Quantal Response Equilibria and Stochastic Best Response Dynamics*

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Abstract

This paper provides an evolutionary interpretation of quantal response equilibria of an n-player game. Consider a stochastic best response process in the corresponding n-population random matching game. The main result states that if a stochastic best response process satisfies the detailed balance condition then the support of the stationary distribution converges to the set of quantal response equilibria as the population size goes to infinity.

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1. INTRODUCTION

Quantal response models are regression models in which dependent variables take discrete values (Amemiya, 1985). They have numerous applications in economics because many behavioral responses are qualitative in nature. A class of quantal response models including logit models and probit models are defined by means of random payoff maximization.

In order to analyze data of behavioral responses under strategic interaction, Mckelvey and Palfrey (1995) investigated the use of quantal response models in a game theoretical setting. Consider an n-player game with payoffs subject to random error and suppose that a player maximizes the realization of random payoffs. The stochastic response of a player is described by a quantal response model and observed as a mixed strategy. Mckelvey and Palfrey (1995) defined a quantal response equilibrium as a mixed strategy profile such that, for every player, a mixed strategy is the stochastic response to competitors' mixed strategies. In other words, a quantal response equilibrium is a statistical version of a Nash equilibrium. Mckelvey and Palfrey (1995) demonstrated that quantal response equilibria are unique when the error variance is sufficiently large. The intuition behind this is that if the error variance is sufficiently large then players' responses are nearly uniformly distributed whatever competitors' strategies are, which results in the uniqueness.

An n-player game with payoffs subject to random error induces not only quantal response equilibria but also stochastic best response dynamics. Imagine that n populations of players are randomly and repeatedly matched to play the game. In each period, one randomly chosen player revises his strategy to maximize the realization of random payoffs. We call this stochastic process a stochastic best response process. It is well known that, as the error variance goes to zero, the support of the stationary distribution converges to a unique Nash equilibrium for some class of games (Kirkpatrick *et al.*, 1983; Foster and Young, 1990; Kandori *et al.*, 1993; Young, 1993; Blume, 1993). The unique Nash equilibrium is called a stochastically stable equilibrium or a long run equilibrium.

The purpose of this paper is to provide an evolutionary interpretation of quantal response equilibria in terms of stochastic best response processes. The main result states that if a stochastic best response process satisfies the detailed balance condition for every population size, then, as the population size goes to infinity, the support of the stationary distribution converges to the set of quantal response equilibria. For example, if an n-player game is a potential game (Monderer and Shapley, 1996) and quantal responses are logistic, then the stochastic best response process satisfies the detailed balanced condition. This class of stochastic best response processes include

the finite version of the model considered by Blume (1993).

The main result leads us to the following finding. By the uniqueness of quantal response equilibria with large error variance, the support of the stationary distribution with a large population size is close to a singleton when the error variance is very large. By the uniqueness of stochastically stable equilibria with infinitesimally small error variance, the support of the stationary distribution is close to a singleton when the error variance is very small. Combining the above observations, we find that if a stochastic best response process satisfies the detailed balance condition then the support of the stationary distribution with a large population size is close to a singleton not only when the error variance is very small but also when the error variance is very large. In other words, both small noise and large noise generate uniqueness in stochastic best response dynamics. Myatt and Wallace (1998) considered a single population 2×2 coordination game with normal random error and noted that uniqueness arises for both small noise and large noise. Ui (2001b) demonstrated that a similar statement is true in incomplete information games with payoffs subject to correlated random error. The class of games provided a unified view on two distinct equilibrium selection results: the result in global games by Carlsson and van Damme (1993) where correlated small noise generates uniqueness, and the result in quantal response equilibria where independent large noise generates uniqueness. See Morris and Shin (2002) who further investigated this issue.

The organization of this paper is as follows. Section 2 defines quantal response equilibria. Section 3 defines stochastic best response processes. Section 4 provides the main result. Section 5 discusses an implication for equilibrium selection.

2. QUANTAL RESPONSE EQUILIBRIA

An n-player game consists of a finite set of players $N = \{1, \dots, n\}$, a finite strategy space $A = \prod_{i \in N} A_i$, and a payoff function $g_i : A \to \mathbb{R}$ for $i \in N$. We write $A_{-i} = \prod_{j \neq i} A_j$. We simply denote an n-player game by $\mathbf{g} = (g_i)_{i \in N}$. For a finite set X, the set of probability distributions on X is denoted by $\Delta(X)$. Let $\mathcal{A}_i = \Delta(A_i)$ be the set of mixed strategies of player $i \in N$. We write $\mathcal{A} = \prod_{i \in N} \mathcal{A}_i \subset \Delta(A)$ and $\mathcal{A}_{-i} = \prod_{j \neq i} \mathcal{A}_j \subset \Delta(A_{-i})$. The domain of g_i is extended to $A_i \times \mathcal{A}_{-i}$ by the rule $g_i(a_i, p_{-i}) = \sum_{a_{-i} \in A_{-i}} g_i(a_i, a_{-i}) p_{-i}(a_{-i})$. We regard $g_i(\cdot, p_{-i}) \in \mathbb{R}^{A_i}$.

For $i \in N$, let $\varepsilon_i = (\varepsilon_i(a_i))_{a_i \in A_i}$ be an error vector of player i where $\varepsilon_i(a_i)$ is a random variable taking real numbers with $\mathrm{E}[\varepsilon_i(a_i)] = 0$ for all $a_i \in A_i$. Assume that ε_i is distributed according to a joint distribution with an absolutely continuous probability density function f_i. An error vector ε_i induces a function $\pi_i : \mathbb{R}^{A_i} \to \mathcal{A}_i$

such that, for each $c_i \in \mathbb{R}^{A_i}$, a probability of $a_i \in A_i$ assigned by $\pi_i(c_i) \in \mathcal{A}_i$ is

$$\pi_i(a_i|c_i) \equiv \Pr\left[c_i(a_i) + \varepsilon_i(a_i) \geq c_i(a_i') + \varepsilon_i(a_i') \text{ for all } a_i' \in A_i\right].$$

Note that $\sum_{a_i \in A_i} \pi_i(a_i|c_i) = 1$ by the absolute continuity of f_i. We call π_i a quantal response function of player i.

For example, let $\varepsilon_i(a_i)$ be identically and independently distributed according to a log Weibull distribution.[1] Then, we have a quantal response function such that

$$\pi_i(a_i|c_i) = \frac{\exp[\beta c_i(a_i)]}{\sum_{a_i' \in A_i} \exp[\beta c_i(a_i')]} \tag{1}$$

for all $a_i \in A_i$ (McFadden, 1974). We call π_i a logistic quantal response function. It is known that large β corresponds to the small error variance. In addition, $\lim_{\beta \to \infty} \pi_i(a_i|c_i) > 0$ if and only if $c(a_i) \geq c(a_i')$ for all $a_i' \in A_i$, and $\lim_{\beta \to 0} \pi_i(a_i|c_i) = 1/|A_i|$ for all $a_i \in A_i$.

Suppose that payoffs of player i are subject to random error ε_i and that he maximizes the realization of random payoffs $g_i(a_i, p_{-i}) + \varepsilon_i(a_i)$. Then, the response of player i to p_{-i} is observed as a mixed strategy $\pi_i(g_i(\cdot, p_{-i})) \in \mathcal{A}_i$.

Mckelvey and Palfrey (1995) defined a quantal response equilibrium as a mixed strategy profile such that, for every player, a mixed strategy is the response to competitors' mixed strategies given by his quantal response function.

Definition 1 A mixed strategy profile $p = (p_i)_{i \in N} \in \mathcal{A}$ is a *quantal response equilibrium* of $\mathbf{g}$ with $(\pi_i)_{i \in N}$ if

$$p_i = \pi_i(g_i(\cdot, p_{-i}))$$

for all $i \in N$ where $p_{-i} = (p_j)_{j \neq i}$.

In symmetric and one population settings, a similar concept is introduced by Daganzo and Sheffi (1977) for traffic congestion models, and by Brock and Durlauf (2001) for social interaction models with binary choices.

When quantal response functions are logistic, a quantal response equilibrium is called a logit equilibrium. Mckelvey and Palfrey (1995) defined the logit equilibrium correspondence to be the correspondence $\Pi : \mathbb{R}_+ \to 2^{\mathcal{A}}$ given by

$$\Pi(\beta) = \left\{ p \in \mathcal{A} \;\middle|\; p_i(a_i) = \frac{\exp[\beta g_i(a_i, p_{-i})]}{\sum_{a_i' \in A_i} \exp[\beta g_i(a_i', p_{-i})]} \text{ for all } a_i \in A_i \text{ and } i \in N \right\}.$$

Note that $\Pi(\beta)$ is the set of logit equilibria. It is obvious that $\Pi(0)$ is a singleton, which consists of $p \in \mathcal{A}$ such that $p_i(a_i) = 1/|A_i|$. Mckelvey and Palfrey (1995, Lemma 1 and Theorem 3) showed the following property of Π.

[1] The probability density function is $\exp(-\beta\varepsilon_i(a_i)) \exp(-\exp(\beta\varepsilon_i(a_i)))$ where $\beta > 0$.

THEOREM 1 *For almost all* $\mathbf{g}$*, the following is true.*

1. *For sufficiently small* $\beta > 0$*,* $\Pi(\beta)$ *is a singleton.*

2. *The graph of* Π *contains a unique branch which starts at* $p \in \mathcal{A}$ *such that* $p_i(a_i) = 1/|A_i|$*, for* $\beta = 0$*, and converges to a unique Nash equilibrium, as* $\beta \to \infty$*.*

If the error variance is sufficiently large, then a logit equilibrium is unique. By tracing the unique branch of the graph containing the unique logit equilibrium, we have a unique Nash equilibrium. Mckelvey and Palfrey (1995) called the unique Nash equilibrium a limiting logit equilibrium.

3. STOCHASTIC BEST RESPONSE PROCESSES

Let an n-player game $\mathbf{g}$ be given. We define an n-population game with population size k. For each $i \in N$, let $\{(i,1),(i,2),\dots,(i,k)\}$ be a population of players. We call it population i. The set of players of an n-population game is a union of the n populations, which is denoted by

$$N^k \equiv \bigcup_{i \in N} \{(i,1),(i,2),\dots,(i,k)\}.$$

Let A_{ij} be a strategy set of player $(i,j) \in N^k$ with a generic element a_{ij}. Assume that $A_{ij} = A_i$ for all $j \in \{1,\dots,k\}$. Thus, every player in population i has the same strategy set A_i. The strategy space of an n-population game is $A^k = \prod_{(i,j)\in N^k} A^k_{ij}$ with a generic element $a^k = (a_{ij})_{(i,j)\in N^k}$. For $i \in N$, the set of strategy profiles of population i is $A^k_i = \prod_{j=1}^k A_{ij}$ with a generic element $a^k_i = (a_{ij})_{j\in\{1,\dots,k\}}$, and the set of strategy profiles of the other populations is $A^k_{-i} = \prod_{h\neq i} A^k_h$ with a generic element $a^k_{-i} = (a^k_h)_{h\neq i}$.

For $a^k_i \in A^k_i$, let $p_i^{a^k_i} \in \mathcal{A}_i$ be such that $p_i^{a^k_i}(a_i)$ is a relative frequency of $a_i \in A_i$ in $a^k_i = (a_{i1},\dots,a_{ik})$:

$$p_i^{a^k_i}(a_i) = \frac{|\{(i,j) \,|\, a_{ij} = a_i \text{ for } j \in \{1,\dots,k\}\}|}{k}.$$

For $a^k \in A^k$, let $p^{a^k} \in \mathcal{A}$ be such that $p^{a^k}(a) = \prod_{i\in N} p_i^{a^k_i}(a_i)$. For $a^k_{-i} \in A^k_{-i}$, let $p_{-i}^{a^k_{-i}} \in \mathcal{A}_{-i}$ be such that $p_{-i}^{a^k_{-i}}(a_{-i}) = \prod_{h\neq i} p_h^{a^k_h}(a_h)$.

Let $g^k_{ij} : A^k \to \mathbb{R}$ be a payoff function of player $(i,j) \in N^k$. Assume that

$$g^k_{ij}(a^k) = g_i(a_{ij}, p_{-i}^{a^k_{-i}}) = \sum_{a_{-i}\in A_{-i}} g_i(a_{ij}, a_{-i}) p_{-i}^{a^k_{-i}}(a_{-i})$$

for all $a^k \in A^k$. Thus, payoffs of player (i,j) are calculated as expected payoffs of player i when n players are randomly drawn from the respective n populations to play $\mathbf{g}$. We simply denote an n-population game by $\mathbf{g}^k = (g^k_{ij})_{(i,j)\in N^k}$.

For $i \in N$, let an error vector ε_i and a quantal response function π_i be given. Suppose that payoffs of player $(i,j) \in N^k$ are subject to random error ε_i and that he maximizes the realization of random payoffs $g^k_{ij}(a^k) + \varepsilon_i(a_{ij}) = g_i(a_{ij}, p_{-i}^{a^k_{-i}}) + \varepsilon_i(a_{ij})$. Then, he chooses $a_{ij} \in A_{ij} = A_i$ with probability $\pi_i(a_{ij}|g_i(\cdot, p_{-i}^{a^k_{-i}}))$.

We consider a discrete time strategy revision process in $\mathbf{g}^k$ with $(\pi_i)_{i\in N}$ described as follows.[2]

- In each period, one player is randomly chosen with probability $1/nk$. Let the player be (i,j).
- Player (i,j) chooses $a_{ij} \in A_{ij} = A_i$ with probability $\pi_i(a_{ij}|g_i(\cdot, p_{-i}^{a^k_{-i}}))$ when players in the other populations choose a^k_{-i} in the previous period.
- The other players do not change their strategies for the period.

This process is a Markov chain with a state space A^k. Let $\{a^k(t) \in A^k\}_{t=1}^{\infty}$ be the Markov chain, which we call a stochastic best response process in $\mathbf{g}^k$ with $(\pi_i)_{i\in N}$. Let $\Pr[a^k \to b^k]$ be the transition probability from $a^k = (a_{ij})_{(i,j)\in N^k}$ to $b^k = (b_{ij})_{(i,j)\in N^k}$. Then, we have

$$\Pr[a^k \to b^k] = \begin{cases} \dfrac{1}{nk}\pi_i(b_{ij}|g_i(\cdot, p_{-i}^{a^k_{-i}})) & \text{if } b^k = a^k \backslash b_{ij} \text{ and } a_{ij} \neq b_{ij}, \\ 1 - \displaystyle\sum_{(i,j)\in N^k} \sum_{b_{ij}\neq a_{ij}} \frac{1}{nk}\pi_i(b_{ij}|g_i(\cdot, p_{-i}^{a^k_{-i}})) & \text{if } b^k = a^k, \\ 0 & \text{otherwise} \end{cases}$$

where $a^k \backslash b_{ij} = (a_{i1}, \ldots, a_{ij-1}, b_{ij}, a_{ij+1}, \ldots, a_{ik}, a^k_{-i}) \in A^k$.

Because $\{a^k(t)\}_{t=1}^{\infty}$ is a finite irreducible Markov chain, its stationary distribution is unique.[3] Let $q^k \in \Delta(A^k)$ be the stationary distribution of $\{a^k(t)\}_{t=1}^{\infty}$. Note that q^k is the unique solution of

$$\sum_{b^k \in A^k} q^k(b^k) \Pr[b^k \to a^k] = q^k(a^k) \tag{2}$$

[2]It is straightforward to replace the discrete time process with a continuous time Poisson process without changing the main result.

[3]See Seneta (1981), for example.

for all $a^k \in A^k$.

We consider an example of stochastic best response processes studied by Blume (1993, 1997) and Ui (1997). Let $\mathbf{g}$ be a potential game[4] (Monderer and Shapley, 1996) and let $(\pi_i)_{i\in N}$ be logistic.

Definition 2 An n-player game $\mathbf{g}$ is a *potential game* if it has a potential function $f : A \to \mathbb{R}$ such that

$$g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i}) = f(a_i, a_{-i}) - f(a'_i, a_{-i})$$

for all $a_i, a'_i \in A_i$, $a_{-i} \in A_{-i}$, and $i \in N$.

If $\mathbf{g}$ is a potential game, then $\mathbf{g}^k$ is also a potential game.

Lemma 1 *If $\mathbf{g}$ has a potential function $f : A \to \mathbb{R}$, then $\mathbf{g}^k$ has a potential function $f^k : A^k \to \mathbb{R}$ such that*

$$f^k(a^k) = k \sum_{a\in A} f(a) p^{a^k}(a)$$

for all $a^k \in A^k$.

Proof. For $a^k, a^k \backslash a'_{ij} \in A^k$,

$$\begin{aligned}
g^k_{ij}(a^k) - g^k_{ij}(a^k \backslash a'_{ij}) &= g_i(a_{ij}, p^{a^k_{-i}}_{-i}) - g_i(a'_{ij}, p^{a^k_{-i}}_{-i}) \\
&= \sum_{a_{-i}\in A_{-i}} \left(g_i(a_{ij}, a_{-i}) - g_i(a'_{ij}, a_{-i})\right) p^{a^k_{-i}}_{-i}(a_{-i}) \\
&= \sum_{a_{-i}\in A_{-i}} \left(f(a_{ij}, a_{-i}) - f(a'_{ij}, a_{-i})\right) p^{a^k_{-i}}_{-i}(a_{-i}) \\
&= \sum_{a_{-i}\in A_{-i}} \sum_{a_i\in A_i} f(a_i, a_{-i}) \left(k p_i^{a^k_i}(a_i) - k p_i^{a^k_i \backslash a'_{ij}}(a_i)\right) p^{a^k_{-i}}_{-i}(a_{-i}) \\
&= k \sum_{a\in A} f(a) p^{a^k}(a) - k \sum_{a\in A} f(a) p^{a^k \backslash a'_{ij}}(a) \\
&= f^k(a^k) - f^k(a^k \backslash a'_{ij}),
\end{aligned}$$

which completes the proof. ∎

If $\mathbf{g}$ is a potential game and $(\pi_i)_{i\in N}$ is logistic, then a stochastic best response process in $\mathbf{g}^k$ with $(\pi_i)_{i\in N}$ has the following stationary distribution.

[4]Examples of potential games include the incomplete contract model of Hart and Moore (1990), the endogenous coalition formation model of Myerson (1977, 1991), and the congestion model of Rosenthal (1973). See Monderer and Shapley (1996) and Ui (2000).

Lemma 2 *Let* $\mathbf{g}$ *be a potential game and let* $(\pi_i)_{i\in N}$ *be logistic. Then, the stationary distribution of a stochastic best response process in* $\mathbf{g}^k$ *with* $(\pi_i)_{i\in N}$ *is* $q^k \in \Delta(A^k)$ *such that*

$$q^k(a^k) = \frac{\exp[\beta f^k(a^k)]}{\sum_{b^k\in A^k}\exp[\beta f^k(b^k)]}. \tag{3}$$

Proof. For a^k and $b^k = a^k\backslash b_{ij}$ with $a_{ij} \neq b_{ij}$, we have

$$\Pr[a^k \to b^k] = \frac{1}{nk}\pi_i(b_{ij}|g_i(\cdot, p_{-i}^{a^k_{-i}})) = \frac{1}{nk}\frac{\exp[\beta g_i(b_{ij}, p_{-i}^{a^k_{-i}})]}{\sum_{a'_i\in A_i}\exp[\beta g_i(a'_i, p_{-i}^{a^k_{-i}})]}.$$

Thus,

$$\begin{aligned}
\frac{\Pr[a^k \to b^k]}{\Pr[b^k \to a^k]} &= \exp\left[\beta\left(g_i(b_{ij}, p_{-i}^{a^k_{-i}}) - g_i(a_{ij}, p_{-i}^{a^k_{-i}})\right)\right] \\
&= \exp\left[\beta\left(g^k_{ij}(b^k) - g^k_{ij}(a^k)\right)\right] \\
&= \exp\left[\beta\left(f^k(b^k) - f^k(a^k)\right)\right] \\
&= \frac{\exp\left[\beta f^k(b^k)\right]}{\exp\left[\beta f^k(a^k)\right]} = \frac{q^k(b^k)}{q^k(a^k)}
\end{aligned}$$

where $q^k \in \Delta(A^k)$ is given in (3). This implies that

$$q^k(b^k)\Pr[b^k \to a^k] = q^k(a^k)\Pr[a^k \to b^k] \tag{4}$$

for a^k and $b^k = a^k\backslash b_{ij}$ with $a_{ij} \neq b_{ij}$. Note that (4) is also true when $b^k = a^k$ and when $b^k \neq a^k\backslash b_{ij}$ for any $b_{ij} \in A_i$ and $(i,j) \in N^k$. Thus, (4) is true for all $a^k, b^k \in A^k$. Therefore,

$$\sum_{b^k\in A^k} q^k(b^k)\Pr[b^k \to a^k] = q^k(a^k)\sum_{b^k\in A^k}\Pr[a^k \to b^k] = q^k(a^k)$$

for all $a^k \in A^k$, which implies that q^k is a stationary distribution of the stochastic best response process. ∎

Let $a^* \in A$ be a potential maximizer such that $f(a^*) > f(a)$ for all $a \neq a^*$. Let $a^{*k} \in A^k$ be such that $a^*_{ij} = a^*_i$ for all $(i,j) \in N^k$. Then, $f^k(a^{*k}) > f^k(a^k)$ for all $a^k \neq a^{*k}$. Thus, it is straightforward to see that $\lim_{\beta\to\infty} q^k(a^{*k}) = 1$. In other words, as the error variance goes to zero, the support of the stationary distribution converges to a singleton $\{a^{*k}\}$. The strategy profile $a^* \in A$ satisfying this property is said to be stochastically stable.[5] We will discuss this example again in Section 5.

[5] Monderer and Shapley (1996) remarked that, at least technically, a potential function defines a refinement concept by a potential maximizer. This example provides one justification for it. For another justification, see Ui (2001a).

4. THE MAIN RESULT

In the stationary situation of a Markov chain $\{a^k(t)\}_{t=1}^{\infty}$, the probability of transition into a^k must balance the probability of transition out of a^k:

$$\sum_{b^k \neq a^k} q^k(b^k) \Pr[b^k \to a^k] = \sum_{b^k \neq a^k} q^k(a^k) \Pr[a^k \to b^k]$$

for all $a^k \in A^k$ where $q^k \in \Delta(A^k)$ is a stationary distribution of $\{a^k(t)\}_{t=1}^{\infty}$. We call it the balance condition. Note that the balance condition is an immediate consequence of (2). In fact, (2) implies that

$$\sum_{b^k \neq a^k} q^k(b^k) \Pr[b^k \to a^k] = q^k(a^k) \Big(1 - \Pr[a^k \to a^k]\Big) = \sum_{b^k \neq a^k} q^k(a^k) \Pr[a^k \to b^k].$$

In the main result, we assume a stronger balance condition. We say that a Markov chain $\{a^k(t)\}_{t=1}^{\infty}$ satisfies the detailed balance condition[6] if, in the stationary situation, the probability of transition from b^k to a^k balances the probability of transition from a^k to b^k for all $a^k, b^k \in A^k$.

Definition 3 A Markov chain $\{a^k(t)\}_{t=1}^{\infty}$ satisfies the *detailed balance condition* if

$$q^k(b^k) \Pr[b^k \to a^k] = q^k(a^k) \Pr[a^k \to b^k] \tag{5}$$

for all $a^k, b^k \in A^k$.

The example discussed in the previous section satisfies the detailed balance condition because (4) is true for all $a^k, b^k \in A^k$.

A stationary distribution q^k induces a probability distribution Q^k over a mixed strategy space $\mathcal{A}$ such that, for $\mathcal{O} \subseteq \mathcal{A}$, $Q^k(\mathcal{O})$ is a probability of an event $p^{a^k} \in \mathcal{O}$:

$$Q^k(\mathcal{O}) = \sum_{a^k:\, p^{a^k} \in \mathcal{O}} q^k(a^k).$$

The main result states that if a stochastic best response process $\{a^k(t)\}_{t=1}^{\infty}$ satisfies the detailed balance condition, then the support of Q^k converges to the set of quantal response equilibria as k goes to infinity.

Theorem 2 *Let $\{a^k(t)\}_{t=1}^{\infty}$ be a stochastic best response process in $\mathbf{g}^k$ with $(\pi_i)_{i \in N}$. Suppose that $\{a^k(t)\}_{t=1}^{\infty}$ satisfies the detailed balance condition for all $k \geq 1$. Then, for any open set $\mathcal{O} \subset \mathcal{A}$ containing all the quantal response equilibria of $\mathbf{g}$ with $(\pi_i)_{i \in N}$,*

$$\lim_{k \to \infty} Q^k(\mathcal{O}) = 1.$$

[6]The detailed balance condition is usually defined for continuous time Markov processes where a transition probability is replaced by a transition probability rate. See Risken (1984), for example.

The proof is presented in four steps. For the first step, consider a conditional stationary distribution $q^k(\cdot|a^k_{-i}) \in \Delta(A^k_i)$ such that

$$q^k(a^k_i|a^k_{-i}) = \frac{q^k(a^k_i, a^k_{-i})}{\sum_{b^k_i \in A^k_i} q^k(b^k_i, a^k_{-i})}$$

for all $a^k_i \in A^k_i$. We claim that if $a^k_i = (a_{i1}, \dots, a_{ik})$ is a random variable and distributed according to $q^k(\cdot|a^k_{-i}) \in \Delta(A^k_i)$, then $a_{i1}, \dots, a_{ik}$ are independently and identically distributed according to $\pi_i(g_i(\cdot, p^{a^k_{-i}}_{-i})) \in \mathcal{A}_i$.

Lemma 3 *If $\{a^k(t)\}_{t=1}^{\infty}$ satisfies the detailed balance condition, then*

$$q^k(a^k_i|a^k_{-i}) = \prod_{j \in \{1,\dots,k\}} \pi_i(a_{ij}|g_i(\cdot, p^{a^k_{-i}}_{-i}))$$

for all $a^k_i \in A^k_i$, $a^k_{-i} \in A^k_{-i}$, and $i \in N$.

Proof. Let $a^k = (a_{ij})_{(i,j)\in N^k} \in A^k$ and $b^k_i = (b_{ij})_{j\in\{1,\dots,k\}} \in A^k_i$. For $j \in \{0, \dots, k\}$, let $x^k(j) \in A^k$ be such that $x^k(0) = a^k$ and $x^k(j) = x^k(j-1)\backslash b_{ij}$ for $j \geq 1$. Note that $x^k(j) = (b_{i1}, \dots, b_{ij}, a_{ij+1}, \dots, a_{ik}, a^k_{-i})$ for $j \geq 1$.

We first show that, for $j \geq 1$,

$$q^k(x^k(j-1))\pi_i(b_{ij}|g_i(\cdot, p^{a^k_{-i}}_{-i})) = q^k(x^k(j))\pi_i(a_{ij}|g_i(\cdot, p^{a^k_{-i}}_{-i})). \tag{6}$$

If $a_{ij} = b_{ij}$, then (6) is clearly true. If $a_{ij} \neq b_{ij}$, then we have transition probabilities

$$\begin{aligned} \Pr[x^k(j-1) \to x^k(j)] &= \frac{1}{nk}\pi_i(b_{ij}|g_i(\cdot, p^{a^k_{-i}}_{-i})), \\ \Pr[x^k(j) \to x^k(j-1)] &= \frac{1}{nk}\pi_i(a_{ij}|g_i(\cdot, p^{a^k_{-i}}_{-i})). \end{aligned}$$

Plugging these equations into the detailed balance condition (5), we have (6).

Consider the following summation of (6):

$$\sum_{h=1}^{j} \sum_{b_{ih}\in A_i} q^k(x^k(j-1))\pi_i(b_{ij}|g_i(\cdot, p^{a^k_{-i}}_{-i})) = \sum_{h=1}^{j} \sum_{b_{ih}\in A_i} q^k(x^k(j))\pi_i(a_{ij}|g_i(\cdot, p^{a^k_{-i}}_{-i})).$$

Then,

$$\begin{aligned} \pi_i(a_{ij}|g_i(\cdot, p^{a^k_{-i}}_{-i})) &= \frac{\sum_{h=1}^{j}\sum_{b_{ih}\in A_i} q^k(x^k(j-1))\pi_i(b_{ij}|g_i(\cdot, p^{a^k_{-i}}_{-i}))}{\sum_{h=1}^{j}\sum_{b_{ih}\in A_i} q^k(x^k(j))} \\ &= \begin{cases} \dfrac{q^k(x^k(0))}{\sum_{b_{i1}\in A_i} q^k(x^k(1))} & \text{if } j = 1, \\ \dfrac{\sum_{h=1}^{j-1}\sum_{b_{ih}\in A_i} q^k(x^k(j-1))}{\sum_{h=1}^{j}\sum_{b_{ih}\in A_i} q^k(x^k(j))} & \text{if } j \geq 2. \end{cases} \end{aligned}$$

Therefore,

$$\prod_{j\in\{1,\dots,k\}} \pi_i(a_{ij}|g_i(\cdot,p_{-i}^{a_{-i}^k})) = \frac{q^k(x^k(0))}{\sum_{h=1}^k \sum_{b_{ih}\in A_i} q^k(x^k(k))}$$
$$= \frac{q^k(a^k)}{\sum_{b_i^k\in A_i^k} q^k(b_i^k, a_{-i}^k)} \qquad = q^k(a_i^k|a_{-i}^k),$$

which completes the proof. ∎

For the second step, define $d_i^{a_i}(p) = |p_i(a_i) - \pi_i(a_i|g_i(\cdot,p_{-i}))|$ for $p \in \mathcal{A}$. Note that if $\max_{i\in N}\max_{a_i\in A_i} d_i^{a_i}(p) = 0$ then p is a quantal response equilibrium.

By the law of large numbers, if $a_{i1},\dots,a_{ik}$ are independently and identically distributed according to $\pi_i(g_i(\cdot,p_{-i}^{a_{-i}^k}))$, then $p_i^{a_i^k}(a_i)$ converges to $\pi_i(a_i|g_i(\cdot,p_{-i}^{a_{-i}^k}))$ in probability as k goes to infinity. In other words, $d_i^{a_i}(p^{a^k}) = |p_i^{a_i^k}(a_i) - \pi_i(a_i|g_i(\cdot,p_{-i}^{a_{-i}^k}))|$ converges to zero in probability.

Lemma 4 *Fix $i \in N$ and $a_i \in A_i$. If $\{a^k(t)\}_{t=1}^\infty$ satisfies the detailed balance condition, then, for $\delta > 0$,*

$$\sum_{a_i^k\in A_i^k:\, d_i^{a_i}(p^{a^k})>\delta} q^k(a_i^k|a_{-i}^k) \le \frac{1}{4k\delta^2} \tag{7}$$

for all $a_{-i}^k \in A_{-i}^k$.

Proof. If a_i^k is distributed according to $q^k(\cdot|a_{-i}^k) \in \Delta(A_i^k)$, then $a_{i1},\dots,a_{ik}$ are independently and identically distributed according to $\pi_i(g_i(\cdot,p_{-i}^{a_{-i}^k})) \in \mathcal{A}_i$ by Lemma 3. Thus,

$$\mathrm{E}[p_i^{a_i^k}(a_i)] = \pi_i(a_i|g_i(\cdot,p_{-i}^{a_{-i}^k})),$$
$$\mathrm{Var}[p_i^{a_i^k}(a_i)] = \pi_i(a_i|g_i(\cdot,p_{-i}^{a_{-i}^k}))\left(1-\pi_i(a_i|g_i(\cdot,p_{-i}^{a_{-i}^k}))\right)/k \le \frac{1}{4k}.$$

Plugging the above into the Chebyshev inequality:

$$\Pr\left[|p_i^{a_i^k}(a_i) - \mathrm{E}[p_i^{a_i^k}(a_i)]| > \delta\right] \le \frac{\mathrm{Var}[p_i^{a_i^k}(a_i)]}{\delta^2},$$

we have

$$\sum_{a_i^k\in A_i^k:\, d_i^{a_i}(p^{a^k})>\delta} q^k(a_i^k|a_{-i}^k) \le \frac{1}{4k\delta^2},$$

which completes the proof. ∎

For the third step, consider a subset of $\mathcal{A}$ such that

$$\mathcal{D}^\delta = \left\{ p \in \mathcal{A} \;\middle|\; \max_{i \in N} \max_{a_i \in A_i} d_i^{a_i}(p) > \delta \right\}$$

where $\delta > 0$. Note that $\mathcal{D}^\delta$ does not contain any quantal response equilibria. Using Lemma 4, we claim that, in the stationary situation, a probability of an event $p^{a^k} \in \mathcal{D}^\delta$ goes to zero as k goes to infinity.

Lemma 5 *If $\{a^k(t)\}_{t=1}^\infty$ satisfies the detailed balance condition, then*

$$\lim_{k \to \infty} Q^k(\mathcal{D}^\delta) = 0.$$

Proof. Plugging $q^k(a_i^k | a_{-i}^k) = q^k(a^k) / \sum_{b_i^k \in A_i^k} q^k(b_i^k, a_{-i}^k)$ into (7) in Lemma 4,

$$\sum_{a_i^k \in A_i^k : d_i^{a_i}(p^{a^k}) > \delta} q^k(a^k) \leq \frac{1}{4k\delta^2} \sum_{b_i^k \in A_i^k} q^k(b_i^k, a_{-i}^k).$$

Taking summation over all $a_{-i}^k \in A_{-i}^k$,

$$\sum_{a^k \in A^k : d_i^{a_i}(p^{a^k}) > \delta} q^k(a^k) \leq \frac{1}{4k\delta^2}.$$

Note that if $p^{a^k} \in \mathcal{D}^\delta$ then $d_i^{a_i}(p^{a^k}) > \delta$ for some $a_i \in A_i$ and $i \in N$. Thus, we have

$$\begin{aligned} Q^k(\mathcal{D}^\delta) &= \sum_{a^k : p^{a^k} \in \mathcal{D}^\delta} q^k(a^k) \\ &\leq \sum_{i \in N} \sum_{a_i \in A_i} \left(\sum_{a^k \in A^k : d_i^{a_i}(p^{a^k}) > \delta} q^k(a^k) \right) \\ &\leq \sum_{i \in N} \sum_{a_i \in A_i} \frac{1}{4k\delta^2} \\ &= \frac{\sum_{i \in N} |A_i|}{4k\delta^2}. \end{aligned}$$

Therefore, $\lim_{k \to \infty} Q^k(\mathcal{D}^\delta) \leq \lim_{k \to \infty} \frac{\sum_{i \in N} |A_i|}{4k\delta^2} = 0.$ ∎

We now report the forth and final step.

Proof of Theorem 2. By the absolute continuity of a probability density function f_i, it is straightforward to see that $\pi_i : \mathbb{R}^{A_i} \to \mathcal{A}_i$ is continuous. Since $g_i(a_i, p_{-i})$ is continuous in p_{-i}, $\pi_i(g_i(\cdot, p_{-i}))$ is continuous in p_{-i}. Since $p_i(a_i)$ is continuous in p_i,

it must be true that $d_i^{a_i}(p) = |p_i(a_i) - \pi_i(a_i|g_i(\cdot, p_{-i}))|$ is continuous in p. Therefore, $\max_{i\in N} \max_{a_i\in A_i} d_i^{a_i}(p)$ is continuous in p.

Let $\mathcal{O} \subset \mathcal{A}$ be an open set containing all the quantal response equilibria. Then, $\mathcal{A}\backslash\mathcal{O}$ is a closed and compact set containing no quantal response equilibria. Define

$$\delta' = \min_{p\in\mathcal{A}\backslash\mathcal{O}} \max_{i\in N} \max_{a_i\in A_i} d_i^{a_i}(p).$$

It must be true that $\delta' > 0$. To see this, suppose that $\delta' = 0$. Then, there exists $p^* \in \mathcal{A}\backslash\mathcal{O}$ such that $\max_{i\in N} \max_{a_i\in A_i} d_i^{a_i}(p^*) = 0$. This implies that $p^* \in \mathcal{A}\backslash\mathcal{O}$ is a quantal response equilibrium, which is a contradiction.

If $p \in \mathcal{A}\backslash\mathcal{O}$, then $\max_{i\in N} \max_{a_i\in A_i} d_i^{a_i}(p) \geq \delta'$, and thus $p \in \mathcal{D}^\delta$ for $0 < \delta < \delta'$. This implies that $Q^k(\mathcal{A}\backslash\mathcal{O}) \leq Q^k(\mathcal{D}^\delta)$. By Lemma 5,

$$\lim_{k\to\infty} Q^k(\mathcal{O}) = \lim_{k\to\infty} \left(1 - Q^k(\mathcal{A}\backslash\mathcal{O})\right) \geq \lim_{k\to\infty} \left(1 - Q^k(\mathcal{D}^\delta)\right) = 1,$$

which completes the proof. ∎

5. DISCUSSIONS

We discuss one implication of Theorem 2 in the light of Theorem 1 using the example in Section 3. Let $\mathbf{g}$ be a potential game with a potential function f and let $(\pi_i)_{i\in N}$ be logistic. Suppose that $f(a^*) > f(a)$ for all $a \neq a^*$. Let $p^* \in \mathcal{A}$ be such that $p_i^*(a_i^*) = 1$ for all $i \in N$. Let $\{a^k(t)\}_{t=1}^\infty$ be a stochastic best response process in $\mathbf{g}^k$ with $(\pi_i)_{i\in N}$. The stationary distribution q^k is given by Lemma 2.

Assume that β is very small. By Theorem 1, a quantal response equilibrium is unique. Let p^{**} be the unique quantal response equilibrium. By Theorem 2, for any open set $\mathcal{O} \subset \mathcal{A}$ containing p^{**}, $Q^k(\mathcal{O})$ is close to 1 if k is very large. Thus, when the error variance is very large, the support of Q^k is close to a singleton $\{p^{**}\}$ for very large k. Fix k.

Assume that β is very large. Then, $Q^k(\{p^*\})$ is close to 1 since $Q^k(\{p^*\}) = q^k(a^{*k}) \to 1$ as $\beta \to \infty$. Thus, when the error variance is very small, the support of Q^k is close to a singleton $\{p^*\}$.

Combining the above observations, if k is very large, then the support of Q^k is close to a singleton not only when the error variance is very small but also when the error variance is very large. In other words, both small noise and large noise generate uniqueness.

We have an open question whether or not p^* and p^{**} are on the same branch of the graph of the logit equilibrium correspondence defined in Section 2. That is, we do

not know whether or not the potential maximizer and the limiting logit equilibrium coincide.

The above observations also raise the question of how two different approaches to equilibrium selection, the evolutionary game approach and the incomplete information game approach, are related. As noted in the introduction, also in incomplete information games, both small noise and large noise generate uniqueness (Ui, 2001b; Morris and Shin, 2002). This suggests some connection between the evolutionary game approach and the incomplete information game approach.

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