A Note on the Lucas Model:  
Iterated Expectations and the Neutrality of Money

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Abstract

We generalize the Lucas island model and re-examine the Phelps-Lucas hypothesis by allowing agents to have private as well as public ex ante information about monetary shocks. We demonstrate that the real effects can be decomposed into the unanticipated part and the anticipated part. We then study the relationship between the real effects and public information about monetary shocks. We show that the real effects are small not only when the public signal contains very precise information but also when it contains very imprecise information.

1 Introduction

The Phelps-Lucas hypothesis (Phelps, 1970; Lucas, 1972; Lucas, 1973) is that monetary shocks have the real effects because of imperfect information about prices. In the Lucas formulation, it is unanticipated monetary shocks that have the real effects, which is the consequence of the assumption that ex ante information about monetary shocks is common knowledge.

This paper re-examines the Phelps-Lucas hypothesis using the model of Lucas (1973). In so doing, we allow agents to have ex ante private information about monetary shocks, following Phelps (1993) and Woodford (2003). We first demonstrate that monetary shocks have the real effects not only when they are unanticipated but also when they are anticipated with imperfect common knowledge. Then, we decompose the real effects into the unanticipated shock effect and the imperfect common knowledge effect, each of which is obtained in closed forms. Finally, we study the relationship between the real effects and public information. We consider imperfect information consisting of normally distributed private signals and public signals. We show that the real effects are small not only when the public signal contains very precise information but also when it contains very imprecise information. The result is closely
related to the welfare effects of public information studied by Morris and Shin (2002).\footnote{The connection to the Lucas model is suggested by Morris and Shin (2002). See also Amato \textit{et al.} (2002) who derived another connection.}

Woodford (2003) already re-examined the Phelps-Lucas hypothesis allowing agents to have ex ante private information about monetary shocks. The model is a dynamic model consisting of firms in monopolistic competition, which is different from the Lucas formulation. The focus of Woodford (2003) is to show persistence of the real effects in the dynamic setup. Building on the model of Woodford (2003), Amato and Shin (2003) studied the implication of public information when agents have private information.

In contrast to Woodford (2003) and Amato and Shin (2003), the model of this paper is based upon the static setup of Lucas (1973), and we are interested in what factors of imperfect information induce the real effects and to what extent they are.

In section 2, we derive the consequence of the Lucas formulation when agents have private information and show that the real effects consist of the unanticipated part and the anticipated part. In section 3, we assume normal distributions for information about monetary shocks and identify both parts of the real effects. In section 4, we study the relationship between the real effects and public information.

\section{The Lucas Model with Private Information}

The economy consists of many separate competitive markets, each of which is called an island. The supply curve in island \( i \in \{1, \ldots, n\} \) is

\[ y^i_t = b(p^i_t - E[p_t | I^i_t, p^i_t]) \]  \hspace{1cm} (1)

where \( b > 0 \) is a constant, \( y^i_t \) and \( p^i_t \) are the logarithms of output and the nominal price of output in island \( i \) in period \( t \), and \( I^i_t \) is the information available in island \( i \) about the economy up to and including time \( t - 1 \). Note that \( E[\cdot | I^i_t, p^i_t] \) is the expectation operator conditional on \( I^i_t \) and \( p^i_t \). In the original formulation of Lucas (1973), it is assumed that \( I^i_t \) are the same for all \( i \). In our formulation, \( I^i_t \) may differ and contain private information in island \( i \).

As in Lucas (1973), we assume that \( p_t, E[p_t | I^i_t] \), and \( p^i_t \) are jointly normally distributed with the same expected values, and there exists \( \theta > 0 \) such that

\[ E[p_t | I^i_t, p^i_t] = (1 - \theta)p^i_t + \theta E[p_t | I^i_t] \]  \hspace{1cm} (2)

Plugging (2) into (1),

\[ y^i_t = b(1 - \theta)(p^i_t - E[p_t | I^i_t]) \]
\[ = \beta(p^i_t - E[p_t | I^i_t]) \]  \hspace{1cm} (3)

where \( \beta = b(1 - \theta) \). In the remainder of the paper, we simply write \( E^i_t = E[\cdot | I^i_t] \). Thus (3) is written as

\[ y^i_t = \beta(p^i_t - E^i_t p_t) \]  \hspace{1cm} (4)
Taking aggregation of (4) over $i$, we have the aggregate supply curve

$$y_t = \beta (p_t - \bar{E}_t p_t) \tag{5}$$

where \( y_t = n^{-1} \sum_i y_t^i \), \( p_t = n^{-1} \sum_i p_t^i \), and \( \bar{E}_t = n^{-1} \sum_i E_t^i \). The aggregate demand curve is

$$y_t = m_t - p_t. \tag{6}$$

Market clearing condition with (5) and (6) gives

$$m_t - p_t = \beta (p_t - \bar{E}_t p_t) \tag{7}$$

Solving this for $p_t$,

$$p_t = \frac{1}{1 + \beta} m_t + \frac{\beta}{1 + \beta} \bar{E}_t p_t = (1 - r)m_t + r \bar{E}_t p_t$$

where $r = \beta / (1 + \beta)$. Operating $\bar{E}_t$ on both sides,

$$\bar{E}_t p_t = (1 - r) \bar{E}_t m_t + r \bar{E}_t^{(2)} p_t \tag{8}$$

where $\bar{E}_t^{(2)} = \bar{E}_t \bar{E}_t$. Plugging (8) into (7),

$$p_t = (1 - r)m_t + (1 - r)r \bar{E}_t m_t + r^2 \bar{E}_t^{(2)} p_t.$$ 

Repeating this,

$$p_t = (1 - r) \sum_{k=0}^{\infty} r^k \bar{E}_t^{(k)} m_t \tag{9}$$

Plugging (9) into (5),

$$y_t = \beta (1 - r) \sum_{k=0}^{\infty} r^k (\bar{E}_t^{(k)} m_t - \bar{E}_t^{(k+1)} m_t)$$

$$= r \sum_{k=0}^{\infty} r^k (\bar{E}_t^{(k)} m_t - \bar{E}_t^{(k+1)} m_t)$$

$$= r \left( m_t - (1 - r) \bar{E}_t m_t - (r - r^2) \bar{E}_t^{(2)} m_t - (r^2 - r^3) \bar{E}_t^{(3)} m_t \cdots \right)$$

$$= r \left( m_t - (1 - r) \sum_{k=1}^{\infty} r^{k-1} \bar{E}_t^{(k)} m_t \right) \tag{10}$$

This equation is the consequence of the generalized Lucas formulation. It should be noted that the real effects are generated by the difference between the realized value of $m_t$ and the weighted average of iterated expectations of $m_t$.

If $I_t^i = I_t^j$ for all $i \neq j$, then $E_t^i = E_t^j$ and thus $\bar{E}_t = E_t^i$. In this case, the law of iterated expectations holds: $E_t m = E_t^i m = E_t \bar{E}_t m = \bar{E}_t \bar{E}_t m$. Then, (10) is reduced to

$$y_t = r (m_t - \bar{E}_t m_t).$$
implying that $E_yt = 0$. The real effects arise because of unanticipated monetary shocks only. However, if $I_i^t \neq I_i^r$, then $E_t m_t \neq E_t E_t m_t$, as we will see in the next section.\footnote{Interesting implication of the failure of the law of iterated expectations is discussed by Morris and Shin (2002) and Allen et al. (2002).}

Equation (10) is decomposed into the following three terms:

$$y_t = r \left( m_t - E_t^1 m_t \right) + r \left( E_t m_t - (1 - r) \sum_{k=1}^{\infty} r^{k-1} E_t^{(k)} m_t \right) + r \left( E_t^r m_t - E_t m_t \right).$$

(11)

We call the first term the unanticipated shock effect because $E_t^1 (m_t - E_t^1 m_t) = 0$. We call the second term the imperfect common knowledge effect because this term is zero if information about $m_t$ is common knowledge and thus $E_t m_t = E_t E_t m_t$. We call the third term the private noise effect. The real effects induced by the second and the third term is in fact anticipated, as we will see in the next section.

3 Normally Distributed Signals

For the notational simplicity, we drop the subscript $t$. Assume that information available to residents of island $i$ is a random vector $x^i$ taking values in $\mathbb{R}^L$. Assume that the random vector $(m, x^1, \ldots, x^n)$ is normally distributed and every component has mean zero. Then, there exists a $1 \times L$ matrix $M^i$ and a $L \times L$ matrix $A^{ij}$ with $i \neq j$ such that $E^i m = M^i x^i$ and $E^i x^j = A^{ij} x^i$. These matrices are derived from the covariance matrix of $(m, x^1, \ldots, x^n)$. We assume symmetry such that $M = M^i$ and $A = A^{ij}$ for all $i \neq j$. Thus

$$E^i m = M^i x^i, \quad E^i x^j = A^{ij} x^i \quad \text{for } j \neq i, \quad E^i x^i = x^i.$$

Let $\bar{x} = n^{-1} \sum_i x^i$. Then,

$$E^i \bar{x} = (n-1)n^{-1} A x^i + n^{-1} x^i = A_n x^i$$

where $A_n = (n-1)n^{-1} A + n^{-1} I$ and $I$ is a unit matrix. Since $E^i m = M^i x^i$, we have $\bar{E}m = M \bar{x}$. Then $E^i \bar{E}m = E^i M \bar{x} = ME^i \bar{x} = MA_n x^i$, and thus $\bar{E} \bar{E} m = MA_n \bar{x}$. In general,

$$E^{(k)} m = MA_n^{k-1} \bar{x}.$$

(12)

Plugging (12) into (10),

$$y = r \left( m - (1 - r) \sum_{k=1}^{\infty} r^{k-1} E^{(k)} m \right)$$

$$= r \left( m - (1 - r) \sum_{k=1}^{\infty} r^{k-1} MA_n^{k-1} \bar{x} \right)$$

$$= r \left( m - M (1 - r) (I - r A_n)^{-1} \bar{x} \right).$$

(13)

By (11),

$$y = r \left( m - M x^i \right) + r M (I - (1 - r) (I - r A_n)^{-1}) \bar{x} + r \left( M x^i - M \bar{x} \right).$$

(14)
The expected value of $y$ held by residents in island $i$ is

$$E^i y = r M (I - (1 - r)(I - r A_n))^{-1} E^i \bar{x} + r (M E^i x^i - M E^i \bar{x})$$

$$= r M (I - (1 - r)(I - r A_n))^{-1} A_n x^i + r (M x^i - M A_n x^i)$$

$$= r M (I - (1 - r)(I - r A_n))^{-1} A_n x^i.$$  

Thus, the real effects are anticipated by residents in island $i$ as far as $x^i \neq 0$ and the matrix $M (I - (1 - r)(I - r A_n))^{-1} A_n$ is regular.

4 Real Effects of Public Information

Assume that $m$ is normally distributed with mean zero and variance $\sigma^2_m = \infty$, i.e., an improper prior for $m$. Let

$$x^i = \begin{pmatrix} v^i \\ w \end{pmatrix}$$

where $v^i$ is a private signal and $w$ is a public signal. The private signal is given by

$$v^i = m + \xi + \varepsilon^i$$

where $\xi$ and $\varepsilon^i$ are normally distributed with mean zero and variance $\sigma^2_\xi$ and $\sigma^2_\varepsilon$, respectively. Note that the noise term $\xi$ is common to all the islands. The public signal is given by

$$w = m + \eta$$

where $\eta$ is normally distributed with mean zero and variance $\sigma^2_\eta$.

Assume that $m$, $\xi$, $\eta$, $\varepsilon^1$, \ldots, $\varepsilon^n$ are independent. We consider the limit $n \to \infty$:

$$\bar{x} = \lim_{n \to \infty} n^{-1} \sum_i x^i = \begin{pmatrix} v \\ w \end{pmatrix}$$

where $v \equiv \text{plim}_{n \to \infty} n^{-1} \sum_i v^i = m + \xi$ by the law of large numbers. If $\sigma^2_\xi = 0$, then $v = m$, i.e., there is no aggregate uncertainty in the private signals.

We have

$$M = \begin{pmatrix} \alpha & 1 - \alpha \\ \beta & 1 - \beta \end{pmatrix}$$

$$A_n = A = \begin{pmatrix} \beta & 1 - \beta \\ 0 & 1 \end{pmatrix}$$

$$\bar{x} = \begin{pmatrix} v \\ w \end{pmatrix}$$

where $v = m + \xi$ and $w = m + \eta$.

Plugging all of the above into (13), we have

$$y = r (m - (\lambda v + (1 - \lambda) w))$$

where $\lambda = \frac{(1 - r)A}{1 - rA}$.  

(15)
By (14), we can rewrite (15) as

\[ y = r(m - E'm) + r^2 \alpha (1 - \beta) \frac{(v - w) + r\alpha(v' - v)}. \]

The first term is the unanticipated shock effect, the second term is the imperfect common knowledge effect, and the third term is the private noise effect. Note that the imperfect common knowledge effect is the difference between the common knowledge \( w \) and the aggregate private knowledge (or distributed knowledge) \( v \). The expected value of \( y \) held by residents in island \( i \) is

\[
E_i' y = r^2 \alpha (1 - \beta) \frac{(v' - w)}{1 - r\beta} + r\alpha (v' - w)
\]

Thus, \( v' - w \) generates the anticipated real effects. The anticipated real effects are not equal to zero almost surely if \( \alpha \neq 0 \), \( \beta \neq 1 \), and \( \beta \neq 1/r(1 - r) \).

We now consider when money is neutral. Let \( V \) be the variance of \( y \):

\[
V = V(r\lambda \xi + r(1 - \lambda)\eta)
\]

\[
= r^2 \lambda^2 \sigma_x^2 + r^2 (1 - \lambda)^2 \sigma_\eta^2
\]

\[
= r^2 \frac{(1 - r)^2 \sigma_x^2 + \sigma_x^2 + \sigma_\eta^2}{(1 - r)(\sigma_x^2 + \sigma_\eta^2) + \sigma_\eta^2}
\]

Note that if \( V = 0 \) then \( y = 0 \) almost surely and thus money is neutral. For example, consider the following cases.

- If \( 0 < \sigma_\eta^2 < \infty \) and \( \sigma_x^2 + \sigma_\eta^2 > 0 \), then \( V \neq 0 \), i.e., money is not neutral. Since \( \lim_{\sigma_\eta^2 \to \infty} V = \lim_{\sigma_x^2 \to \infty} V = 0 \), we have \( \lim_{\sigma_\eta^2 \to \infty} y = \lim_{\sigma_x^2 \to \infty} y = r(m - w) \). This is the case closest to the conclusion of the original Lucas model.

- If \( \sigma_x^2 = 0 \), then \( V = 0 \). Thus, if \( m \) is publicly announced, then money is neutral.

- If \( \sigma_x^2 = 0 \), then \( \lim_{\sigma_\eta^2 \to \infty} V = 0 \). Thus, if the public signal contains no information and there is no aggregate uncertainty in the private signals, then money is neutral. If \( \sigma_\eta^2 = 0 \) and \( 0 < \sigma_x^2 < \infty \), however, we have \( V \neq 0 \), and thus money is not neutral. In other words, if the public signal contains some information and there is no aggregate uncertainty in the private signals, then money is not neutral.

From the last observation, we can say that, if there is no aggregate uncertainty in the private signals, less uncertainty in the public signal may result in the real effects, though no
uncertainty in the public signal results in the neutrality. This is closely related to the finding of Morris and Shin (2002) about the welfare effects of public information. To see this, note first that larger $V$ implies more real effects. We have

$$\frac{\partial V}{\partial \sigma^2_\xi} = 2r^2 \left(1 - r\right) \sigma^2_\xi \left(1 - r\right) \sigma^2_\eta \geq 0,$$

$$\frac{\partial V}{\partial \sigma^2_\eta} = r^2 \left(1 - r\right)^2 \sigma^4_\eta \left(1 - r\right) \sigma^2_\xi + 3 \sigma^2_\xi \geq 0.$$

Thus, more uncertainty in the private signal always results in more real effects. We also have

$$\frac{\partial V}{\partial \sigma^2_\eta} = r^2 \left(1 - r\right) \sigma^2_\xi + \sigma^2_\eta \left(1 - r\right) \sigma^4_\eta \left(1 - r\right) \sigma^2_\xi + 3 \sigma^2_\xi \geq 0$$

if and only if

$$\left(1 - r\right) \sigma^2_\xi + \sigma^2_\eta \geq 0.$$

Thus, if $\sigma^2_\xi$ is small enough such that $\sigma^2_\xi < \sigma^2_\eta / (1 - r)$ and $\sigma^2_\eta$ is large enough such that

$$\sigma^2_\eta > \frac{(\sigma^2_\xi + (1 - r) \sigma^2_\eta)^2}{(1 - r) \sigma^2_\eta \sigma^2_\xi},$$

then $\partial V(y)/\partial \sigma^2_\eta < 0$, i.e., more uncertainty in the public signal results in less real effects.

Especially, when there is no aggregate uncertainty in the private signals ($\sigma^2_\xi = 0$) and the public signal contains no information ($\sigma_\eta = \infty$), money has no real effects as discussed above.

**References**


