Abstract

If players have ambiguous beliefs about the state in global games, rational behavior depends on ambiguity not only about the state but also about others’ behavior. The impact of ambiguity is most evident when one of the actions yields a constant payoff, which we call a safe action. Ambiguous information makes more players choose a safe action, whereas low-quality information makes more players choose an ex ante best response to the uniform belief over the opponents’ actions. If the safe action and the ex ante best response coincide, sufficiently ambiguous information generates a unique equilibrium, whereas sufficiently low-quality information generates multiple equilibria. In any case, the two types of information have opposite effects. In applications to financial crises, we demonstrate that news of more ambiguous quality triggers a liquidity crisis, whereas news of less ambiguous quality triggers a currency crisis.

JEL classification numbers: C72, D81, D82.

Keywords: strategic ambiguity; global game; currency crisis; liquidity crisis.

*I thank seminar participants for valuable discussions at Bank of Japan, GRIPS, Hitotsubashi University, Hokkaido University, ISEG, Kobe University, Nanzan University, the University of Tokyo, and the University of Warwick. I acknowledge financial support by MEXT, Grant-in-Aid for Scientific Research.
1 Introduction

Incomplete information games typically assume that players assign common probabilities to all uncertain events. In reality, however, they may have little confidence regarding the true probabilities and their beliefs may be ambiguous. If all players have ambiguous beliefs about the state, they also have ambiguous beliefs about the opponents’ beliefs. Thus, even if players know the opponents’ strategies, which assign an action to each possible private signal, their beliefs about the opponents’ actions are ambiguous. That is, rational behavior depends not only on structural ambiguity (ambiguity about the state) but also on strategic ambiguity (ambiguity about others’ actions).

Strategic ambiguity can have a substantial impact on equilibrium outcomes. To see this, consider the following example analogous to the bank run model of Diamond and Dybvig (1983). Imagine that a large number of players decide whether to invest money in a project. When players invest, they expect to earn a fixed return if the project succeeds but lose the money if it fails. The latter case arises if the state of the economy is bad or if more than one third of the players do not invest. The state is either good or bad with equal probability 1/2. Players receive private signals, G or B, which are conditionally independent given the state over the players. When the state is good, a player receives G with probability \( p \in [1/2, 1] \) and B with probability \( 1 - p \). Symmetrically, when the state is bad, a player receives B with probability \( p \) and G with probability \( 1 - p \). Thus, when players receive G (B), they believe that the state is good (bad) with probability \( p \) and that the opponents receive the same signal with probability \( p^2 + (1 - p)^2 \).

We assume that private signals are ambiguous and that players are ambiguity averse. To be more specific, players do not know \( p \) and make a decision by the most pessimistic beliefs over \( p \in [1/2, 1] \), as assumed in the maxmin expected utility (MEU) model axiomatized by Gilboa and Schmeidler (1989). When a player receives B, he regards it as very reliable and behaves as if \( p = 1 \). Hence, this player does not invest because he believes that the state is bad. When a player receives G, he regards it as unreliable and behaves as if \( p = 1/2 \). Hence, this player does not invest because he believes that half of the opponents receive B (by the law of large numbers). Consequently, ambiguous signals cause all the players not to invest. This effect of ambiguous signals is in clear contrast to that of low-quality signals with \( p = 1/2 \). When private signals are uninformative, players behave solely on
the basis of the prior belief that the good and bad states are equally likely. Thus, there is an equilibrium in which all players invest if the return is sufficiently large.

The purpose of this paper is to present a formal analysis of such an impact of strategic ambiguity using global games of Carlsson and van Damme (1993) (henceforth, CvD) and demonstrate a distinction between ambiguity (where some probabilities are unknown) and risk (where all probabilities are known) under incomplete information. A global game is a binary-action game of incomplete information with strategic complementarities, where players receive noisy private signals about the state. Although multiple equilibria exist under perfect or no information, sufficiently high-quality signals generate a unique equilibrium. In this equilibrium, most players choose a best response to the uniform belief over the opponents’ actions, which Morris and Shin (2002) call a Laplacian action. However, low-quality signals cause some players to choose an ex ante best response to the same uniform belief, which we call an ex ante Laplacian action. Moreover, sufficiently low-quality signals generate multiple equilibria.

We consider a global game with multiple priors, where players have MEU preferences. Players have little confidence regarding the true probability distribution of the state and private signals. Instead, they know that the probability distribution is in a given set of priors and evaluate their actions in terms of the minimum interim expected payoffs, where the minimum is taken over the set of priors. If the set of priors is a singleton, our model is reduced to CvD’s global game.

We propose a tractable procedure to analyze our multiple-priors model. Because players evaluate each action by the most pessimistic beliefs, they eventually behave as if they use different priors to evaluate different actions. Thus, we can analyze our model using a fictitious game with a pair of priors included in the set of priors, where each prior is separately assigned to each action and used to evaluate it. In particular, we show that if each fictitious game has a unique equilibrium, then a multiple-priors game also has a unique equilibrium, where each strategy is a unique strategy surviving iterated deletion of interim-dominated strategies. Our procedure is especially effective if one of the actions yields a constant payoff (regardless of the opponents’ actions and the state), in which case a fictitious game is reduced to a standard single-prior game. We call such an action a safe action, which exists in many applications of global games including currency crises.
(Morris and Shin, 1998) and liquidity crises (Morris and Shin, 2004).

When one of the actions is a safe action, a multiple-priors game has a unique equilibrium if each single-prior game has a unique equilibrium, as stated above. However, even when each single-prior game has multiple equilibria, a multiple-priors game has a unique equilibrium if a safe action is ex-ante Laplacian and quality of private signals is sufficiently ambiguous. In each case, the unique equilibrium coincides with the equilibrium of the single-prior game that maximizes the range of signals to which the safe action is assigned, where the maximum is taken over the set of all equilibria and the set of priors.

The above results imply that information of ambiguous quality sharply contrasts with information of low quality in the following respects. If a safe action is not ex ante Laplacian, ambiguous quality causes more players to choose this action, whereas low quality causes less players to choose it.\(^1\) If a safe action is ex ante Laplacian, sufficiently ambiguous quality generates a unique equilibrium, whereas sufficiently low quality generates multiple equilibria. In each case, ambiguous quality and low quality have opposite effects. In a multiple-priors game, ambiguity-averse players exhibit strong preferences for a safe action. Such preferences together with strategic complementarities make a safe action more survivable in the iterated deletion of interim-dominated strategies. Contrastingly, a safe action does not matter in a single-prior game because the difference of payoffs to the actions completely determines a best response.

As applications, we study the effects of ambiguous information in the models of currency crises (Morris and Shin, 1998) and liquidity crises (Morris and Shin, 2004). Because there is a safe action, MEU preferences can be directly incorporated into these models by means of our procedure. In the model of currency crises, speculators must decide whether to attack the currency, where a currency crisis occurs if sufficiently many speculators attack the currency. In this model, news of less ambiguous quality can trigger a currency crisis because not to attack is a safe action. On the other hand, in the model of liquidity crises, creditors must decide whether to roll over the debt, where a liquidity crisis occurs if a sufficiently many creditors do not roll over the debt. In this model, news of more ambiguous quality can trigger a liquidity crisis because not to roll over the debt is a safe action.

\(^1\)Kawagoe and Ui (2013) conducted a laboratory experiment using a two-player global game and obtained data supporting this opposite effects.
Therefore, the effects of ambiguous information are quite different in the two models of financial crises.

There is a growing literature on incomplete information games with ambiguity-averse players. Most studies have focused on auctions and mechanism design (Salo and Weber, 1995; Lo, 1998; Bose et al., 2006; Turocy, 2008; Bose and Daripa, 2009; Bodoh-Creed, 2012; Lopomo et al., 2009; Di Tillio et al., 2012; Bose and Renou, 2014), where ambiguous information has a special role. For example, Bose and Renou (2014) consider a mechanism with an ambiguous communication device, where a designer can send ambiguous information to individuals with MEU preferences. They demonstrate that ambiguous mediated communication can allow implementation of social choice functions that are not incentive compatible with respect to prior beliefs.

This paper contributes to the literature by identifying the role of ambiguous information in iterated deletion of interim-dominated strategies under ambiguous belief hierarchies. The type space is a special case of the following models. Epstein and Wang (1996) construct a type space consisting of hierarchies of general preferences rather than beliefs, which include MEU preferences. Epstein (1997) and Epstein and Wang (1996) give a formal analysis of iterative deletion of dominated strategies on the basis of general preferences. Iterated deletion in this paper is based upon hierarchies of MEU preferences, whose construction follows Kajii and Ui (2005). They introduce a class of incomplete information games with multiple priors by generalizing Harsanyi (1967–1968), where players have MEU preferences, and propose a couple of equilibrium concepts. A unique equilibrium in this paper satisfies both definitions.

This paper is presented as follows. After giving an illustrative example in Section 2, we introduce a general model in Section 3 and report the main results in Section 4. Section 5 is devoted to applications. We give some concluding remarks in Section 6.

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2Complete information games with ambiguity-averse players are also studied (Dow and Werlang, 1994; Klibanoff, 1996; Lo, 1996, 1999; Eichberger and Kelsey, 2000; Marinacci, 2000), where players are assumed to have ambiguous beliefs about the opponents’ strategies even though there is no ambiguity about the state. This is in clear contrast to the case of incomplete information, where players know the opponents’ strategies but have ambiguous beliefs about the state.

3A few exceptions are Stauber (2011) and Azrieli and Teper (2011).

4See also Ahn (2007), Di Tillio (2008), and Chen (2010).
2 A linear example

As an illustrative example, we consider a linear-normal global game discussed in Morris and Shin (2001). A continuum of players decide whether to invest. The payoff to investing is $\theta + l - 1$, where $\theta \in \mathbb{R}$ is a normally distributed random variable with mean $y \in (0, 1)$ and precision $\eta > 0$ and $l \in [0, 1]$ is the proportion of the opponents investing. The payoff to not investing is a constant 0.

If $0 < \theta < 1$ is common knowledge or if there is no information about $\theta$, two symmetric pure-strategy equilibria exist: all players invest and no players invest. If a player knows $\theta > 1$ then investing is a dominant action; if a player knows $\theta < 0$ then not investing is a dominant action. A Laplacian action at $\theta$ (Morris and Shin, 2002) is defined as a best response to the uniform belief over the opponents’ actions when a player knows $\theta$: it is investing if $\theta \geq 1/2$ and not investing if $\theta \leq 1/2$. An ex ante Laplacian action is defined as an ex ante best response to the same uniform belief: it is investing if $y \geq 1/2$ and not investing if $y \leq 1/2$.

Player $i$ observes a private signal $x_i = \theta + \epsilon_i$, where the noise term $\epsilon_i$ is independently normally distributed with mean 0 and precision $\xi > 0$ (i.e. variance $1/\xi$). We say that information quality is low (high) if $\xi$ is small (large).

Players have a set of interim beliefs and they are ambiguity averse with MEU preferences (Gilboa and Schmeidler, 1989). More specifically, players know $y$ and $\eta$, but do not know $\xi$. Instead, they know that $\xi$ is an element of a closed interval $\Xi \equiv [\underline{\xi}, \overline{\xi}]$. Each player with each private signal has a set of conditional probability distributions indexed by $\xi \in \Xi$ as a set of interim beliefs (Fagin and Halpern, 1990; Jaffray, 1992; Pires, 2002) and evaluates an action in terms of the minimum expected payoff to the action, where the minimum is taken over all $\xi \in \Xi$. When $\Xi$ is a singleton, this game is reduced to a standard single-prior game. We say that information quality is ambiguous when $\Xi$ is not a singleton.

Consider a strategy where a player invests if and only if a private signal $x$ is above a cutoff point $\kappa \in \mathbb{R} \cup \{-\infty, \infty\}$. This strategy is referred to as a switching strategy with

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5This is a stripped down version of the bank run model of Goldstein and Pauzner (2005), which is based upon Diamond and Dybvig (1983).

6Players are assumed to have a common set of priors. See Kajii and Ui (2005, 2009) for an implication of this assumption.
cutoff $\kappa$ and denoted by

$$s(\kappa)(x) \equiv \begin{cases} 
\text{to invest} & \text{if } x > \kappa, \\
\text{not to invest} & \text{if } x \leq \kappa.
\end{cases}$$

It can be readily shown that a best response to a switching strategy is also a switching strategy. To obtain the best response, imagine that a player receives a private signal $x$ and all the opponents follow $s[\kappa]$. When the true precision is $\xi$, the expected payoff to investing is

$$\pi_{\xi}(x, \kappa) \equiv E_{\xi}[\theta|x] + \text{Prob}_{\xi}[x' > \kappa|x] - 1 - \frac{\eta y + \xi x}{\eta + \xi} - \Phi\left(\sqrt{\frac{\xi(\eta + \xi)}{\eta + 2\xi}(\kappa - \frac{\eta y + \xi x}{\eta + \xi})}\right),$$

where $\Phi$ is the cumulative distribution function of the standard normal distribution. Thus, this player prefers investing if and only if

$$\min_{\xi \in \Xi} \pi_{\xi}(x, \kappa) \geq 0.$$ 

Because $\min_{\xi \in \Xi} \pi_{\xi}(x, \kappa)$ is strictly increasing in $x$, we conclude that the best response to $s[\kappa]$ is $s[\beta_{\Xi}(\kappa)]$, where $\beta_{\Xi}(\kappa)$ is a unique value satisfying

$$\min_{\xi \in \Xi} \pi_{\xi}(\beta_{\Xi}(\kappa), \kappa) = 0.$$ 

We write $\beta_{\Xi}(\kappa)$ rather than $\beta_{\{\xi\}}(\kappa)$ when $\Xi$ is a singleton with some abuse of notation.

Using $\beta_{\Xi}$, we study a symmetric equilibrium where all players follow $s[\kappa]$. This equilibrium, referred to as a switching equilibrium with cutoff $\kappa$, exists if and only if $\beta_{\Xi}(\kappa) = \kappa$ (or equivalently, $\min_{\xi \in \Xi} \pi_{\xi}(\kappa, \kappa) = 0$). Moreover, a unique switching equilibrium exists if and only if $\beta_{\Xi}(\kappa) = \kappa$ has a unique solution $\kappa^*$, in which case $\beta_{\Xi}(\kappa) \equiv \kappa$ if and only if $\kappa \leq \kappa^*$. Thus, it follows that $\lim_{n \to \infty} \beta_{\Xi}^{n}(\kappa) = \kappa^*$, where $\beta_{\Xi}^{1}(\kappa) = \kappa$ and $\beta_{\Xi}^{n}(\kappa) = \beta_{\Xi}(\beta_{\Xi}^{n-1}(\kappa))$ for each $n \geq 1$. Using this property, we can iteratively delete interim-dominated strategies, all but $s[\kappa^*]$. 

**Claim 1.** If a switching equilibrium with cutoff $\kappa^*$ is a unique switching equilibrium, $s[\kappa^*]$ is the unique strategy surviving iterated deletion of interim-dominated strategies.

As a benchmark, we consider a single-prior game with $\Xi = \{\xi\}$. Morris and Shin (2001) show that if information quality is sufficiently high, there exists a unique switching equilibrium, where most players choose a Laplacian action. Lower quality, however,
causes some players to choose an ex ante Laplacian action because they put more emphasis on the prior belief about the state. Moreover, sufficiently low quality generates multiple equilibria.

**Claim 2.** Suppose that $\mathcal{X} = \{\xi\}$. If $\xi > q \equiv \eta \left(\eta - 2\pi + (\eta^2 + 12\pi \eta + 4\pi^2)^{1/2}\right)/(8\pi)$, then $s[k(\xi)]$ is the unique strategy surviving iterated deletion of interim-dominated strategies, where $k(\xi)$ is a unique solution of $\beta_\xi(\kappa) = \kappa$ with

$$k(\xi) \equiv \lim_{\xi' \to \infty} k(\xi') = 1/2 \iff y \geq 1/2.$$ 

In the limit as $\xi$ goes to infinity, the cutoff equals 1/2. Thus, a player observing a private signal $x$ invests if $x > 1/2$ and does not invest if $x < 1/2$; that is, he chooses a Laplacian action at $x$. However, lower quality of information decreases the cutoff if $y > 1/2$ and increases the cutoff if $y < 1/2$. Thus, if $y > 1/2$, then a player observing a private signal $x \in (k(\xi), 1/2)$ chooses an ex ante Laplacian action (i.e. investing) rather than a Laplacian action at $x < 1/2$ (see Figure 1a). Similarly, if $y < 1/2$, then a player observing a private signal $x \in (1/2, k(\xi))$ chooses an ex ante Laplacian action (i.e. not investing) rather than a Laplacian action at $x > 1/2$ (see Figure 1b).

On the basis of the above benchmark, we study a multiple-priors game with $\Xi = [\xi, \xi']$. The following properties of $\beta_\Xi$ are essential. Because $\min_{\xi \in \Xi} \pi_\xi(x, \kappa)$ is strictly increasing in $x$, it follows that the best-response cutoff in a multiple-priors game equals the maximum

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**Figure 1:** Graphs of $\pi_\xi(\kappa, \kappa)$ with $\eta = 2$, $\xi = 10^6$ in the solid line, $\xi = 1.1$ in the dashed line, $\xi = 0.56$ in the dash-dot line, and $\xi = 0.37$ in the dotted line. Cutoffs are given by $\kappa$-intercepts.
best-response cutoff in the single-prior games:

\[
\beta_\Xi(\kappa) = \max_{\xi \in \Xi} \beta_\xi(\kappa). \tag{1}
\]

Moreover, the best-response cutoff can be arbitrarily large if not investing is ex-ante Laplacian and information quality is sufficiently low:

\[
\lim_{\xi \to \infty} \beta_\Xi(\kappa) \geq \lim_{\xi \to \infty} \beta_\xi(\kappa) = \infty. \tag{2}
\]

This is because \(\lim_{\xi \to 0} \pi_\xi(x, \kappa) = y - 1/2 < 0\): as the variance of the noise term goes to infinity, the proportion of the opponents investing converges to 1/2 and the conditional expected value of \(\theta\) converges to \(y\), whereby the best response is an ex ante Laplacian action for any private signal.

The property (1) implies that if each single-prior game with \(\{\xi\} \subseteq \Xi\) has a unique switching equilibrium, then the multiple-priors game also has a unique switching equilibrium, whose cutoff equals the maximum cutoff in the single-prior games with \(\{\xi\} \subseteq \Xi\).

**Claim 3.** Suppose that \(\Xi = [\xi, \bar{\xi}]\) and \(\xi > q\). Then, \(s[k^*]\) is the unique strategy surviving iterated deletion of interim-dominated strategies, where \(k^* = \max_{\xi \in \Xi} k(\xi)\).

To see this, recall that \(\beta_\xi(\kappa) \geq \kappa\) if and only if \(\kappa \leq k(\xi)\) for each \(\xi \in \Xi\). Let \(\xi^* \in \arg \max_{\xi \in \Xi} k(\xi)\). If \(\kappa < k^* = k(\xi^*)\), then \(\beta_\Xi(\kappa) \geq \beta_{\xi^*}(\kappa) > \kappa\) by (1). If \(\kappa > k^*\), then \(\beta_\Xi(\kappa) < \kappa\) for all \(\xi \in \Xi\) and thus \(\beta_\Xi(\kappa) < \kappa\) by (1). Hence, the cutoff must be \(k^*\).

If investing is ex ante Laplacian, ambiguous quality of information increases the cutoff by Claim 3, whereas low quality of information decreases the cutoff by Claim 2. That is, if \(\xi_H > \xi_L\) and \(\xi_H \in \Xi\), then

\[
k(\xi_L) < k(\xi_H) \leq \max_{\xi \in \Xi} k(\xi),
\]

as illustrated in Figure 2. Therefore, ambiguous quality and low quality have the opposite effects.

If not investing is ex ante Laplacian, both ambiguous quality and low quality increase the cutoff. Nonetheless, they can have the opposite effects on the number of equilibria. That is, although sufficiently low quality generates multiple equilibria, sufficiently ambiguous quality generates a unique equilibrium. More specifically, given the maximum precision \(\bar{\xi}\), if the minimum precision \(\xi\) is sufficiently small, then a multiple-priors game
Figure 2: Graphs of $\minur_{\xi \in \Xi} \pi_{\xi}(\kappa, \kappa)$ with $y = 0.55$, $\eta = 2$, and $\Xi = \{2\}$ in the dashed line.

with $\Xi = [\xi, \bar{\xi}]$ has a unique switching equilibrium even if each single-prior game with $\{\xi\} \subseteq \Xi$ has multiple switching equilibria (see Figure 3). The cutoff in a multiple-priors game equals the maximum cutoff in the single-prior games with $\{\xi\} \in \Xi$, where the maximum is taken over the set of all switching equilibria and the set of all single-prior games.

**Claim 4.** Suppose that $\Xi = [\xi, \bar{\xi}]$ and $y < 1/2$. For each $\bar{\xi} > 0$, there exists $\delta > 0$ such that, for any $\xi \in (0, \delta)$, $s[\kappa^*]$ is the unique strategy surviving iterated deletion of interim-dominated strategies, where $\kappa^* = \max_{\xi \in \Xi} \max_{\xi \subseteq \Xi} \beta_\xi(\kappa) = \kappa$. Moreover, $\lim_{\xi \to 0} \kappa^* = \infty$.

In the limit as $\xi$ goes to zero, no players invest for the following reasons. First, $s[-\infty]$ is not an equilibrium strategy. When all the opponents follow $s[-\infty]$ and a private signal $x$ is sufficiently small, not investing is a dominant action because $\min_{\xi \in \Xi} \pi_{\xi}(x, -\infty) \leq \pi_{\xi}(x, -\infty) = (\eta y + \bar{\xi}x)/(\eta + \bar{\xi}) < 0$ for $x < -\eta y/\bar{\xi}$. In other words, when ambiguity-averse players receive bad news, they behave as if its precision is highest, whereby they do not invest believing that the state is bad. Next, $s[\kappa]$ with $\kappa \neq \pm \infty$ is not an equilibrium strategy. When all the opponents follow $s[\kappa]$ and a private signal $x$ is greater than $\kappa$, not investing is a best response because $\beta_\Xi(\kappa) \geq \beta_\xi(\kappa) > x$ for sufficiently small $\xi$ by (2). In other words, when ambiguity-averse players receive good news, they behave as if its precision is lowest, whereby they do not invest believing that about half of the opponents do not invest.

By Claim 4, there exists $\Xi_n = [\xi_n, \bar{\xi}_n]$ with $\lim_{n \to \infty} \bar{\xi}_n = 0$ such that, for each $n$, a multiple-priors game with $\Xi_n$ has a unique switching equilibrium with cutoff $\kappa^*_n$. Note that $\lim_{n \to \infty} \kappa^*_n = \infty$ and $\lim_{n \to \infty} \Xi_n = \{0\}$ (with respect to the Hausdorff distance). When
Figure 3: Graphs of $\min_{\xi \in \mathcal{E}} \pi(\kappa, \kappa)$ with $y = 0.4$, $\eta = 2$, $\mathcal{E} = [1/10^4, 1/10]$ in the solid line, and $\mathcal{E} = \{1/20\}$ in the dashed line.

$\mathcal{E} = \{0\}$, however, two symmetric pure-strategy equilibria exist: all players invest and no players invest. This is because $s[-\infty]$ is not dominated when $\mathcal{E} = \{0\}$, whereas $s[-\infty]$ is dominated when $\mathcal{E} \neq \{0\}$. Therefore, the equilibrium correspondence that maps $\mathcal{E}$ to the set of equilibria is not lower hemicontinuous at $\mathcal{E} = \{0\}$.

3 The model

There is a continuum of players. Each player has an action set $\{0, 1\}$ and a payoff function $u : \{0, 1\} \times [0, 1] \times \mathbb{R} \to \mathbb{R}$, where $u(a, l, \theta)$ is a payoff to action $a \in \{0, 1\}$ when proportion $l \in [0, 1]$ of the opponents choose action 1 and the state is $\theta \in \mathbb{R}$. An action $a \in \{0, 1\}$ is called a safe action if $u(a, l, \theta)$ is independent of $(l, \theta)$.

The state $\theta$ is randomly drawn and player $i$ observes a noisy private signal $x_i = \theta + \varepsilon_i$, where $\varepsilon_i$ is an independent noise term. Each player has a set of interim beliefs indexed by a set of parameters $\mathcal{E}$, which is a compact and connected set. A parameter $\xi \in \mathcal{E}$ can be any parameter of the type space besides the precision of private signals. For $\xi \in \mathcal{E}$, let $p_\xi(\theta|x)$ be a probability density function of the state $\theta$ held by a player observing a private signal $x$ and let $q_\xi(\varepsilon)$ be a probability density function of the noise term $\varepsilon$, where the mappings $(\xi, \theta, x) \mapsto p_\xi(\theta|x)$ and $(\xi, \varepsilon) \mapsto q_\xi(\varepsilon)$ are assumed to be continuous. The interim belief $p_\xi(\theta|x)$ is typically the conditional probability density function. In this case, players obtain a set of posteriors by updating each prior in the set of priors. This updating

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\footnote{We consider a continuum of players because many applications of global games assume so. It is straightforward to translate our results for a symmetric two-player case as in Cvd.}
rule of multiple priors is called the full Bayesian updating rule (Fagin and Halpern, 1990; Jaffray, 1992) and axiomatized by Pires (2002).

A global game with multiple priors, or simply a game, is a pair $(u, \Xi)$. A player in $(u, \Xi)$ does not know the true parameter $\xi$ but knows that it is an element of $\Xi$. We assume that each player with each set of interim beliefs indexed by $\Xi$ has MEU preferences axiomatized by Gilboa and Schmeidler (1989). That is, a player evaluates an action in terms of the minimum expected payoff, where the minimum is taken over all $\xi \in \Xi$.

A strategy is a function $\sigma : \mathbb{R} \to \{0, 1\}$, which assigns an action to each possible private signal. A switching strategy with cutoff $\kappa$, denoted by $s[\kappa]$, assigns action 1 to a private signal if and only if it is above a cutoff point $\kappa \in \mathbb{R} \cup \{-\infty, \infty\}$:

$$
s[\kappa](x) = \\
\begin{cases} 
1 & \text{if } x > \kappa, \\
0 & \text{if } x \leq \kappa.
\end{cases}
$$

To give formal definitions of preferences and equilibria, consider a player who observes a private signal $x$ and believes that all the opponents follow a strategy $\sigma$. When the state is $\theta$ and the true parameter is $\xi$, the proportion of the opponents taking action 1 is

$$E_\xi[\sigma|\theta] = \int \sigma(x) q_\xi(x - \theta) dx$$

by the law of large numbers. Thus, when the true parameter is $\xi$, the conditional expected payoff to action $a \in \{0, 1\}$ given a private signal $x$ is

$$E_\xi[u(a, E_\xi[\sigma|\theta], \theta)|x] = \int u(a, E_\xi[\sigma|\theta], \theta) p_\xi(\theta|x) d\theta.$$ 

Then, this player prefers action $a$ to action $a'$ if and only if

$$\min_{\xi \in \Xi} E_\xi[u(a, E_\xi[\sigma|\theta], \theta)|x] \geq \min_{\xi \in \Xi} E_\xi[u(a', E_\xi[\sigma|\theta], \theta)|x]. \quad (3)$$

A strategy profile in which all players follow $\sigma$ is an equilibrium if $\sigma(x) = a$ implies (3) with $a' \neq a$.

A switching equilibrium with cutoff $\kappa$ is an equilibrium where all players follow $s[\kappa]$. Let $\pi^a(x, \kappa) = E_\xi[u(a, E_\xi[s[\kappa]|\theta], \theta)|x]$ denote the expected payoff to action $a \in \{0, 1\}$ when a player observes a private signal $x$, all the opponents follow $s[\kappa]$, and the true parameter is $\xi$. Then, a switching equilibrium with cutoff $\kappa$ exists if and only if

$$\min_{\xi \in \Xi} \pi^1_\xi(x, \kappa) - \min_{\xi \in \Xi} \pi^0_\xi(x, \kappa) \begin{cases} 
\geq 0 & \text{if } x > \kappa, \\
\leq 0 & \text{if } x \leq \kappa.
\end{cases} \quad (4)$$
If the left-hand side of (4) is continuous and increasing in $x$, (4) is equivalent to

$$
\min_{\xi \in \Xi} \pi^1_\xi(x, \kappa) - \min_{\xi \in \Xi} \pi^0_\xi(x, \kappa) = 0.
$$

We consider $(u, \Xi)$ satisfying the following conditions, which guarantee the equivalence of (4) and (5) among others.

**A1 (Action Monotonicity)** $u(1, l, \theta)$ is increasing in $l$. $u(0, l, \theta)$ is decreasing in $l$.

**A2 (State Monotonicity)** $u(1, l, \theta)$ is increasing in $\theta$. $u(0, l, \theta)$ is decreasing in $\theta$.

**A3 (Stochastic Dominance)** If $x > x'$, $p_\xi(\theta|x)$ first-order stochastically dominates $p_\xi(\theta|x')$.

**A4 (Continuity)** The function $(x, \kappa, \xi) \mapsto \pi^0_\xi(x, \kappa)$ is continuous for each $\alpha \in \{0, 1\}$.

**A5 (Limit Dominance)** There exist $\bar{\theta}, \tilde{\theta} \in \mathbb{R}$ satisfying

$$
\begin{align*}
&u(1, 1, \theta) - u(0, 1, \theta) < 0, \lim_{x \to -\infty} \int_{-\infty}^{\theta} p_\xi(\theta|x) d\theta = 1, \\
u(1, 0, \theta) - u(0, 0, \theta) > 0, \lim_{x \to -\infty} \int_{\theta}^{\infty} p_\xi(\theta|x) d\theta = 1.
\end{align*}
$$

Condition A1 states that the incentive to choose action 1 (action 0) is increasing in the proportion of the opponents choosing action 1 (action 0). Condition A2 states that the incentive to choose action 1 (action 0) is increasing (decreasing) in the state. Condition A3 requires that high (low) signals convey good news for action 1 (action 0). If $q_\xi$ satisfies the monotone likelihood ratio property, A3 holds (Milgrom, 1981). Many probability distributions including the normal, the exponential, and the uniform distributions satisfy the monotone likelihood ratio property. Condition A4 is a technical assumption, which is satisfied if payoff functions are continuous and bounded. In some applications of global games, payoff functions are discontinuous or unbounded, but A4 is satisfied. Condition A5 requires that action 1 (action 0) be a dominant strategy for sufficiently high (low) signals.

Note that A1 and A2 are stronger than the standard assumption of global games. In fact, Morris and Shin (2002) assume that the payoff differential $u(1, l, \theta) - u(0, l, \theta)$ is increasing in both $l$ and $\theta$. To study a single-prior game, it is enough to consider $u(1, l, \theta) - u(0, l, \theta)$ because it completely determines best responses. To study a multiple-priors game, however, we must deal with $u(1, l, \theta)$ and $u(0, l, \theta)$ separately because players evaluate an action using a worst-case belief, which may differ depending upon actions. Such a difference gives rise to a distinction between risk and ambiguity in global games.
4 Main results

We establish that there exists a unique strategy surviving iterated deletion of interim-dominated strategies if and only if (5) has a unique solution, which generalizes Claim 1.

Proposition 1. Consider \((u, \Xi)\) satisfying A1, A2, A3, A4, and A5. A switching strategy \(s[\kappa^*]\) is the unique strategy surviving iterated deletion of interim-dominated strategies if and only if there exists a unique value \(\kappa = \kappa^* \in \mathbb{R}\) solving (5).

Proof. See Appendix A. □

Dominance solvability of \((u, \Xi)\) on the basis of hierarchies of MEU preferences (Epstein, 1997; Epstein and Wang, 1996; Kajii and Ui, 2005) is a consequence of strategic complementarities, as in CvD’s argument. In fact, \((u, \Xi)\) exhibits strategic complementarities in terms of MEU preferences, and the unique switching equilibrium is a unique symmetric equilibrium in \((u, \Xi)\), whose existence implies dominance solvability (Milgrom and Roberts, 1990; Vives, 1990).

To study under what conditions (5) has a unique solution and a unique switching equilibrium exists, we consider a fictitious game with a pair of priors indexed by \((\xi_0, \xi_1) \in \Xi \times \Xi\), where players evaluate action 0 using \(\xi_0\) and action 1 using \(\xi_1\). The next proposition, whose special case is Claim 3, states that if each fictitious game has a unique switching equilibrium, then \((u, \Xi)\) also has a unique switching equilibrium. Moreover, the equilibrium cutoff equals the “maxmin” of the equilibrium cutoffs in the fictitious games.

Proposition 2. Consider \((u, \Xi)\) satisfying A1, A2, A3, A4, and A5. Suppose that there exists a unique value \(\kappa = k(\xi_0, \xi_1)\) solving

\[
\pi^1_{\xi_1}(k, k) - \pi^0_{\xi_0}(k, k) = 0
\]

for each \((\xi_0, \xi_1) \in \Xi \times \Xi\) and that \(k : \Xi \times \Xi \rightarrow \mathbb{R}\) is bounded. Then, \(s[\kappa^*]\) is the unique strategy surviving iterated deletion of interim-dominated strategies, where

\[
\kappa^* = \min_{\xi_0 \in \Xi} \max_{\xi_1 \in \Xi} k(\xi_0, \xi_1) = \max_{\xi_1 \in \Xi} \min_{\xi_0 \in \Xi} k(\xi_0, \xi_1).
\]

Proof. See Appendix B. □

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Each fictitious game has a unique switching equilibrium in the following two typical cases as argued by CvD and Morris and Shin (2002). In the first case, the minimum precision of the noise term in a private signal is sufficiently large. Claim 3 in Section 2 corresponds to this case. In the second case, $\theta$ is drawn from an improper uniform distribution over the real line. Applications in Section 5 correspond to this case.

If either $\pi_\xi^0(\kappa, \kappa)$ or $\pi_\xi^1(\kappa, \kappa)$ is independent of $\xi$, a fictitious game is reduced to a single-prior game $(u,\{\xi\})$ because (6) is rewritten as $\pi_\xi^1(\kappa, \kappa) - \pi_\xi^0(\kappa, \kappa) = 0$. In this case, if each single-prior game $(u,\{\xi\})$ with $\xi \in \Xi$ has a unique switching equilibrium, then a multiple-prior game $(u,\Xi)$ also has a unique switching equilibrium, whose cutoff equals the maximum (resp. minimum) equilibrium cutoff in the single-prior games if $\pi_\xi^0(\kappa, \kappa)$ (resp. $\pi_\xi^1(\kappa, \kappa)$) is independent of $\xi$.

For $\pi_\xi^a(\kappa, \kappa)$ to be independent of $\xi$, it suffices that $a \in \{0, 1\}$ is a safe action, as assumed in Section 2. It also suffices that $u(a,l,\theta)$ is independent of $\theta$, and $\theta$ and $\epsilon_i$ are drawn from uniform distributions (or $\theta$ is drawn from an improper uniform distribution), as the next lemma shows. The bank run model studied by Goldstein and Pauzner (2005) satisfies this condition.

**Lemma 1.** For $a \in \{0, 1\}$, suppose that $u(a,l,\theta) = u(a,l,\theta')$ for all $l \in [0,1]$ and $\theta, \theta' \in \mathbb{R}$. If $\theta$ is drawn from an improper uniform distribution over the entire real line, $\pi_\xi^a(\kappa, \kappa)$ is independent of $\xi$. If both $\theta$ and $\epsilon_i$ are drawn from uniform distributions, $\pi_\xi^a(\kappa, \kappa)$ is independent of $\xi$ unless $\kappa$ is too small or too large.

**Proof.** See Appendix C. \qed

In general, a multiple-priors game can have a unique switching equilibrium even when some fictitious single-prior game does not have a unique switching equilibrium. We consider such a case on the basis of the next proposition.

**Proposition 3.** Consider $(u,\Xi)$ satisfying A1, A2, A3, A4, and A5. Suppose that $\pi_\xi^a(\kappa, \kappa) = c(\kappa)$ is independent of $\xi$ for $a \in \{0, 1\}$. Then, $s[\kappa^*]$ is the unique strategy surviving iterated deletion of interim-dominated strategies if and only if the following condition holds.

- If $a = 0$, $\min_{\xi \in \Xi} \pi_\xi^1(\kappa, \kappa) < c(\kappa)$ for all $\kappa < \kappa^* = \max_{\xi \in \Xi} \max\{\kappa : \pi_\xi^1(\kappa, \kappa) = c(\kappa)\}$.
- If $a = 1$, $\min_{\xi \in \Xi} \pi_\xi^0(\kappa, \kappa) < c(\kappa)$ for all $\kappa > \kappa^* = \min_{\xi \in \Xi} \min\{\kappa : \pi_\xi^0(\kappa, \kappa) = c(\kappa)\}$.
Proof. See Appendix D.

If $\pi^0_\xi(\kappa, \kappa) = c(\kappa)$ (resp. $\pi^1_\xi(\kappa, \kappa) = c(\kappa)$) is independent of $\xi$, then the unique equilibrium cutoff equals the maximum (resp. minimum) equilibrium cutoff in the single-prior games, where the maximum (resp. minimum) is taken over the set of all switching equilibria in a single-prior game and the set of all single-prior games with $\{\xi\} \subseteq \Xi$. When each single-prior game has a unique switching equilibrium, Proposition 3 is a special case of Proposition 2.

Using Proposition 3, we demonstrate that a unique switching equilibrium can exist even if each single-prior game has multiple equilibria. To this end, let $\xi \in \Xi \equiv [\xi, \bar{\xi}] \subset \mathbb{R}_+$ be the precision of the noise term in a private signal. That is, the probability density function of the noise term is $q_\xi(\varepsilon_i) = \sqrt{\xi} q(\sqrt{\xi} \varepsilon_i)$, where $q$ is a continuous probability density function with a support $\mathbb{R}$ such that $\int q(\varepsilon) d\varepsilon = 0$, $\int \varepsilon^2 q(\varepsilon) d\varepsilon = 1$, and $q(\varepsilon) = q(-\varepsilon)$.

The interim belief $p_\xi(\theta|x)$ is a conditional probability density function:

$$p_\xi(\theta|x) = \frac{p(\theta) q_\xi(\theta - x)}{\int p(\theta) q_\xi(\theta - x) d\theta},$$

where $p(\theta)$ is a continuous probability density function of $\theta$. The next proposition, whose special case is Claim 4, shows that sufficiently ambiguous information can generate a unique switching equilibrium if one of the actions is safe and ex ante Laplacian.

**Proposition 4.** Let $\xi > 0$ be the precision of private signals. Assume the following conditions.

1. $(u, \{\xi\})$ satisfies A1, A2, A3, A4, and A5 for each $\xi > 0$.

2. $a \in \{0, 1\}$ is a safe and ex ante Laplacian action; that is, $u(a, 1, \theta) = c \in \mathbb{R}$ is constant and $c > \int_{-\infty}^{\infty} u(a', 1/2, \theta) p(\theta) d\theta$ for $a' \neq a$.

3. There exists $\bar{\kappa} \in \mathbb{R}$ such that each $(u, \{\xi\})$ has at most one switching equilibrium with cutoff $\kappa > \bar{\kappa}$ and it holds that $d\pi^a_\xi(\kappa, \kappa)/d\kappa \neq 0$.

Then, for each $\xi > 0$, there exists $\delta > 0$ such that, for each $\xi \in (0, \delta]$, $s[\kappa^*]$ is the unique strategy surviving iterated deletion of interim-dominated strategies in $(u, \bar{\xi})$ with $\Xi = [\xi, \bar{\xi}]$, where $\kappa^*$ is given in Proposition 3. Moreover, $\lim_{\xi \to 0} \kappa^* = \infty$ if $a = 0$ and $\lim_{\xi \to 0} \kappa^* = -\infty$ if $a = 1$.
Proof. See Appendix E. \hfill \Box

Given the maximum precision $\xi$, if the minimum precision $\xi$ is sufficiently low, then a multiple-priors game $(u, \Xi)$ with $\Xi = [\xi, \xi]$ has a unique switching equilibrium, whose cutoff is given by Proposition 3. Moreover, in the limit as $\xi \to 0$, all players choose a safe action. To see the intuition in the limit case, assume that action 0 is a safe action. When a private signal $x$ is sufficiently small, the worst case for action 1 is the highest precision $\xi = \xi$, in which case action 0 is a dominant action by A3 and A5. This implies that $s[-\infty]$ is not an equilibrium strategy. When all the opponents follow $s[\kappa]$ with $\kappa \neq \pm \infty$ and a private signal $x$ is greater than $\kappa$, the worst case for action 1 is the lowest precision $\xi = \xi$, in which case about half of the opponents choose action 0 and the best response is also action 0, an ex ante Laplacian action. This implies that $s[\kappa]$ is not an equilibrium strategy. Therefore, all players choose action 0.

5 Applications

MEU preferences can be directly incorporated into many applications of global games by Proposition 2. This section discusses applications to currency crises and liquidity crises.\footnote{Financial crises under ambiguity are also studied by Caballero and Krishnamurthy (2008) and Uhlig (2010). Caballero and Krishnamurthy (2008) focus on a flight to quality and Uhlig (2010) focuses on systemic bank runs. For more applications of MEU preferences to financial markets, see surveys by Epstein and Schneider (2010), Guidolin and Rinaldi (2013), and Gilboa and Marinacci (2013).}

Mathematically, the models of a currency crisis and a liquidity crisis are very close. However, the economic implications of strategic ambiguity are quite different. That is, more ambiguous information can trigger a liquidity crisis, whereas less ambiguous information can trigger a currency crisis.

5.1 Currency crises

We study the model of self-fulfilling currency attacks introduced by Morris and Shin (1998). The following formulation conforms to that of Corsetti et al. (2004), by which we can obtain a unique equilibrium in a closed form.
A continuum of players, who are speculators, must decide whether to attack the currency by selling it short. The current value of the currency is 1. If an attack is successful, the currency collapses and floats to the shadow rate 0. There is a fixed transaction cost $t \in (0,1)$ of attacking. Thus, the net payoff to a successful attack is $1 - t$, while that to an unsuccessful attack is $-t$. An attack is successful if and only if the proportion of players attaching the currency is greater than $\theta$. Writing 1 for the action not to attack and 0 for the action to attack, we have the following payoff function:

$$u(a,l,\theta) = \begin{cases} 0 & \text{if } a = 1, \\ 1 - t & \text{if } a = 0 \text{ and } 1 - l \geq \theta, \\ -t & \text{if } a = 0 \text{ and } 1 - l < \theta, \end{cases}$$

where $l$ is the proportion of the opponents not attacking the currency. If $0 < \theta < 1$ is common knowledge, there are two symmetric equilibria: all to attack and all not to attack.

Assume that $\theta$ is drawn from an improper uniform distribution and that $\varepsilon_i$ is drawn from a normal distribution with mean zero and precision $\xi$. Then, $p_\xi(\theta|x) = \sqrt{\xi} \phi(\sqrt{\xi}(x - \theta))$ and $q_\xi(\varepsilon) = \sqrt{\xi} \phi(\sqrt{\xi}\varepsilon)$, where $\phi$ is the probability density function of the standard normal distribution. Thus, when the state is $\theta$ and each player follows $s[k]$, the currency collapses if and only if $\Phi(\sqrt{\xi}(k - \theta)) \geq \theta$.


**Proposition 5.** In $(u,\{\xi\})$, $s[k(\xi)]$ is the unique strategy surviving iterated deletion of interim-dominated strategies, where $k(\xi) \equiv 1 - t + \xi^{-1/2}\Phi^{-1}(1-t)$. The currency collapses if and only if $\theta \leq 1 - t$.

The equilibrium cutoff $k(\xi)$ is increasing in $\xi$ if and only if $t > 1/2$, but the critical state $1 - t$ is independent of $\xi$. Thus, low-quality information decreases or increases the cutoff if $t > 1/2$ or $t < 1/2$, respectively, but it has no influence on whether the currency collapses.

The next corollary shows that the equilibrium cutoff in a multiple-priors game is the minimum of the equilibrium cutoffs in the single-prior games, which is a straightforward consequence of Propositions 2 and 5.
Corollary 6. In \((u, \Xi)\) with \(\Xi = [\xi, \bar{\xi}]\), \(s(\kappa^*(\Xi))\) is the unique strategy surviving iterated deletion of interim-dominated strategies, where

\[
\kappa^*(\Xi) = \min_{\xi \in \Xi} k(\xi) = \begin{cases} 
1 - t + \Phi^{-1}(1 - t)/\xi^{1/2} & \text{if } t \leq 1/2, \\
1 - t + \Phi^{-1}(1 - t)/\Xi^{1/2} & \text{if } t > 1/2.
\end{cases}
\]  

When the true precision is \(\xi \in \Xi\), the currency collapses if and only if \(\theta \leq \theta^*(\Xi)\), where \(\theta^*(\Xi)\) is the unique value of \(\theta\) solving

\[
\Phi(\sqrt{\xi}(\kappa^*(\Xi) - \theta)) = \theta.
\]

Moreover, \(\Xi \supset \Xi'\) implies \(\theta^*(\Xi) \leq \theta^*(\Xi') \leq \theta^*(\{\xi\}) = 1 - t\).

By Corollary 6, we can partition the space of fundamentals into three intervals when the true precision is \(\xi \in \Xi\).

- If \(\theta > 1 - t\), fundamentals are so strong that the currency does not collapse in \((u, \Xi)\) for any \(\Xi\).
- If \(\inf_{\Xi, \xi \in \Xi} \theta^*(\Xi) < \theta \leq 1 - t\), the currency collapses in \((u, \Xi)\) with \(\theta \leq \theta^*(\Xi)\), but it does not collapse in \((u, \Xi)\) for \(\Xi\) with \(\theta > \theta^*(\Xi)\). Thus, less ambiguous information can trigger a currency crisis because \(\theta^*(\Xi)\) is larger under less ambiguous information with smaller \(\Xi\).

- If \(\theta \leq \inf_{\Xi, \xi \in \Xi} \theta^*(\Xi)\), fundamentals are so weak that the currency collapses in \((u, \Xi)\) for any \(\Xi\).

By (8) and (9), if \(t \in (0, 1/2]\) then \(\inf_{\Xi, \xi \in \Xi} \theta^*(\Xi)\) is the unique value of \(\theta\) solving

\[
\Phi(\sqrt{\xi}(1 - t - \theta)) = \theta, \text{ and if } t \in (1/2, 1) \text{ then } \inf_{\Xi, \xi \in \Xi} \theta^*(\Xi) = 0.
\]

5.2 Liquidity crises

We study the model of creditor coordination introduced by Morris and Shin (2004). A continuum of players, who are creditors, hold collateralized debt and decide whether to roll over the debt. A player who rolls over the debt receives 1 if an underlying investment project succeeds, and receives 0 if the project fails. A player who does not roll over the debt receives the value of the collateral \(\lambda \in (0, 1)\). The project succeeds if and only if
the proportion of players not rolling over the debt is less than \( \theta \in \mathbb{R} \); that is, sufficient liquidity is available. A project failure due to a lack of liquidity is referred to as a liquidity crisis. Writing 1 for the action to roll over and 0 for the action not to roll over, we have the following payoff function:

\[
u(a, l, \theta) = \begin{cases} 
\lambda & \text{if } a = 0, \\
1 & \text{if } a = 1 \text{ and } 1 - l < \theta, \\
0 & \text{if } a = 1 \text{ and } 1 - l \geq \theta,
\end{cases}
\]

where \( l \) is the proportion of players rolling over the debt. If \( 0 < \theta < 1 \) is common knowledge, there are two symmetric equilibria: all to roll over and all not to roll over.

The information structure is the same as that in Section 5.1. Thus, when the state is \( \theta \) and each player follows \( s[\kappa] \), the project fails if and only if \( \Phi(\sqrt{\xi}(\kappa - \theta)) \geq \theta \).

Morris and Shin (2004) obtain the equilibrium cutoff in a single-prior game.

**Proposition 7.** In \((u, \{\xi\})\), \( s[k(\xi)] \) is the unique strategy surviving iterated deletion of interim-dominated strategies, where \( k(\xi) \equiv \lambda + \xi^{-1/2}\Phi^{-1}(\lambda) \). The project fails if and only if \( \theta \leq \lambda \).

Propositions 5 and 7 illustrate a close connection between the model of creditor coordination and that of self-fulfilling currency attacks. That is, the equilibria in single-prior games are the same when \( \lambda = 1 - t \) because the payoff differentials are the same. However, the equilibria in multiple-priors games are different because safe actions are different. In the model of creditor coordination, the equilibrium cutoff in a multiple-priors game is the maximum of the equilibrium cutoffs in the single-prior games. The next corollary is a straightforward consequence of Propositions 2 and 7.

**Corollary 8.** In \((u, \Xi)\) with \( \Xi = [\underline{\xi}, \overline{\xi}] \), \( s[k^*(\Xi)] \) is the unique strategy surviving iterated deletion of interim-dominated strategies, where

\[
k^*(\Xi) = \max_{\xi \in \Xi} k(\xi) = \begin{cases} 
\lambda + \Phi^{-1}(\lambda)/\overline{\xi}^{1/2} & \text{if } \lambda \leq 1/2, \\
\lambda + \Phi^{-1}(\lambda)/\underline{\xi}^{1/2} & \text{if } \lambda > 1/2.
\end{cases}
\]
When the true precision is \( \xi \in \Xi \), the project fails if and only if \( \theta \leq \theta^*(\Xi) \), where \( \theta^*(\Xi) \) is the unique value of \( \theta \) solving

\[
\Phi(\sqrt{\xi}(\kappa^*(\Xi) - \theta)) = \theta.
\]  

(11)

Moreover, \( \Xi \supset \Xi' \) implies \( \theta^*(\Xi) \geq \theta^*(\Xi') \geq \theta^*(\{\xi\}) = \lambda \).

By Corollary 8, we can partition the space of fundamentals into three intervals when the true precision is \( \xi \in \Xi \).

- If \( \theta < \lambda \), fundamentals are so weak that the project fails in \((u, \Xi)\) for any \( \Xi \).

- If \( \lambda \leq \theta < \sup_{\Xi, \xi \in \Xi} \theta^*(\Xi) \), the project succeeds in \((u, \Xi)\) for \( \Xi \) with \( \theta > \theta^*(\Xi) \), but it fails in \((u, \Xi)\) for \( \Xi \) with \( \theta \leq \theta^*(\Xi) \). Thus, more ambiguous information can trigger a liquidity crisis because \( \theta^*(\Xi) \) is larger when information is more ambiguous with larger \( \Xi \).

- If \( \theta \geq \sup_{\Xi, \xi \in \Xi} \theta^*(\Xi) \), fundamentals are so strong that the project succeeds in \((u, \Xi)\) for any \( \Xi \).

By (10) and (11), if \( t \in (0, 1/2] \) then \( \sup_{\Xi, \xi \in \Xi} \theta^*(\Xi) \) is the unique value of \( \theta \) solving

\[
\Phi(\sqrt{\xi}(\lambda - \theta)) = \theta, \text{ and if } \lambda \in (1/2, 1) \text{ then } \sup_{\Xi, \xi \in \Xi} \theta^*(\Xi) = 1.
\]

6 Concluding remarks

This paper analyzes global games with multiple priors by iterative deletion of interim-dominated strategies on the basis of a hierarchy of MEU preferences. This hierarchy is a special case of a hierarchy of general preferences introduced by Epstein and Wang (1996). Their paper is “motivated in part by the presumption that uncertainty or vagueness are important in strategic situations and therefore that it is worthwhile generalizing received game theory so that the effects of uncertainty can be studied in principle” (Epstein and Wang, 1996, p. 1134).

We find that ambiguity (i.e. uncertainty) contrasts sharply with risk in global games, thus confirming the above presumption. First, ambiguous information and low-quality information can have the opposite effects. If a safe action is not ex ante Laplacian, ambiguous
information causes more players to choose this action, whereas low-quality information causes less players to choose it. If a safe action is ex ante Laplacian, sufficiently ambiguous information generates a unique rationalizable strategy, whereas sufficiently low-quality information generates multiple equilibria. Next, ambiguous information plays different roles in different types of financial crises. More ambiguous information can trigger a liquidity crisis, whereas less ambiguous information can trigger a currency crisis. On the other hand, low-quality information does not always have such an effect.

These results are explained by the fact that ambiguity-averse players exhibit strong preferences for a safe action. Such preferences together with strategic complementarities make a safe action more survivable in the iterated deletion of interim-dominated strategies. In fact, we can obtain similar results by adopting other models of ambiguity aversion such as the smooth model axiomatized by Klibanoff et al. (2005).

In the smooth model, players have not only a possible set of “first order” probability distributions indexed by $\xi \in \Xi$ but also a “second order” probability distribution $\mu$ over $\Xi$. When the opponents follow $s(\kappa)$, a player with a private signal $x$ prefers action $a$ to action $a'$ if and only if

$$\int \Xi \psi \left( \pi^a_{\xi}(x, \kappa) \right) d\mu(\xi) \geq \int \Xi \psi \left( \pi^{a'}_{\xi}(x, \kappa) \right) d\mu(\xi),$$

where $\psi$ is an increasing and concave function. More specifically, assume that $\mu$ is a uniform distribution over $\Xi$ and $\psi(x) = -\frac{1}{\alpha} e^{-\alpha x}$ with $\alpha > 0$. The function $\psi$ is said to display constant ambiguity aversion and $\alpha$ is called the coefficient of ambiguity aversion. Note that $\mu$ and $\psi$ represent ambiguity perception and ambiguity attitude, respectively.

As shown by Klibanoff et al. (2005), this preference relation converges to that in the MEU model as the coefficient of ambiguity aversion goes to infinity. This implies that iterated deletion of interim-dominated strategies works similarly if the coefficient of ambiguity aversion is large enough, whereby a safe action is also more survivable in the case of the smooth model. Because low-quality or high-quality information is defined by a first-order probability distribution and ambiguous information is defined by a second-order probability distribution, the above discussion suggests that first-order and second-order probability distributions can have the opposite effects in global games.
Appendix

A Proof of Proposition 1

We use the following three lemmas.

Lemma A. Regard the set of strategies as partially ordered in the sense that \( \sigma \geq \sigma' \) if and only if \( \sigma(x) \geq \sigma'(x) \) for all \( x \in \mathbb{R} \). A game \((u, \Xi)\) satisfies strategic complementarities in the following sense: if \( \sigma \geq \sigma' \) then, for all \( x \in \mathbb{R} \)

\[
\min_{\xi \in \Xi} E_\xi[u(1, E_\xi[\sigma|\theta], \theta)|x] - \min_{\xi \in \Xi} E_\xi[u(0, E_\xi[\sigma|\theta], \theta)|x] \geq \min_{\xi \in \Xi} E_\xi[u(1, E_\xi[\sigma'|\theta], \theta)|x] - \min_{\xi \in \Xi} E_\xi[u(0, E_\xi[\sigma'|\theta], \theta)|x].
\]

Proof. Because \( E_\xi[\sigma|\theta] \geq E_\xi[\sigma'|\theta] \), it holds that \( u(1, E_\xi[\sigma|\theta], \theta) \geq u(1, E_\xi[\sigma'|\theta], \theta) \) and \( u(0, E_\xi[\sigma|\theta], \theta) \leq u(0, E_\xi[\sigma'|\theta], \theta) \) by A1, which implies the above. \( \square \)

Lemma B. The function \((x, \kappa) \mapsto \min_{\xi \in \Xi} \pi^1_\xi(x, \kappa) - \min_{\xi \in \Xi} \pi^0_\xi(x, \kappa)\) is continuous, increasing in \( x \), and decreasing in \( \kappa \).

Proof. This function is continuous by A4, increasing in \( x \) by A1, A2, and A3, and decreasing in \( \kappa \) by A1. \( \square \)

Lemma C. There exist \( x, \bar{x} \in \mathbb{R} \) such that \( \min_{\xi \in \Xi} \pi^1_\xi(x, \kappa) - \min_{\xi \in \Xi} \pi^0_\xi(x, \kappa) < 0 \) for all \( x \leq x \) and \( \kappa \in \mathbb{R} \) and \( \min_{\xi \in \Xi} \pi^1_\xi(x, \kappa) - \min_{\xi \in \Xi} \pi^0_\xi(x, \kappa) > 0 \) for all \( x \geq \bar{x} \) and \( \kappa \in \mathbb{R} \).

Proof. We prove the existence of \( x \). We can prove the existence of \( \bar{x} \) similarly. For \( \xi^* \in \arg \min_{\xi \in \Xi} \pi^0_\xi(x, \kappa) \),

\[
\min_{\xi \in \Xi} \pi^1_\xi(x, \kappa) - \min_{\xi \in \Xi} \pi^0_\xi(x, \kappa) \leq \pi^1_{\xi^*}(x, \kappa) - \pi^0_{\xi^*}(x, \kappa) \leq \max_{\xi \in \Xi} E_\xi[u(1, 1, \theta) - u(0, 1, \theta)|x]
\]

by A1. Thus, it is enough to show that

\[
\lim_{x \to -\infty} \max_{\xi \in \Xi} E_\xi[u(1, 1, \theta) - u(0, 1, \theta)|x] < 0,
\]

which is true if

\[
\lim_{x \to -\infty} E_\xi[u(1, 1, \theta) - u(0, 1, \theta)|x] < 0 \tag{A1}
\]

for each \( \xi \) by Dini’s theorem because \( E_\xi[u(1, 1, \theta) - u(0, 1, \theta)|x] \) is increasing in \( x \) by A2 and A3 and \( \xi \mapsto E_\xi[u(1, 1, \theta) - u(0, 1, \theta)] \) is continuous on a compact set \( \Xi \).
Let $\varepsilon = u(0, 1, \theta) - u(1, 1, \theta) > 0$, where $\theta \in \mathbb{R}$ is given in A5. Note that $u(1, 1, \theta) - u(0, 1, \theta) \leq -\varepsilon$ for all $\theta \leq \underline{\theta}$ by A2 and

$$E_\varepsilon[u(1, 1, \theta) - u(0, 1, \theta)|x] \leq -\varepsilon \int_{-\infty}^{\theta} p_\varepsilon(\theta|x)d\theta + \int_{\theta}^{\infty} (u(1, 1, \theta) - u(0, 1, \theta))p_\varepsilon(\theta|x)d\theta.$$ 

Thus, (A1) holds for the following reason. First, by A5,

$$\lim_{x \to -\infty} \left(-\varepsilon \int_{-\infty}^{\theta} p_\varepsilon(\theta|x)d\theta\right) = -\varepsilon.$$  

(A2)

Next, for arbitrary $x' \in \mathbb{R}$, there exists $\hat{\theta} > \underline{\theta}$ such that, for all $x < x'$,

$$\int_{\hat{\theta}}^{\infty} (u(1, 1, \theta) - u(0, 1, \theta))p_\varepsilon(\theta|x)d\theta < \varepsilon/2$$

by A2 and A3. Because

$$\lim_{x \to -\infty} \int_{\hat{\theta}}^{\theta} (u(1, 1, \theta) - u(0, 1, \theta))p_\varepsilon(\theta|x)d\theta = 0,$$

we have

$$\lim_{x \to -\infty} \int_{\hat{\theta}}^{\infty} (u(1, 1, \theta) - u(0, 1, \theta))p_\varepsilon(\theta|x)d\theta < -\varepsilon/2.$$

(A3)

Therefore, (A1) holds by (A2) and (A3).

The above three lemmas together with CvD’s discussion imply Proposition 1. We give a proof for completeness.

**Proof of Proposition 1.** By Lemma B, if there exists a unique switching equilibrium, its cutoff is the unique solution of (5), which implies the “only if” part.

To prove the “if” part, let $\Sigma_n$ be the set of strategies satisfying

$$\sigma(x) = \begin{cases} 0 & \text{if } x < \kappa_n, \\ 1 & \text{if } x > \overline{\kappa}_n, \end{cases}$$

where $\kappa_0 = -\infty$ and $\overline{\kappa}_0 = \infty$, and $\kappa_n$ and $\overline{\kappa}_n$ are defined inductively by

$$\kappa_{n+1} = \min\{x \in \mathbb{R} : \min_{\xi \in \Xi} \pi^1_\xi(x, \kappa_n) - \min_{\xi \in \Xi} \pi^0_\xi(x, \kappa_n) = 0\},$$

$$\overline{\kappa}_{n+1} = \max\{x \in \mathbb{R} : \min_{\xi \in \Xi} \pi^1_\xi(x, \overline{\kappa}_n) - \min_{\xi \in \Xi} \pi^0_\xi(x, \overline{\kappa}_n) = 0\}.$$ 

Note that $\Sigma_0$ is the set of all strategies. Because $\min_{\xi \in \Xi} \pi^1_\xi(x, \kappa_n) - \min_{\xi \in \Xi} \pi^0_\xi(x, \kappa_n)$ is continuous and increasing in $x$ by Lemma B, $\kappa_{n+1}$ is the lowest signal where action 1 is a best response to $s[\kappa_n]$; $\overline{\kappa}_{n+1}$ is the highest signal where action 0 is a best response to $s[\overline{\kappa}_n]$. 

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We show by induction that $\Sigma_n$ is the set of strategies surviving $n$ rounds of iterated deletion of interim-dominated strategies. As an induction hypothesis, suppose that $\Sigma_n$ is the set of strategies surviving $n$ rounds of iterated deletion for $n \geq 0$. By strategic complementarities shown in Lemma A, when action 1 is a player’s best response to $\sigma \in \Sigma_n$, it is also a best response to $s[\kappa_n]$, which is the maximum strategy in $\Sigma_n$, and thus his private signal must be greater than $\kappa_{n+1}$. In other words, when a player receives a signal below $\kappa_{n+1}$, his best response to $\sigma \in \Sigma_n$ cannot be action 1. Similarly, when action 0 is a player’s best response to $\sigma \in \Sigma_n$, it is also a best response to $s[\bar{\kappa}_n]$, which is the minimum strategy in $\Sigma_n$, and thus his private signal must be less than $\bar{\kappa}_{n+1}$. In other words, when a player receives a signal above $\bar{\kappa}_{n+1}$, his best response to $\sigma \in \Sigma_n$ cannot be action 0. Therefore, $\Sigma_{n+1}$ is the set of strategies surviving $n+1$ rounds of iterated deletion.

Note that $\kappa_n$ is increasing in $n$ and $\bar{\kappa}_n$ is decreasing in $n$ because $\kappa_0 = -\infty < \bar{\kappa} < 1$, $\bar{\kappa}_0 = \infty > \bar{\kappa} > 1$, and $\min_{\xi \in \Xi} \pi_1^1(\kappa, \kappa) - \min_{\xi \in \Xi} \pi_0^0(\kappa, \kappa)$ is increasing in $\kappa$ and decreasing in $\kappa$ by Lemma B. Thus, there exist $\kappa = \lim_{n \to \infty} \kappa_n$ and $\bar{\kappa} = \lim_{n \to \infty} \bar{\kappa}_n$. By continuity, we must have

$$\min_{\xi \in \Xi} \pi_1^1(\kappa, \kappa) - \min_{\xi \in \Xi} \pi_0^0(\kappa, \kappa) = \min_{\xi \in \Xi} \pi_1^1(\bar{\kappa}, \bar{\kappa}) - \min_{\xi \in \Xi} \pi_0^0(\bar{\kappa}, \kappa) = 0,$$

which implies that $\kappa = \bar{\kappa} = \kappa^*$ by the assumption. Therefore, $s[\kappa^*]$ is the unique strategy surviving iterated deletion of interim-dominated strategies. □

**B Proof of Proposition 2**

We use the following lemma.

**Lemma D.** Let $f : \mathbb{R} \times \Xi \to \mathbb{R}$ be a continuous function. Assume the following.

- For each $\xi \in \Xi$, an equation $f(x, \xi) = 0$ has a unique solution $x^*(\xi)$ and $x^*(\xi)$ is bounded over $\Xi$.

- $f(x, \xi) < 0$ if and only if $x < x^*(\xi)$.

Then, $\max_{\xi \in \Xi} x^*(\xi)$ and $\min_{\xi \in \Xi} x^*(\xi)$ exist and they are the unique solutions of $\min_{\xi \in \Xi} f(x, \xi) = 0$ and $\max_{\xi \in \Xi} f(x, \xi) = 0$, respectively.
Proof. To prove the existence, it is enough to show that \( x^*(\xi) \) is continuous in \( \xi \) because \( \Xi \) is compact. Seeking a contradiction, suppose that \( x^*(\xi) \) is not continuous at \( \bar{\xi} \). Then, there exists a sequence \( \{\xi_k\}_{k=1}^{\infty} \) such that \( \lim_{k \to \infty} \xi_k = \bar{\xi} \) and \( \lim_{k \to \infty} x^*(\xi_k) = \bar{x} \neq x^*(\bar{\xi}) \). Then, \( \lim_{k \to \infty} f(x^*(\xi_k), \xi_k) = f(\bar{x}, \bar{\xi}) = 0 \) by the continuity of \( f \), implying that \( \bar{x} = x^*(\bar{\xi}) \), a contradiction.

We show that \( \max_{\xi \in \Xi} x^*(\xi) \) is the unique solution of \( \min_{\xi \in \Xi} f(x, \xi) = 0 \). Note that \( f(x, \xi) > 0 \) for all \( x > \max_{\xi \in \Xi} x^*(\xi') \) and \( \xi \in \Xi \), which implies that \( \min_{\xi \in \Xi} f(x, \xi) > 0 \) for all \( x > \max_{\xi \in \Xi} x^*(\xi) \) because \( f(x, \xi) \) is continuous in \( \xi \) and \( \Xi \) is compact. Note that \( f(x, \xi) < 0 \) for all \( x < \max_{\xi \in \Xi} x^*(\xi') \) and \( \xi \in \arg \max_{\xi \in \Xi} x^*(\xi') \), which implies that \( \min_{\xi \in \Xi} f(x, \xi) < 0 \) for all \( x < \max_{\xi \in \Xi} x^*(\xi') \). Hence, \( \max_{\xi \in \Xi} x^*(\xi) \) is the unique solution of \( \min_{\xi \in \Xi} f(x, \xi) = 0 \) because \( \min_{\xi \in \Xi} f(x, \xi) \) is continuous in \( x \).

We can show the case of \( \min_{\xi \in \Xi} x^*(\xi) \) similarly. \( \square \)

We are ready to prove Proposition 2.

Proof of Proposition 2. By Proposition 1, it suffices to prove that (7) is the unique value of \( \kappa \in \mathbb{R} \) solving (5). We write \( f(\kappa, \xi_0, \xi_1) = \pi^1_{\xi_1}(\kappa, \kappa) - \pi^0_{\xi_0}(\kappa, \kappa) \).

We show that \( f(\kappa, \xi_0, \xi_1) < 0 \) if \( \kappa < k(\xi_0, \xi_1) \) for all \( \xi_0, \xi_1 \in \Xi \). Seeking a contradiction, suppose that there exist \( \xi'_0, \xi'_1 \in \Xi \) and \( \kappa < k(\xi'_0, \xi'_1) \) such that \( f(\kappa, \xi'_0, \xi'_1) \geq 0 \). Then, it follows that \( f(\kappa, \xi'_0, \xi'_1) > 0 \) for all \( \kappa < k(\xi'_0, \xi'_1) \) because \( k(\xi'_0, \xi'_1) \) is a unique solution of \( f(\kappa, \xi'_0, \xi'_1) = 0 \). For \( \hat{k} < \min\{\Xi, \min_{\xi_0, \xi_1} k(\xi_0, \xi_1)\} \), there exist \( \xi'_0, \xi'_1 \in \Xi \) such that \( f(\hat{k}, \xi'_0, \xi'_1) < 0 \) by Lemma C. Thus, there exist \( \xi''_0, \xi''_1 \in \Xi \) such that \( f(\hat{k}, \xi''_0, \xi''_1) = 0 \) by the intermediate value theorem, which contradicts \( \hat{k} < \min_{\xi_0, \xi_1} k(\xi_0, \xi_1) \leq k(\xi''_0, \xi''_1) \). Therefore, \( f(\kappa, \xi_0, \xi_1) < 0 \) if \( \kappa < k(\xi_0, \xi_1) \). Similarly, we can show that \( f(\kappa, \xi_0, \xi_1) > 0 \) if \( \kappa > k(\xi_0, \xi_1) \).

From the above discussion, \( f(\kappa, \xi_0, \xi_1) < 0 \) if and only if \( \kappa < k(\xi_0, \xi_1) \). Thus, we can apply Lemma D to \( \min_{\xi_1} f(\kappa, \xi_0, \xi_1) = 0 \) for each fixed \( \xi_0 \); that is, \( \max_{\xi_1} k(\xi_0, \xi_1) \) is the unique solution of this equation. Because \( \min_{\xi_1} f(\kappa, \xi_0, \xi_1) < 0 \) if and only if \( \kappa < \max_{\xi_1} \kappa^*(\xi_0, \xi_1) \), we can apply Lemma D to \( \max_{\xi_0} \min_{\xi_1} f(\kappa, \xi_0, \xi_1) = 0 \); that is, \( \min_{\xi_0} \max_{\xi_1} \kappa^*(\xi_0, \xi_1) \) is the unique solution of this equation. Similarly, we can show that \( \max_{\xi_1} \min_{\xi_0} k(\xi_0, \xi_1) \) is the unique solution of \( \min_{\xi_1} \max_{\xi_0} f(\kappa, \xi_0, \xi_1) = 0 \). Because \( \max_{\xi_0} \min_{\xi_1} f(\kappa, \xi_0, \xi_1) = \min_{\xi_1} \max_{\xi_0} f(\kappa, \xi_0, \xi_1) \), (7) is the unique value of \( \kappa \in \mathbb{R} \) solving (5). \( \square \)
C Proof of Lemma 1

Proof of Lemma 1. Let \( u(a, l, \theta) = f(l) \) be independent of \( \theta \). Assume that \( \theta \) is drawn from an improper uniform distribution over the real line. Then, \( p_\xi(\theta|x) = q_\xi(x - \theta) \) and thus

\[
E_\xi[s(\kappa)|\theta] = \int_{-\infty}^{\infty} q_\xi(x - \theta) dx = 1 - Q_\xi(x - \theta) dx,
\]

where \( Q_\xi \) is the cumulative distribution function. Therefore,

\[
\pi^a_\xi(\kappa, \kappa) = \int_{-\infty}^{\infty} u(a, 1 - Q_\xi(\kappa - \theta), \theta) q_\xi(\kappa - \theta) d\theta
= \int_{0}^{1} u(a, 1 - Q_\xi^{-1}(1 - l))(1 - l) dl = \int_{0}^{1} f(l) dl,
\]

(C1)

where we use the substitution \( l = 1 - Q_\xi(\kappa - \theta) \).

Assume that \( \theta \) and \( \xi \) are drawn from uniform distributions over the intervals \([g(\xi), h(\xi)]\) and \([-d(\xi), d(\xi)]\), respectively. Then, it can be readily shown that \( p_\xi(\theta|x) = q_\xi(x - \theta) \) for each \( x \in [g(\xi) + d(\xi), h(\xi) - d(\xi)] \). This implies that if \( \kappa \in [g(\xi) + d(\xi), h(\xi) - d(\xi)] \) then (C1) holds.

□

D Proof of Proposition 3

Proof of Proposition 3. We prove the case of \( a = 0 \). We can prove the other case similarly.

To prove the “only if” part, suppose that \( s[\kappa^{**}] \) is the unique strategy surviving iterated deletion of interim-dominated strategies. By Proposition 1, \( \kappa^{**} \) is the unique solution of (5), which is rewritten as \( \min_{\xi \in \Xi} \pi^1_\xi(\kappa, \kappa) - c(\kappa) = 0 \). By Lemma C, \( \min_{\xi \in \Xi} \pi^1_\xi(\kappa, \kappa) \geq c(\kappa) \) if and only if \( \kappa \geq \kappa^{**} \). If there exists \( \kappa' > \kappa^{**} \) and \( \xi' \in \Xi \) such that \( \pi^1_\xi(\kappa', \kappa') = c(\kappa') \), it follows that \( \min_{\xi \in \Xi} \pi^1_\xi(\kappa', \kappa') \leq c(\kappa') \), which is a contradiction. This implies that \( \kappa^{**} = \max_{\xi \in \Xi} \max\{\kappa : \pi^1_\xi(\kappa, \kappa) = c(\kappa)\} = \kappa^* \) because \( \pi^1_\xi(\kappa^{**}, \kappa^{**}) = c(\kappa^{**}) \) for \( \xi^{**} \in \arg\min_{\xi \in \Xi} \pi^1_\xi(\kappa^{**}, \kappa^{**}) \).

To prove the “if” part, suppose that \( \min_{\xi \in \Xi} \pi^1_\xi(\kappa, \kappa) < c(\kappa) \) for all \( \kappa < \kappa^* \). By Lemma C, \( \min_{\xi \in \Xi} \pi^1_\xi(\kappa, \kappa) \geq c(\kappa) \) if and only if \( \kappa \geq \kappa^* \). Thus, \( \kappa^* \) is the unique solution of (5) and \( s[\kappa^*] \) is the unique strategy surviving iterated deletion of interim-dominated strategies by Proposition 1.

□
E Proof of Proposition 4

Proof of Proposition 4. We prove the case of $a = 0$. We can prove the other case similarly.

Because (5) is rewritten as $\min_{\xi \in \Xi} \pi^1_\xi(\kappa, \kappa) = c$, it is enough to show that, for each $\bar{\xi} > 0$, there exists $\delta > 0$ such that, for each $\bar{\xi} \in (0, \delta]$, $\kappa^*$ is the unique solution of $\min_{\xi \in \Xi} \pi^1_\xi(\kappa, \kappa) = c$.

For each $\kappa \neq \pm \infty$, we have

$$\lim_{\xi \to 0} E_\xi[s(\kappa)|\theta] = \lim_{\xi \to 0} \int_\kappa^\infty \sqrt{\xi}q(\sqrt{\xi}(\theta - x))dx = 1 - \lim_{\xi \to 0} Q(\sqrt{\xi}\kappa) = 1/2$$

because $q(\varepsilon) = q(-\varepsilon)$, where $Q$ is the cumulative distribution function. We also have

$$\lim_{\xi \to 0} p_\xi(\theta|x) = \lim_{\xi \to 0} \frac{p(\theta)\sqrt{\xi}q(\sqrt{\xi}(\theta - x))}{\int p(\theta)\sqrt{\xi}q(\sqrt{\xi}(\theta - x))d\theta} = \lim_{\xi \to 0} \frac{p(\theta)q(\sqrt{\xi}(\theta - x))}{\int p(\theta)q(\sqrt{\xi}(\theta - x))d\theta} = p(\theta).$$

Therefore, it follows that

$$\lim_{\xi \to 0} \pi^1_\xi(\kappa, \kappa) = \lim_{\xi \to 0} \int u(1, E_\xi[s(\kappa)|\theta], \theta)p_\xi(\theta|x)d\theta = \int u(1, 1/2, \theta)p(\theta)d\theta < c \quad (E1)$$

by A4 and the second condition. This implies that, for any closed interval $[\kappa, \bar{\kappa}]$, there exists $\delta > 0$ such that $\pi^1_\xi(\kappa, \kappa) < c$ for all $\kappa \in [\kappa, \bar{\kappa}]$ and $\xi \leq \delta$.

Let $\kappa$ be such that $\min_{\xi \in \Xi} \pi^1_\xi(\kappa, \kappa) < c$ for all $\kappa < \kappa$, which exists by Lemma C. Let $\bar{\kappa}$ be given in the third condition. For $\delta$ given above, choose arbitrarily $\bar{\xi} < \delta$ and set $\Xi = [\bar{\xi}, \bar{\xi}]$. Then, $\min_{\xi \in \Xi} \pi^1_\xi(\kappa, \kappa) \leq \pi^1_\bar{\xi}(\kappa, \kappa) < c$ for all $\kappa \leq \bar{\kappa}$. We show that the solution of $\min_{\xi \in \Xi} \pi^1_\xi(\kappa, \kappa) = c$ is unique.

Seeking a contradiction, suppose that there are two solutions $\kappa^*$ and $\kappa_*$ with $\kappa^* > \kappa_*$ > $\bar{\kappa}$. For $\bar{\xi}^* \in \arg \min \pi^1_\bar{\xi}(\kappa^*, \kappa^*)$), we have $\pi^1_\bar{\xi}(\kappa^*, \kappa^*) = c$ and thus $\pi^1_\bar{\xi}(\kappa^*, \kappa_*) < c$ by the third condition, which contradicts $\min_{\xi \in \Xi} \pi^1_\xi(\kappa_*, \kappa_*) = c$. Therefore, $\kappa^*$ given in Proposition 3 is the unique solution of $\min_{\xi \in \Xi} \pi^1_\xi(\kappa, \kappa) = c$.

For arbitrary $\bar{\kappa} > \bar{\kappa}$, there exists $\delta > 0$ such that $\pi^1_\xi(\kappa, \kappa) < c$ for all $\kappa \in [\kappa, \bar{\kappa}]$ and $\xi \leq \delta$ by (E1). This implies that $\lim_{\xi \to \infty} \kappa^* = \infty$. $\square$
References


