Online Appendix to
"Characterizing Social Value of Information"

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This document is an online appendix to "Characterizing Social Value of Information" by Takashi Ui and Yasunori Yoshizawa.

1 A large Cournot game

Corollary 14 states that a large Cournot game studied by Vives [3] can be type +III; that is, expected total profits can decrease with the precision of both public and private information. Angeletos and Pavan [1] (henceforth AP) consider the same large Cournot game and their Corollary 10 states that expected total profits necessarily increase with the precision of private information, but can decrease with that of public information. Thus, Corollary 14 is inconsistent with AP's Corollary 10.

To explain what induces this inconsistency, we adopt AP's notation in p. 1128 and write the payoff function as follows:

$$U = (a_0 - c_1 + a_1\theta - a_3K)k - (a_2 + c_2)k^2,$$

where $a_0, a_1, a_2, a_3, c_1, c_2 > 0$ are constants, $k \in \mathbb{R}$ is an action, and $K \in \mathbb{R}$ is its mean over all the players. This payoff function is the same as (18) when $a_0 = a_2 = c_1 = 0$, $a_1 = 1$, $a_3 = -r$, and $c_2 = 1/2$. AP define

$$\alpha \equiv -(\partial^2 U/\partial k\partial K)/(\partial^2 U/\partial k^2) = -a_3/(2(a_2 + c_2)),$$

which is the same as α in this paper and assumed to be strictly less than 1. In AP's proof, they directly calculate the partial derivative of the welfare loss $\mathcal{L}$ due to incomplete information given by (36) in AP, where their parameter ϕ plays a key role. They obtain $\phi = \alpha/(2(1-\alpha))$, but this includes an error. By correcting it, we obtain $\phi = \alpha/(1-2\alpha)$.

Using the correct value of ϕ, we calculate $\mathcal{L}$ and its partial derivatives based upon (36) in AP. We write $\sigma_x^2 \equiv 1/\tau_x$ and $\sigma_z^2 \equiv 1/(\tau_y + \tau_\theta)$, following AP. Then, we have

$$\mathcal{L} = a_1^2/(a_2+c_2) \times \sigma_x^2\sigma_z^2(\sigma_x^2 + (1-\alpha^2)\sigma_z^2)/(4(1-\alpha)^2(\sigma_x^2 + (1-\alpha)\sigma_z^2)^2),$$
$$\partial\mathcal{L}/\partial\sigma_x^2 = a_1^2/(a_2+c_2) \times \sigma_z^4(\sigma_x^2 + (1+\alpha)\sigma_z^2)/(4(\sigma_x^2 + (1-\alpha)\sigma_z^2)^3),$$
$$\partial\mathcal{L}/\partial\sigma_z^2 = a_1^2/(a_2+c_2) \times \sigma_x^4(\sigma_x^2 + (1-\alpha)(2\alpha+1)\sigma_z^2)/(4(1-\alpha)^2(\sigma_x^2 + (1-\alpha)\sigma_z^2)^3).$$

Thus, $\partial\mathcal{L}/\partial\sigma_x^2 < 0$ if and only if $\sigma_x^2 + (1+\alpha)\sigma_z^2 < 0$, which is rewritten as $\sigma_x^2/\sigma_z^2 = (\tau_y+\tau_\theta)/\tau_x < -(\alpha+1)$, and $\partial\mathcal{L}/\partial\sigma_z^2 < 0$ if and only if $\sigma_x^2 + (1-\alpha)(2\alpha+1)\sigma_z^2 < 0$, which is rewritten as $\sigma_x^2/\sigma_z^2 = (\tau_y+\tau_\theta)/\tau_x < -(1-\alpha)(2\alpha+1)$. That is, expected total profits decrease with the precision of private information if and only if $\alpha < -1$ and $\tau_x > -(\tau_y+\tau_x)/(\alpha+1)$, and decrease with that of public information if and only if $\alpha < -1/2$ and $\tau_x > -(\tau_y+\tau_\theta)/((1-\alpha)(2\alpha+1))$. This result is consistent with Corollary 14.

2 A large Bertrand game

We consider a large Bertrand game studied by AP (see also Vives [2]) and revise their result. Player i produces good i and chooses its price a_i. The demand function is $\theta - a_i + \rho\int a_j dj$, where $\rho > 0$ and θ is normally distributed. The cost function is cq^2 with $c > 0$. Then, player i's profit is

$$\begin{aligned}\Big(\theta - a_i + \rho\int a_j dj\Big)a_i - c\Big(\theta - a_i + \rho\int a_j dj\Big)^2 &= \\ &- (c+1)a_i^2 + \rho(2c+1)a_i\int a_j dj + (2c+1)\theta a_i \\ &- \rho^2 c\Big(\int a_j dj\Big)^2 - 2\rho c\theta\int a_j dj - c\theta^2. \qquad (1)\end{aligned}$$

The type of this game is summarized as follows by Corollary 3.

Corollary 15. *Suppose that $\rho < 2(c+1)/(2c+1)$. This game is type $-IV$ if $c > (-1+\sqrt{2})/2$ and $(2c+1)/(4c) < \rho < 2(c+1)/(2c+1)$ and type $+I$ otherwise.*

Proof. Dividing the payoff function by $c+1$, we have $\alpha = \rho(2c+1)/(2(c+1))$, $\zeta = (2(1-2\rho)c+1)/(2c+1))$, and $\eta = (c(2c+1)\rho^2 - 4c(c+1)\rho + 2c^2 + 3c +$

$1)/((c+1)(2c+1))$. The condition $\alpha < 1$ guarantees the uniqueness of equilibrium, i.e., $\rho = 2(c+1)\alpha/(2c+1) < 2(c+1)/(2c+1)$.

Note that $\zeta < 0$ if and only if $\rho > (2c+1)/(4c)$. Thus, if $\zeta < 0$, then we must have $(2c+1)/(4c) < 2(c+1)/(2c+1)$ because $\rho < 2(c+1)/(2c+1)$, which is rewritten as $c > (-1+\sqrt{2})/2$. Note also that $\eta > 0$ because the discriminant of the numerator of η as a quadratic function of ρ is $-4c^2 - 4c < 0$.

Therefore, this game is type $-$IV if $c > (-1+\sqrt{2})/2$ and $(2c+1)/(4c) < \rho < 2(c+1)/(2c+1)$. Otherwise, this game is type +I. To see this, suppose that $\rho \leq \min\{(2c+1)/(4c), 2(c+1)/(2c+1)\}$. Then, $\zeta, \eta > 0$ and

$$(1-\alpha)\zeta - 3\eta/2 = ((2c^2+c)\rho^2 - \rho - 2c^2 - 3c - 1)/(2(c+1)(2c+1)) < 0.$$

In fact, the numerator of the fraction above is strictly negative for all $\rho \in (0, 2(c+1)/(2c+1))$ because it is so at the endpoints of this interval. Therefore, $(1-\alpha)\zeta/\eta < 3/2$. □

For example, if we set $\rho = 4/5$ and $c = 1$, then $\zeta = -1/20 < 0$, $\eta = 19/100 > 0$, and $X = 8$. Thus, expected total profits decrease with the precision of private information if and only if $\tau_x < (\tau_y + \tau_\theta)/8$.

AP's Corollary 11 states that expected total profits necessarily increase with the precision of both public and private information. This implies that a large Bertrand game is type +I for all $\rho, c > 0$, which is inconsistent with the above result.

AP consider the following payoff function in p. 1129:

$$U = (\theta - k + bK)k - c(\theta - k + bK)^2,$$

where $b, c \in \mathbb{R}$ are constants with $0 < b < 1$, k is an action, and $K \in \mathbb{R}$ is its mean over all the players. This payoff function is the same as (1) by the replacement of b with ρ.

In AP's proof, they directly calculate the partial derivative of the welfare loss $\mathcal{L}$ due to incomplete information given by (36) in AP. They then show that $\partial\mathcal{L}/\partial\sigma_x^2 > 0$, where $\sigma_x^2 \equiv 1/\tau_x$, but this includes an error. To see this, we calculate $\mathcal{L}$ and $\partial\mathcal{L}/\partial\sigma_x^2$ based upon (36) in AP assuming that $b = 4/5$ and $c = 1$. We write $\sigma_z^2 \equiv 1/(\tau_y + \tau_\theta)$, following AP. Then, we have

$$\mathcal{L} = 75\sigma_x^2\sigma_z^2(19\sigma_x^2 + 16\sigma_z^2)/(32(5\sigma_x^2 + 2\sigma_z^2)^2),$$
$$\partial\mathcal{L}/\partial\sigma_x^2 = 75\sigma_z^4(8\sigma_z^2 - \sigma_x^2)/(8(5\sigma_x^2 + 2\sigma_z^2)^3).$$

Thus, $\partial\mathcal{L}/\partial\sigma_x^2 < 0$ if and only if $8\sigma_z^2 - \sigma_x^2 < 0$, which is rewritten as $1/\sigma_x^2 = \tau_x < 1/(8\sigma_z^2) = (\tau_y + \tau_\theta)/8$. That is, expected total profits decrease with the precision of private information if and only if $\tau_x < (\tau_y + \tau_\theta)/8$. This result is consistent with Corollary 15.

3 Games that are efficient under complete information

AP classify games according to the type of inefficiency exhibited by the equilibrium and find the following. In the first class of games, where the equilibrium is efficient under both complete and incomplete information, welfare necessarily increases with both public and private information. In the second class of games, where the equilibrium is inefficient only under incomplete information, welfare can decrease with either public or private information, but not with both. In the third class of games, where the equilibrium is inefficient even under complete information, welfare can decrease with both public and private information.

We reconsider the second class of games. A crucial assumption in AP is the existence of socially optimal strategy profiles. Then, a natural question is whether the above result on the second class of games remains true even without this assumption. We give a negative answer to this question. That is, welfare can decrease with both public and private information in the class of games such that the equilibrium is efficient under complete information if there is no socially optimal strategy profile under incomplete information.

To identify this class of games, assume that players directly observe θ. The equilibrium strategy is to choose $\beta\theta/(1-\alpha)$. When each player chooses $x \in \mathbb{R}$, the payoff is

$$-x^2 + 2\alpha x^2 + 2\beta\theta x + \kappa x^2 + \lambda x^2 + \mu\theta x + \nu x = -(1-2\alpha-\kappa-\lambda)x^2 + ((2\beta+\mu)\theta+\nu)x$$

plus $f(\theta)$, which is maximized at $x^*(\theta) = ((2\beta+\mu)\theta+\nu)/(2(1-2\alpha-\kappa-\lambda))$ if $1-2\alpha-\kappa-\lambda > 0$.

Therefore, the equilibrium is efficient under complete information if and only if $x^*(\theta) = \beta\theta/(1-\alpha)$ for all $\theta \in \mathbb{R}$; that is, $\mu = -2\beta(\alpha+\kappa+\lambda)/(1-\alpha)$, $\nu = 0$, and $1-2\alpha-\kappa-\lambda > 0$. Plugging this into ζ and η, we have

$$\zeta = (1-3\alpha-\alpha\kappa-\kappa-2\lambda)/(1-\alpha), \;\; \eta = 1-2\alpha-\kappa-\lambda > 0.$$

Because $\eta > 0$, possible types are +I, +II, +III, and −IV. Only in type +III can welfare decrease with both public and private information. Note that $(1-\alpha)\zeta/\eta > 2$ if and only if $\kappa > 1$ because $(1-\alpha)\zeta - 2\eta = (1-\alpha)(\kappa - 1)$. Thus, this game is type +III if and only if $\kappa > 1$ by Corollary 3. If $\kappa > 1$, then the expected payoff is unbounded above because it equals $(\kappa-1)\mathrm{var}[\sigma_i] + (2\alpha+\lambda)\mathrm{cov}[\sigma_i, \sigma_j] + (2\beta+\mu)\mathrm{cov}[\theta, \sigma_i]$ plus a constant.

To summarize, welfare can decrease with both public and private information in this class of games if there is no socially optimal strategy profile.

4 The finite case vs. the continuum case

Vives [3] was the first to compare the equilibrium and welfare with a finite number of players and those with a continuum of players, restricting attention to Cournot games. He shows that the former converges to the latter as the number of players goes to infinity. In contrast, we consider a finite model possessing the same equilibrium and welfare as those of a given continuum model, which enables us to study the welfare effects of information in the continuum model based upon that of the finite model. In the following, we illustrate the difference between the two approaches in the case of Cournot games.

In a Cournot game with a continuum of players, player i produces a_i units of a homogeneous product with a cost $a_i^2/2$. The inverse demand function is $\theta + \alpha \int a_j dj$, where $\alpha < 0$ is constant and θ is normally distributed. Then, player i's profit is

$$\Big(\theta + \alpha \int a_j dj\Big) a_i - a_i^2/2, \tag{2}$$

which is the same as that discussed in Corollary 14.

Vives [3] compares the above game and a Cournot game with n players, where the inverse demand function is $\theta + \alpha \sum_{j=1}^n a_j/n$ and player i's profit is

$$\Big(\theta + \alpha \sum_{j=1}^{n} a_j/n\Big) a_i - a_i^2/2. \tag{3}$$

Vives [3] shows that the equilibrium strategy in this game converges to that with a continuum of players as n goes to infinity. Note that we obtain (3) by replacing the integral $\int a_j dj$ in (2) with the average $\sum_{j=1}^n a_j/n$.

In contrast, we consider a fictitious game with n players possessing the same equilibrium strategy as that with a continuum of players, where player i's payoff is

$$\Big(\theta + \alpha \sum_{j \neq 1}^{n} a_j/(n-1)\Big) a_i - a_i^2/2. \tag{4}$$

Note that we obtain (4) by replacing $\int a_j dj$ in (2) with $\sum_{j \neq i} a_j/(n-1)$, i.e., the average of the opponents' actions. Both $\int a_j dj$ and $\sum_{j \neq i} a_j/(n-1)$ are independent of a_i, which results in the same equilibrium strategy.

References

[1] Angeletos, G.-M., Pavan, A., 2007. Efficient use of information and social value of information. Econometrica 75, 1103–1142.

[2] Vives, X., 1990. Trade associations, disclosure rules, incentives to share information and welfare. Rand J. Econ. 21, 409–430.

[3] Vives, X., 1988. Aggregation of information in large Cournot markets. Econometrica 56, 851–876.