium is $U = \pi s(\rho) + \beta (f(k) - k)$ and satisfies $W = \rho + k$.

Next suppose $\rho > \bar{\rho}$. Then the agent’s utility function is written as:

$$U = \pi(q(\min\{\rho x, \bar{\rho}\}) - \beta \min\{\rho x, \bar{\rho}\}) + \beta(Rw - i\rho x) + C'.$$

Here $C = \pi\{\beta \bar{\rho} - c(q(\bar{\rho}))\} + \beta R\tau \rho$. As we show, if $i \geq 0$, the optimal $x$ satisfies $\rho x < \bar{\rho}$.

Since $\rho > \bar{\rho}$ by assumption, the optimal $x$ cannot be equal to 1. Therefore there exists no competitive equilibrium with the total money holding $\rho > \bar{\rho}$.

Let $\rho_{\text{FR}}$ denote a real balance under the Friedman rule with $i = 0$. Since $\beta i = \pi (q'(\rho) - \beta)$, one has

$$q'(\rho_{\text{FR}}) = \beta.$$

(8)

We have the following proposition on the monetary satiation around the Friedman rule.

**Lemma 2** Under the Friedman rule, real balance does not achieve the efficient production at the decentralized market. In other words, $\rho_{\text{FR}} < \bar{\rho}$.

**Proof.** Since $q'(\bar{\rho}) < \beta$, $\rho_{\text{FR}} < \bar{\rho}$.  ■

The real money balance is always insufficient under the Friedman rule. The result differs from LW. Next proposition characterizes the competitive equilibrium.
**Proposition 1** The allocation $(\rho, k)$ is the stationary competitive equilibrium allocation with money growth rate $\tau$ if and only if the following hold:

\begin{align*}
\rho &= W(k) - k, \quad (9) \\
\beta i &= \pi(q'(\rho) - \beta), \quad (10) \\
i &= R(k)(1 + \tau) - 1 \geq 0. \quad (11)
\end{align*}

Here $i$ is the nominal interest rate. At the competitive equilibrium, $\rho \leq \rho_{FR}$.

**Proof.** For $i$ to be greater than 0, we must have $q'(\rho) \geq \beta$. ■

Eq (9) shows the budget constraint of the young, and Eq. (10) shows the first order conditions at the second stage. If the Friedman rule holds, then $1 + \tau = 1/R$.

## 4 Optimal monetary policy

This section derives the optimal monetary policy. Considers the problem of the government who maximizes $U$ subject to the constraint $\rho = \omega(k)(= W(k) - k)$. At the stationary equilibrium with $x = 1$, the utility is

\[ U = \pi s[q(\omega(k))] + \beta(f(k) - k). \]

The utility $\beta(f(k) - k)$ is maximized if the capital is equal to the golden rule of capital $\bar{k}$ satisfying $f'(\bar{k}) = 1$. Since the function $\omega(k) = A(1 - \alpha)k^\alpha - k$ is inverse-U shaped, there are two values of capital, that satisfies equation $\omega(k) = \rho$. Let us call the larger
one $k(\rho)$ and smaller one $k_m(\rho)$. Since the second period consumption is written as $f(k) - k = \alpha f(k) + \rho$, it is higher for larger capital. Since we are deriving the optimal monetary policy, we concentrate on the case with large capital holding $k(\rho)$. Since the function $\omega$ is decreasing if $k = k(\rho)$, one also has $k'(\rho) < 0$. Using $\rho$, we can express the utility $U$ as a function of $\rho$, not $k$:

$$U = \pi s(q(\rho)) + \beta \{ f(k(\rho)) - k(\rho) \} \equiv u(\rho).$$

**Lemma 3** If $\beta < q'(\rho_{\max})$, there exists no competitive equilibrium with the Friedman rule. The lowest possible nominal interest rate $i$ is such that $\beta i = \pi(q'(\rho_{\max}) - \beta)$, which is positive.

**Proof.** See Appendix. 

Next assume that the preference parameter $\beta$ is sufficiently large:

$$\beta > q'(\rho_{\max}). \quad (12)$$

Under the assumption, $\rho_{FR} < \rho_{\max}$ and then $k_{FR}$ always exists. Positive nominal interest rates lower the equilibrium money holding $\rho$ and also lower the utility from the decentralized market transaction. However, the nominal interest rate always increase the equilibrium capital and it may raise the second period utility $\beta(f(k) - k)$ if
\[ f'(k_{FR}) > 1. \] This condition holds if and only if

\[ \rho_{FR} > \omega(\bar{k}). \] (13)

with \( \bar{k} = (\alpha A)^{1/(1 - \alpha)} \). Therefore both inequalities hold if and only if

\[ \beta \in (q'(\rho_{\text{max}}), q'(\omega(\bar{k}))). \]

Such \( \beta \) always exists. Note that \( \omega = [\alpha(1 - \alpha)A]^{1/(1 - \alpha)} < \bar{k} = (\alpha A)^{1/(1 - \alpha)} \).

To compare the utility loss at the decentralized market with the second period utility gain, differentiate the function \( u(\rho) \):

\[ u'(\rho) = \pi q'(\rho)s'(q(\rho)) + \beta k'(\rho)(f'(k) - 1). \]

As long as \( i \) is small, \( q'(\rho) > 0, s'(q(\rho)) > 0, k'(\rho) < 0 \) and \( f'(k(\rho)) > 1 \). Therefore if the welfare gain from deviating the Friedman rule at the old age \( k'(\rho)(f'(k) - 1) \) is large, deviation from the rule is welfare improving. In other words, the Friedman rule is not optimal if

\[ \pi q'(\rho_{FR})s'(q(\rho_{FR})) < \beta(-k'(\rho_{FR}))(f'(k_{FR}) - 1). \]
The capital under the Friedman rule $k_{FR}$ is given by $\rho_{FR} = \omega(k_{FR})$, with $q'(\rho_{FR}) = \beta$. Hence one gets

$$\pi s'(q(\rho_{FR})) < (-k'(\rho_{FR}))(f'(k_{FR}) - 1). \quad (14)$$

We have the following proposition.

**Proposition 2** Suppose Eqs. (12) and (13) hold. For sufficiently small probability of matching $\pi > 0$, the deviation from the Friedman rule is welfare improving.

**Proof.** Here $\rho_{FR}$ and the functions $s, q, k$ and $f$ are independent of $\pi$. Also note that $f'(k_{FR}) > 1$ and $k' < 0$. Hence if $\pi$ is small, positive nominal interest rates are welfare improving. ■

Note that the inequality holds for sufficiently small $\pi$, but not for $\pi = 0$. If $\pi = 0$, money and capital plays exactly the same role and then the Friedman rule is always optimal monetary policy.

**Example 2** In Example 1, suppose $f(k) = 4k^{0.5}$. One has $W(k) = 2k^{0.5}$, $\bar{q} = 1/2$, $\bar{k} = 4$, $\rho_{FR} = \sqrt{3} - 3/2 \simeq 0.2$ and $q(\rho_{FR}) \simeq 0.4 < \bar{q}$. Since $\rho_{max} = 1 > \rho_{FR}$, $k_{FR}$ exists and satisfies $f'(k_{FR}) > 1$ and $s'(q(\rho_{FR})) > 0$. Hence a constant $\phi = -k'(\rho_{FR})(f'(k_{FR}) - 1)/s'(q(\rho_{FR}))$ is positive. If we let $\pi = 0.5\phi$, Eq. (14) hold.

Here the real balance $\rho$ works as a intertemporal transfer from young to old. Under the severe Friedman deflation, the monetary redistribution is too large and reduction of the real money holding via positive nominal interest rate can be welfare improving. Note that the deviation from the Friedman rule reduces the utility at the decentralized market, but the welfare gain at the centralized market dominates the loss.
5 Monetary non-neutrality

This section shows that the expansionary monetary policy can enhance capital accumulation. Aruoba and Wright (2003) finds that their model with infinite horizons exhibits a dichotomy. In their model, one can determine the equilibrium path for the value of money independent of the paths of capital in the centralized market. Here we show that their result does not hold in our OLG model.

Optimality conditions for the real balance $\beta i = \pi(q'(\rho) - \beta)$ is written as

$$1 + \tau = \frac{\pi \beta^{-1} q'(h(k)) + 1 - \pi}{R(k)}.$$  \hspace{1cm} (15)

Obviously the monetary expansion $\tau > 0$ affects the equilibrium capital level.

**Proposition 3** Monetary expansion raises the per capital level $k$ and then lowers the real interest rate $R(k)$. It lowers the total production at the centralized market.

**Proof.** One has $q' < 0$. $h' < 0$ and $R' < 0$. the right hand side of the equation is increasing. Hence $\tau$ lowers the per capital capital $k$. ■

6 Conclusion

In this note, we incorporate capital accumulation into monetary OLG model with search in Zhu (2008). We prove that the Friedman rule may not be optimal and that inflation raises the capital level. These results are different from the recent infinite horizons monetary search model of Lagos and Wright (2005) and Aruoba and Wright
(2005). As a future study, it is interesting to consider the role of banks in our OLG model. Berenseten et al. (2005) consider the role of financial intermediation in the LW model. It is likely that under the existence of the financial intermediation, it may be possible to implement the first best allocation.

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**Appendix**

A Proof of Lemma 1

The inverse function \(m(q)\) satisfies

\[
\beta m'(q) = 1 - \theta \frac{[\theta + (1 - \theta)c'(q)][1 - c'(q)] - (1 - \theta)c''(q)[q - c(q)]}{[\theta + (1 - \theta)c'(q)]^2}.
\]

Hence \(m(0) = 0, m'(0) = 0\) and \(m(\bar{q}) = \bar{m}/\beta\). The solution to the Nash bargaining problem satisfies \(0 = (\rho - c(q)) - c'(q)(q - \rho)\). Hence

\[
\rho = \frac{c'(q)q + c(q)}{1 + c'(q)} = q - \frac{q - c(q)}{1 + c'(q)}.
\]

It is easy to check that when \(1 = c'(q)\), \(\rho'(q) > 1\). Hence \(q'(<\hat{\rho}) < 1\).
B  Proof of Lemma 2

If the inequality holds, $q'$ is decreasing function (i.e. $q$ is concave) and then $\rho_{\text{max}} < \rho_{\text{FR}}$. Then we cannot find $k$ such that $\omega(k) = \rho_{\text{FR}}$. 

C  Proof of Proposition 5

Here $q' (\rho^* ) c' (q (\rho^*)) < 1$ if and only if $\phi' (y^*) > 1$. In what follows, we show that $\phi' (y) > 1$ for all $y < y_{\text{FR}}$. Obviously the function $\phi$ is increasing and strictly concave.

Under the Friedman rule, $c' (q_{\text{FR}}) = 1$. One has

$$\rho' (q_{\text{FR}}) = 1 + q_{\text{FR}} c'' (q_{\text{FR}}) (q_{\text{FR}} - c(q_{\text{FR}})) / 4 > 1.$$  

Hence

$$\phi' (y_{\text{FR}}) = \frac{d \rho (c^{-1} (y))}{dy} = \frac{\rho' (q_{\text{FR}})}{c' (q_{\text{FR}})} > 1.$$  

Hence $\rho' (y) > 1$ for all $y < y_{\text{FR}}$. 

D  Linearity of the utility function

In this note, the utility of buyer $u(q)$ is linear function. Suppose both the utility $\hat{u}$ and the cost function $\hat{c}$ are nonlinear. In this case the bargaining problem is written as

$$\max_{Q \geq 0, \ 0 \leq l \leq \omega} (\hat{u}(Q) - \beta l) \theta (-\hat{c}(Q) + \beta l)^{1-\theta}.$$
If we let $q = \hat{u}(Q)$ and $c(q) = \hat{c}(\hat{u}^{-1}(q))$, we can re-express the bargaining problem as the optimization problem on $q$, not $Q$ itself. These two problems are equivalent since there exists a one-to-one correspondence between $q$ and $Q$.

**References**


