Habit formation, self-deception, and self-control*

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Abstract

Recent research in psychology suggests that successful self-control is attributed to developing adaptive habits rather than resisting temptation. However, developing good habits itself is a self-regulating process, and people often fail to accumulate good habits. This study axiomatically characterizes a dynamic decision model where an agent may form a deceptive belief about his future preference: the agent correctly anticipates his future preference by considering the effect of habits; however, he is also tempted to ignore the habit formation. Self-control must be exerted for resisting such a self-deceptive belief. Our model is flexible enough to accommodate a variety of habit formation and explains behavioral puzzles related to gym attendance, self-control fatigue, and demand for commitment.

Keywords: habit formation, self-deception, projection bias, self-control fatigue, addiction, endogenous time preference

JEL Classification: D15, D91

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1 Introduction

1.1 Objective

In many instances, people cannot exert self-control despite their willingness to control themselves to attain long-term goals, such as healthy living or saving money. Recent research in psychology (e.g., Adriaanse et al. (2014)) suggests that a successful self-control is attributed to developing adaptive habits rather than resisting temptation. Once established, good habits work as effortless strategies to achieve long-term goals.

However, developing good habits itself is a self-regulating process. People often fail to accumulate good habits. Using the date set of U.S. health clubs, Della Vigna and Malmendier (2006) find that consumers tend to choose the more expensive plan because it reduces the marginal cost of attending the gym with presumably believing that this plan will encourage them to attend the gym in the future. Nevertheless, they do not actually attend the gym very often. If drug addicts correctly anticipate their temptation from using drugs in the future, they will prefer commitment, for example, checking in a rehabilitation center. In contrast, there is considerable evidence that drug addicts tend to delay such treatments and fail to break bad habits (Cunningham et al. (1993)).

This paper identifies a temptation associated exclusively with habit formation, which is separated from intrinsic self-control problems (e.g., healthy eating or regular physical exercise). We hypothesize that an agent may form a deceptive belief about his future preference. Meanwhile, the agent correctly anticipates his future preference by considering the effect of habits triggered by the current action. However, he is simultaneously tempted to ignore the effect of the habit formation and continues to believe his future preference is identical with the current one. Self-control must be exerted for resisting such a self-deceptive belief.

We can derive a general insight from our hypothesis. If the agent is tempted to ignore the effect of habits, developing good habits is more inhibited. Moreover, if the effect of bad habits is ignored, bad habits tend to become even worse.

Our hypothesis is motivated by the empirical evidence, called the projection bias. The projection bias is known as the individuals' tendency of exaggerating the degree to which their future tastes will resemble their current tastes; it has been interpreted as people's naiveté or misperception about their own future preferences (see Loewenstein et al. (2003)). We suggest an alternative interpretation. Agents are perfectly sophisticated about anticipating their future preference, but are also tempted to ignore the effect of habit formation on their future preference.

1.2 Outline of the model

To formalize the idea, we provide an axiomatically characterized model of dynamic choice in which the role of temptation is specified as the current status-quo preference motivated by the projection bias. Here let us give a brief overview with some simplifications. Let h denote a history of consumption path up to the current period. Given history h, the agent holds his preference over recursive consumption problems, which are sets of pairs of current consumption and recursive consumption problem to face in the next period. Let x denote such recursive consumption problem. Then, its element is typically denoted by (c, z), where c denotes current consumption and z denotes recursive consumption problem to face in the next period.

The value of decision problem x conditional on history h is represented by $W(x|h)$ having a history-dependent version of Gul and Pesendorfer (2001, 2004):

$$W(x|h) \quad = \quad \max_{(c,z)\in x} \left\{ U(c, z|h) + \alpha(h) \left(V(c, z|h) - \max_{(\widetilde{c},\widetilde{z})\in x} V(\widetilde{c}, \widetilde{z}|h) \right) \right\}$$

where $\alpha(h)$ is the measure of cost of self-control conditional on h, with the specifications of $U(c, z|h)$ and $V(c, z|h)$ as below:

$$U(c, z|h) = u(c|h) + \beta(h)W(z|hc), \tag{1}$$

$$V(c, z|h) = u(c|h) + \beta(h) \max_{(c',z')\in z} V(c', z'|h). \tag{2}$$

On the one hand, $U(c, z|h)$ represents preference over commitment plans for habit formation, which satisfies the recursive form with $u(c|h)$ describing utility per period of consumption c and $\beta(h)$ being the discount factor conditional on h, and $W(z|hc)$ being the continuation value function conditional on updated history hc. On the other hand, the temptation utility term $V(\cdot|h)$ falls in the class of stationary discounted utility due to Koopmans (1960), as it remains invariant after the given history h. Self-control is exerted for resisting this deceptive belief.

1.3 Applications

As motivated in Section 1.1, developing good habits is mainly to alleviate intrinsic self-control issues such as attending the gym or healthy eating. To incorporate this aspect explicitly, in our primary applications, we assume c itself to be a menu of actions, such as regular physical exercise or drugs consumption, and an instantaneous utility function $u(c|h)$ to be a self-control utility of Gul and Pesendorfer (2001). Such a menu c is regarded as a contract allowing for some feasible actions, and x, corresponding to a menu of menus under this specification, is interpreted as a menu of contracts.

As mentioned in Section 1.1, Della Vigna and Malmendier (2006) find that many consumers continue to choose the more expensive plan (called the monthly contract) even though they do not attend the gym very often. A main reason they suggested is overconfidence or naiveté about self-control. We illustrate that although an agent is perfectly sophisticated, he may choose the monthly contract, and nevertheless, end up with low gym attendance afterwards. Before choosing the monthly contract, the agent is assumed to have bad habits (low gym attendance); however, he is tempted to ignore the effect of this bad habits on exercise and believe that his ability to exert self-control is enough for encouraging gym attendance in the future.

Experimental studies report that an increase in cognitive overload leads subjects to lose self-control in decision making (Shiv and Fedorikhin (1999)). This observation suggests that individuals may fail to form good habits (e.g., habits for physical exercise) due to the loss of self-control caused by fatigue. Why do individuals continue to exercise until they lose self-control and consequently fail to form good habits? Our model illustrates that even if an individual is sophisticated and correctly recognizes the effects of exercising beyond the limit, the projection bias will cause him to over-exercise until burnout.

Gul and Pesendorfer (2007) illustrate an intertemporal choice pattern where a drug consumption increases over time up to some period, then jumps down to zero by checking in the rehabilitation center (interpreted as a commitment device), and repeats the same cycle afterward. However, there is considerable evidence that drug addicts delay such treatments (Cunningham et al. (1993)). Compared with the findings of Gul and Pesendorfer (2007), we illustrate that checking in the rehabilitation center tends to be delayed if a self-control problem over the projection bias exists.

Finally, we consider the role of self-control in the endogenous formation of time preference (Becker and Mulligan (1997), Shi and Epstein (1993), and Stern (2006)). Empirical studies have documented a strong correlation between time preference and socioeconomic status (see, e.g., Lawrance (1991), Barsky et al. (1997), Tanaka et al. (2010), Dohmen et al. (2016)). Hence, from a normative perspective, an agent is willing to invest in "patience capital" such as education or health investment, which makes him more patient in the next period. However, the agent may be tempted to ignore the habit formation by the projection bias. We compare this type of self-control problem with present bias or temptation from the immediate consumption, as in Gul and Pesendorfer (2004, 2007). In terms of self-control, whether the temptation from an entire continuation path or the temptation from immediate consumption is more costly remains ambiguous. We verify that the accumulation of the patience capital under the projection bias is reduced compared with no such bias; however it is greater than the temptation from the immediate consumption.

1.4 Related literature

Our model allows that an entire continuation life path plays the role of a tempting item. This contrasts with the model of habit formation by Gul and Pesendorfer (2007), in which the agent is tempted only by the current consumption. Noor (2007) axiomatizes a future temptation (FT) utility function where equations (1) and (2) are history-independent and a temptation utility function is stationary with a possibly different instantaneous utility $v(c)$ and a discount factor γ. Noor (2011) takes a time-invariant choice correspondence as a primitive and rationalizes it as a maximization of $U(\cdot) + V(\cdot)$ where U has a recursive form similar to (1) and V also has a recursive form with a general instantaneous function v and the continuation value being specified as a linear combination of $W(z)$ and $\max_{(c',z')\in z} V(c', z')$.

Ahn et al. (2020) establish a recursive representation with naiveté. Although their primary concern is a recursive model allowing for naiveté, as a special case, they axiomatize a model, called a sophisticated quasi-hyperbolic discounting representation, where (1) and (2) are history-independent and the continuation value of the temptation utility is the same as W with the current discount factor being specified with $\delta\beta$, which exhibits the quasi-hyperbolic discounting only when succumbed to this temptation.

Ikeda and Ojima (2021) treat willpower as a state variable and analyze the change in willpower through time due to self-control behavior and its effect on intertemporal consumption choice and the impact of fatigue due to self-control. This paper also assumes that a consumer is tempted by a consumption path of tempting goods over the future.

Self-deception has been modeled by several papers. For instance, Brunnermeier and Parker (2005) consider an agent who optimally chooses a subjective belief over states, which may be different from the true probability distribution. An optimal belief of increasing anticipated future utilities is a trade-off from poor outcomes resulting from decisions made based on optimistic belief. In their model, this optimization is an unconscious process, whereas in our model, the agent may intentionally form a self-deceptive belief to ignore habit formation. Epstein and Kopylov (2007) take a preference over menus of acts as a primitive and model an agent who is tempted to have a more pessimistic belief about the states of nature compared with his prior. Kopylov and Noor (2018) model an agent with weak resolve by taking preference over menus of menus. When the agent chooses a menu from a menu of menus, he loses confidence on his normative preference in the future and is tempted to deviate from it.

Finally, habit formation as endogenous change in per-period utility function has been extensively studied in much of the literature, as has habit formation in time preference, mentioned in Section 1.3. For theoretical treatments, one can raise the earlier work by Ryder and Heal (1973) and recent works on axiomatization by Rozen (2010) and Tserenjigmid (2020). In these literatures, future preferences are internalized into the current preference, so that the desirable habit formation plan is implemented ex post. As we have explained, a desirable habit formation is not necessarily achieved when habit formation involves a self-regulating process.

2 The model

2.1 Setting

Let C be the consumption space per period, which is a compact metric space. Given a compact metric space Y, let $\Delta(Y)$ denote the set of Borel probability measures, which is again a compact metric space with respect to the Prokhorov metric. Given a compact metric space Y, let $\mathcal{K}(Y)$ denote the set of closed (hence compact) subsets of Y, which is again a compact metric space with regard to the Hausdorff metric.

Let $\mathcal{Z}$ be the recursive domain of consumption problems satisfying the recursive homeomorphism

$$\mathcal{Z} \simeq \mathcal{K}(\Delta(C \times \mathcal{Z})).$$

See Gul and Pesendorfer (2004) for its details. An element of $\mathcal{Z}$ is called *menu*. Because of the homeomorphism, a menu $z \in \mathcal{Z}$ is viewed as a compact set consisting of lotteries. A singleton menu is denoted by $\{l\}$, where $l \in \Delta(C \times \mathcal{Z})$. Also, a lottery degenerate on $(c, x) \in \Delta(C \times \mathcal{Z})$ is simply denoted by (c, x) if no confusion arises.

In each time period, preference is defined on $\mathcal{Z}$ and it depends on past consumption. The set of histories of past consumption can be defined as

$$C^\infty = \{(\cdots, c_{-2}, c_{-1}) \,|\, c_{-t} \in C \text{ for all } t = 1, 2, \cdots\}.$$

The set C^∞ is endowed with the product topology, which is metrizable.[1] Moreover, C^∞ is compact because C is compact. For all $c \in C$ and $h = (\cdots, c_{-2}, c_{-1}) \in C^\infty$, let hc denote the updated history $(\cdots, c_{-2}, c_{-1}, c)$.

Fix an arbitrary history $\bar{h} \in C^\infty$ as an initial history. Define $\bar{h}C^t$ as the set of histories that evolve from $\bar{h}$ up to period t, that is,

$$\bar{h}C^t := \{(\bar{h}, c_{-t}, \cdots, c_{-1}) \,|\, c_{-\tau} \in C \text{ for all } \tau = 1, \cdots, t\} \subset C^\infty$$

with the convention $\bar{h}C^0 := \{\bar{h}\}$. Let

$$H := \bigcup_{t=0}^{\infty} \bar{h}C^t \subset C^\infty$$

denote the set of all histories that can evolve from $\bar{h}$. It is easy to verify that H is a dense subset of C^∞. That is, $\overline{H} = C^\infty$. Given any history $h \in H$, preference relation after h defined over $\mathcal{Z}$ is denoted by $\succsim_h$. Let $\{\succsim_h\}_{h \in H}$ denote the process of such preferences.

2.2 Functional form

Definition 1 We say that $\succsim$ admits a projection-bias temptation (PBT) representation if there exist continuous functions $W : \mathcal{Z} \times H \to \mathbb{R}$, $u : C \times H \to \mathbb{R}$, $\beta : H \to (0, 1)$ and $\alpha : H \to \mathbb{R}_+$ such that for each h, the function $W(\cdot|h)$ represents $\succsim_h$ and satisfies

$$W(x|h) \quad = \quad \max_{l \in x} \left\{ U(l|h) + \alpha(h) \left(V(l|h) - \max_{m \in x} V(m|h) \right) \right\} \tag{3}$$

[1] Let d be a metric on C. A metric on H is defined by $\rho(h, h') = \sum_{t=1}^{\infty} \frac{1}{2^t} \cdot \frac{d(c_{-t}, c'_{-t})}{1 + d(c_{-t}, c'_{-t})}$, where $h = (\cdots, c_{-2}, c_{-1})$, $h' = (\cdots, c'_{-2}, c'_{-1})$. See Aliprantis and Border (1994, p.89).

for all $x \in \mathcal{Z}$, where

$$U(l|h) = \int_{C\times\mathcal{Z}} \{u(c|h) + \beta(h)W(z|hc)\}\,\mathrm{d}l(c,z), \tag{4}$$

$$V(l|h) = \int_{C\times\mathcal{Z}} \left\{u(c|h) + \beta(h)\max_{m\in z} V(m|h)\right\}\mathrm{d}l(c,z). \tag{5}$$

Moreover, there exist $\overline{c}, \underline{c} \in C$ such that $u(\underline{c}|h) = 0$ for all $h \in H$, and $u(\overline{c}|h) = u(\overline{c}|h') > 0$ for all $h, h' \in H$.

A basic structure of the PBT representation $W(x|h)$ is a history-dependent version of the self-control representation of Gul and Pesendorfer (2004). Two component functions $U(l|h)$ and $V(l|h)$ are interpreted as normative and temptation utility functions, respectively, as in Gul and Pesendorfer (2004). The negative term $(V(l|h) - \max_{m\in x} V(m|h))$ is regarded as self-control costs, which are opportunity costs in terms of temptation utilities. The parameter $\alpha(h)$ captures the intensity of self-control costs.

The two functions $U(\cdot|h)$ and $V(\cdot|h)$ have the same utility function from current consumption $u(\cdot|h)$ and discount factor $\beta(h)$ for the continuation utility. The difference is how the menu for the rest of the horizon is evaluated. From the normative perspective, the future menu is evaluated by the recursive utility $W(z|hc)$ at the updated history hc. That is, the normative utility takes into account the habit formation. On the other hand, the temptation utility has a stationary recursive form, and hence, the future menu is evaluated by the utility function and the discount factor determined up to history h over the rest of the horizon. Therefore, a self-control problem in this model is associated with conflicts between habit formation and resistance to it.

We interpret this form of temptation as a reflection of the projection bias. The projection bias is known as the individuals' tendency to exaggerate the degree to which their future tastes will resemble their current tastes (see Loewenstein et al. (2003)). Although the projection bias has been interpreted as naiveté or misperception of one's own future preference, we take an alternative interpretation here. The agent is perfectly sophisticated for anticipating his future utility function. From the normative perspective, the agent correctly anticipates the effect of habit formation, while he is tempted to ignore the effect of habit formation and believe that his future preference will resemble the current preference.

2.3 Different channels of habit formation

In the PBT representation, each of its components, an instantaneous utility $u(\cdot|h)$, a discount factor $\beta(h)$, and an intensity of projection bias $\alpha(h)$, admits history dependence. Depending on each component or combination of them, the representation can explain a variety of habit formation.

The history dependence of $u(\cdot|h)$ describes a situation in which some aspects of utility are more trained through experience, as well as a situation in which bad preference habits are developed through the consumption of addictive goods or indulging oneself. For instance, we can give examples of developing exercise habits by continuing to exercise, losing self control by exercising too much, and developing bad habits by continuing to consume addictive goods.

The history dependence of $\beta(h)$ can capture the effect of consumption history on impatience. In the literature of endogenous time preference, there have been various debates on whether a discount factor is increasing or decreasing with respect to consumption history (see Shi and Epstein (1993)). Also, in Becker and Mulligan (1997), the choice of investment behavior (such as investment in health or education) is considered to affect the discount factor.

The parameter $\alpha(h)$ expresses the degree of self-deceptive belief formation that ignores history dependence. The history dependence of $\alpha(h)$ means that this degree itself can vary with history in general. For example, imagine a situation where good habit formation requires continued exercise for several periods in the future. More formally, $u(\cdot|h)$ does not change with history for a while, but only after a sufficient period of exercise. In this case, the self-deceptive belief will be self-fulfilling while $u(\cdot|h)$ remains unchanged, and hence, the confidence in the belief will be reinforced. This is a situation where $\alpha(h)$ is increasing in history.

Though good habits can enhance people's welfare, accumulating good habits itself is a self-regulating process. The agent may fail to accumulate good habits under the temptation from the projection bias even when he normatively prefers to accumulate good habits. To verify this insight, in the subsequent sections from 3.1 to 3.4, we investigate various implications from the interaction of $u(\cdot|h)$ and $\alpha(h)$. Subsection 3.1 illustrates that the PBT model can explain patterns of the projection bias. Subsections 3.2 and 3.3 show that the existence of $\alpha(h)$ can explain behavioral puzzles related to exercise habits. Finally, in subsection 3.4, we will see that bad habit formation of addictive goods is likely to be exacerbated under $\alpha(h)$.

In subsection 3.5, by adopting the framework of endogenous time preference, we examine how investment in patience is accumulated over time under the projection bias. Our main focus here is the interaction between $\beta(h)$ and $\alpha(h)$. In this analysis, the agent chooses consumption, investment in physical capital, and investment in human capital (corresponding to patience) in each period within the budget constraint. Since the marginal cost of investment in human capital depends on past human capital accumulation, $u(\cdot|h)$ also becomes history-dependent indirectly through the budget constraint.

2.3.1 Specification of $u(\cdot|h)$

To distinguish an intrinsic self-control problem (e.g., self-regulation in exercising) from the associated self-control over the projection bias, we often adopt the following specific setting. Assume that there exists some compact metric space A of consumption and C is written as $C = \mathcal{K}(\Delta(A))$. Consider the following functional form: for each h,

$$W(x|h) = \max_{(c,x')\in x} \left\{ U(c,x'|h) + \alpha(h)\left(V(c,x'|h) - \max_{(\tilde{c},\tilde{x})\in x} V(\tilde{c},\tilde{x}|h) \right) \right\} \tag{6}$$

for all menus $x \in \mathcal{Z}$ of deterministic options, where

$$\begin{aligned} U(c,x'|h) &= u(c|h) + \beta(h)W(x'|hc), \\ V(c,x'|h) &= u(c|h) + \beta(h) \max_{(\tilde{c},\tilde{x})\in x'} V(\tilde{c},\tilde{x}|h), \\ u(c|h) &= \max_{a\in c} \left\{ \mathbf{u}(a) - \lambda(h)\left(\max_{a'\in c} \mathbf{v}(a') - \mathbf{v}(a) \right) \right\}. \end{aligned} \tag{7}$$

We interpret (7) as an intrinsic self-control problem, where $\mathbf{v}$ captures the temptation such as sugar cravings, drug addiction, or neglecting physical exercises, while (6) captures the associated self-control issue resisting the projection bias.

In this setting, a history h is a sequence of menu choices in the past, that is, $h = (c_0, c_1, \cdots, c_t)$ and it affects a magnitude $\lambda(h)$ of the temptation utility $\mathbf{v}$ when the agent faces a menu c. The following is an interpretation. For each history h, a choice from a menu c is given by

$$\mathcal{C}_h(c) = \arg\max_{a\in c} \mathbf{u}(a) + \lambda(h)\mathbf{v}(a).$$

We arbitrarily fix some $a(c,h) \in \mathcal{C}_h(c)$. We assume that the agent correctly anticipates his actual choice from a menu given history h. Then, each menu c is identified with the choice from it, that is, $a(c,h)$. Moreover, $\lambda(h)$ is assumed to depend on h only through the induced history of consumption. For an initial history h_0, let $\lambda(h_0) = \lambda_0$ for some $\lambda_0 > 0$. Then, a history of menus $h = (c_0, c_1, \cdots, c_t)$ is effectively regarded as the corresponding choices of consumption up to period t,

$$a(c_0) \in \arg\max_{a \in c_0} \mathbf{u}(a) + \lambda_0 \mathbf{v}(a),$$

$$a(c_1; c_0) \in \arg\max_{a \in c_1} \mathbf{u}(a) + \lambda(a(c_0))\mathbf{v}(a),$$

$$a(c_t; c_0, \cdots, c_{t-1}) \in \arg\max_{a \in c_t} \mathbf{u}(a) + \lambda(a(c_0), \cdots, a(c_{t-1}; c_0, \cdots, c_{t-2}))\mathbf{v}(a).$$

2.4 Comparison with present bias and future temptation

Our representation has a close relationship with that of Gul and Pesendorfer (2007) and Noor (2007). Gul and Pesendorfer (2007) extend their self-control representation to the setting of history-dependence. The representation and the normative utility function are the same as in (3) and (4). The temptation utility function is specified as

$$V(l|h) = \int v(c|h)\, \mathrm{d}l(c,z).$$

Hence, the agent is tempted from the immediate consumption only.

Noor (2007) axiomatizes an FT utility function where (3) and (4) are history-independent and a temptation utility function is also stationary and specified as

$$V(l) = \int_{C \times \mathcal{Z}} \left\{ v(c) + \gamma \max_{m \in z} V(m) \right\} \mathrm{d}l(c,z).$$

Other than the difference in history-dependence, a PBT representation is a special case of an FT representation in the sense that $u(\cdot|h) = v(\cdot|h)$ and $\beta(h) = \gamma(h)$, which is a necessary restriction for our interpretation.

To illustrate behavioral distinctions among three models, consider two actions; to exercise, denoted by e, or not to do it, denoted by n. Assume that by developing good habits for exercise, the agent will better appreciate a future menu x, but a payoff of e is lower than that of choosing n due to effort costs. Suppose that the agent faces two alternatives of visiting a gym that differs in duration of exercise. For some fixed menu x, let

$$x_0 = \{(e, e, x), (e, n, x)\}.$$

The alternative (e, e, x) means that the agent visits the gym in two periods, and receives x afterward, while in (e, n, x), the agent visits the gym only in the first period, and receives x.[2]

Fix any history h. In terms of the normative utility, the agent may appreciate good habits of exercise and hence, the following ranking may hold:

$$U((e, e, x)|h) > U((e, n, x)|h). \tag{8}$$

However, the temptation utility ignores the development of habits and hence we may have

$$V((e, e, x)|h) < V((e, n, x)|h).$$

Since the choice from x_0 is made by maximizing $U(\cdot|h) + \alpha(h)V(\cdot|h)$, if $\alpha(h)$ is small, the agent may choose $(e, e, x) \in x_0$ with exerting self-control over the deceptive belief, while if $\alpha(h)$ is large, he may choose $(e, n, x) \in x_0$ even though he normatively prefers (e, e, x) to (e, n, x). Accordingly, the agent exhibits either $\{(e, e, x)\} \succ_h x_0 \succ_h \{(e, n, x)\}$ or $\{(e, e, x)\} \succ_h x_0 \sim_h \{(e, n, x)\}$.

In the case of Gul and Pesendorfer (2007), any alternative in x_0 is equally tempting because the temptation utility function only cares about the current consumption. When (8) holds, the agent always chooses $(e, e, x) \in x_0$ without temptation. This reflects preference over menus as $\{(e, e, x)\} \sim_h x_0 \succ_h \{(e, n, x)\}$.

In the case of Noor (2007), both normative and temptation utilities satisfy stationarity. Since habit formation has no effect, we should obtain

$$U((e, e, x)) < U((e, n, x)), \text{ and } V((e, e, x)) < V((e, n, x)).$$

Hence, the agent exhibits $\{(e, n, x)\} \sim x_0 \succ \{(e, e, x)\}$ and always chooses $(e, n, x) \in x_0$.

3 Examples

3.1 Projection bias

The projection bias is known as the tendency of individuals to exaggerate the degree to which their future tastes will resemble their current tastes. Read and van Leeuwen (1998) report the following experimental evidence. There are four states: hungry now (denoted

[2] Formally, the alternative (e, e, x) is identified with $\{(e, \{(e, x)\})\} \in C \times \mathcal{Z}$.

by H_0), satisfied now (S_0), hungry after one week (H_1), and satisfied after one week (S_1). There are two options, unhealthy stack (junk food, denoted by j) and health snack (fruit, denoted by f). Each option is available after one week. The subjects' current state is either H_0 or S_0. They are asked to choose between f and j available in either state H_1 or S_1. Thus, the following four treatments exist: H_0H_1, H_0S_1, S_0H_1, and S_0S_1. For example, H_0H_1 means that subjects who are hungry now are asked to choose between f and j available when they will be hungry one week later.

The following table summarizes the proportion of subjects who choose j in each treatment.

$$\begin{array}{ccc} & H_1 & S_1 \\ H_0 & .78 & .42 \\ S_0 & .56 & .26 \end{array} \tag{9}$$

In each current state, H_0 or S_0, the subjects tend to choose option j when they anticipate being hungry. Moreover, the subjects choose option j more often when they are hungry now. The latter is interpreted as the projection bias.

We illustrate how our PBT model can accommodate the projection bias in the following. Suppose that $C = \{f, j, 0\} \times [0, \overline{c}]$, where $[0, \overline{c}]$ is a set of regular consumption. Moreover, suppose that $\succsim_0$ is the preference when the consumption in the past is $0 = (0, 0)$, which is interpreted as the preference at the hungry state H_0. Similarly, $\succsim_{\overline{c}}$ is the preference when the consumption in the past is $(0, \overline{c})$, which is the preference at the satisfied state S_0.

For notational convenience, let f and j be identified with $(f, 0)$ and $(j, 0)$, respectively. Similarly, let 0 and $\overline{c}$ be identified with $(0, 0)$ and $(0, \overline{c})$, respectively. Consider two menus

$$x^H = \{(0, f, \mathbf{0}), (0, j, \mathbf{0})\}, \text{ and } x^S = \{(\overline{c}, f, \mathbf{0}), (\overline{c}, j, \mathbf{0})\},$$

where $(0, f, \mathbf{0})$ is a commitment stream that gives 0 in the first period, f in the second period, and 0 from the third period onward, and so on. We interpret a choice from x^H at history 0 as the treatment H_0H_1, a choice from x^H at history $\overline{c}$ as the treatment S_0H_1, and so on.

According to $\succsim_0$, the agent is tempted to believe his utility function $u(\cdot|0)$ is stationary, reflecting the projection bias. Assume $u(j|0) > u(f|0)$. Then, at each of two menus x^H and x^S, the self-control cost to choose f is $\alpha(0)\beta(0)(u(f|0) - u(j|0))$. On the other hand, the agent correctly takes into account the effect of habit formation in terms of his normative preference. Assume that $u(j|00) > u(f|00)$ and $u(j|0\overline{c}) < u(f|0\overline{c})$, which are interpreted as preferences at states H_1 and S_1, respectively.

Under these assumptions, j is always chosen from x^H, whereas j is chosen from x^S if

$$\beta(0)u(f|0\overline{c}) + \alpha(0)\beta(0)(u(f|0) - u(j|0)) \leq \beta(0)u(j|0\overline{c}),$$

or

$$u(f|0\overline{c}) - u(j|0\overline{c}) \leq \alpha(0)(u(j|0) - u(f|0)),$$

which is likely to hold when $\alpha(0)$ is large. Thus, j is more likely to be chosen at x^H compared with x^S. Since $\alpha(0)$ captures the intensity of self-control costs over the projection bias, this choice pattern suggests that the projection bias is stronger when the agent is currently hungry.

Next, consider the case of $\succsim_{\overline{c}}$. Assume $u(j|\overline{c}) < u(f|\overline{c})$. Then, at each of two menus x^H and x^S, the self-control cost to choose j is $\alpha(\overline{c})\beta(\overline{c})(u(j|\overline{c}) - u(f|\overline{c}))$. On the other hand, in terms of his normative preference, $u(j|\overline{c}0) > u(f|\overline{c}0)$ and $u(j|\overline{cc}) < u(f|\overline{cc})$. Hence, j is never chosen from x^S, while j is chosen from x^H if

$$\beta(\overline{c})u(j|\overline{c}0) + \alpha(\overline{c})\beta(\overline{c})(u(j|\overline{c}) - u(f|\overline{c})) \geq \beta(\overline{c})u(f|\overline{c}0),$$

or

$$u(j|\overline{c}0) - u(f|\overline{c}0) \geq \alpha(\overline{c})(u(f|\overline{c}) - u(j|\overline{c})),$$

which tends to hold when $\alpha(\overline{c})$ is small. This choice pattern is more likely to occur if the projection bias is weaker when the agent is currently satisfied.

In summary, our model implies a qualitatively similar pattern to (9) as summarized below:

	H_1	S_1
H_0	1	γ_{HS}
S_0	γ_{SH}	0

where $1 > \gamma_{HS} > 0$ and $1 > \gamma_{SH} > 0$. The choice frequency corresponding to γ_{HS} depends on the size of $\alpha(0)$. The larger $\alpha(0)$ is, the closer the frequency is to one. The choice frequency associated with γ_{SH} depends on the size of $\alpha(\overline{c})$, and the smaller $\alpha(\overline{c})$ is, the closer the frequency is to one.

3.1.1 Comparison with other models

Loewenstein et al. (2003) suggest a model accommodating the projection bias. Suppose that $u(c_t, s_t)$ is the agent's true utility function at state s_t, whereas for any $\tau \geq t$, the perceived utility at state s_τ is given by

$$\tilde{u}(c_\tau, s_\tau|s_t) = (1-\alpha)u(c_\tau, s_\tau) + \alpha u(c_\tau, s_t).$$

Based on these, the (true) intertemporal utility is given by

$$U_t(c_t, \cdots, c_T) = \sum_{\tau=t}^{T} \beta^{\tau-t} u(c_\tau, s_\tau),$$

whereas the perceived intertemporal utility function is

$$\begin{aligned} U_t(c_t, \cdots, c_T|s_t) &= \sum_{\tau=t}^{T} \beta^{\tau-t}\tilde{u}(c_\tau, s_\tau|s_t) = \sum_{\tau=t}^{T} \beta^{\tau-t}((1-\alpha)u(c_\tau, s_\tau) + \alpha u(c_\tau, s_t)) \\ &= (1-\alpha)\sum_{\tau=t}^{T} \beta^{\tau-t} u(c_\tau, s_\tau) + \alpha \sum_{\tau=t}^{T} \beta^{\tau-t} u(c_\tau, s_t). \end{aligned} \tag{10}$$

The agent with projection bias maximizes his perceived intertemporal utility.

Note that for each history h, PBT model maximizes $U(\cdot|h) + \alpha(h)V(\cdot|h)$ from a given menu x_0. For simplicity, assume $\beta(h)$ is independent of h. In particular, for a commitment consumption stream $\mathbf{c} = (c_1, c_2, \cdots)$,

$$U(\mathbf{c}|h) + \alpha(h)V(\mathbf{c}|h) = \sum_{\tau \geq 1} \beta^{\tau-1} u(c_\tau|hc_1 \cdots c_{\tau-1}) + \alpha(h) \sum_{\tau \geq 1} \beta^{\tau-1} u(c_\tau|h),$$

which looks similar to (10).

The projection bias has been interpreted as a form of naiveté or a failure to anticipate one's own future utility. Noor (2007) proposes an alternative interpretation such that the agent is perfectly sophisticated for anticipating his future utility but is tempted from the future options. Although Noor (2007)'s formal model satisfies stationarity, he illustrates that a non-stationary extension of his model can accommodate the projection bias given as in (9). PBT model has such a feature because of habit formation.

3.2 Paying not to go to the gym

Using the date set of U.S. health clubs, Della Vigna and Malmendier (2006) find that many consumers continue to choose the monthly contract over the pay-per-visit contract even though they do not visit the gym very often. The main reason they suggest is overconfidence or naiveté about self-control. This subsection illustrates that although an agent is perfectly sophisticated, he may choose the monthly contract and end up with low gym attendance afterward because he is tempted to ignore the effect of (bad) habit formation on exercise and believe that his ability exert self-control is still maintained.

3.2.1 Single-period self-control problem

In this subsection, we adopt the setting of Section 2.3.1. There are two commodities, a regular commodity a and an exercise e at the gym. A monthly contract offers a unit price $p \in (0,1)$ of exercises at a fixed cost r for some $r \in [0,1)$. Thus, it is formalized as a budget set

$$c^m = \{(a,e) \in \mathbb{R}^2_+ \mid a + pe \le 1 - r\}.$$

A pay-per-visit contract can be viewed as the case of $r = 0$ and $p = 1$, that is,

$$c^d = \{(a,e) \in \mathbb{R}^2_+ \mid a + e \le 1\}.$$

When r and p are small, we may interpret c^m as a budget set subsidized for exercise.

Assume that the agent obtains utilities from regular consumption and exercise, but he is tempted not to exercise. For any menu c of pairs (a,e),

$$u(c|h) = \max_{(a,e)\in c} \left\{ \mathbf{u}(a) + \mathbf{u}(e) - \lambda(h) \left(\max_{(a',e')\in c} \mathbf{u}(a') - \mathbf{u}(a) \right) \right\}. \tag{11}$$

Assume also that λ depends on the history of exercises and $\lambda(he)$ is decreasing in e because a good habit for exercise mitigates the self-control problem.

From now on, assume that $\mathbf{u}(a) = \sqrt{a}$, $\mathbf{u}(e) = \sqrt{e}$. First, let us derive an actual choice from c^m and compute the corresponding utility of menu, $u(c^m|h)$. For notational simplicity, $\lambda(h)$ is denoted by λ. By applying the FOC, we can easily obtain

$$a^m = \frac{p(1+\lambda)^2}{p(1+\lambda)^2 + 1}(1-r), \;\; e^m = \frac{1-r}{(p(1+\lambda)^2+1)p}. \tag{12}$$

By substituting these back into (11), we have

$$\begin{aligned} u(c^m|h) &= (1+\lambda)\sqrt{\frac{p(1+\lambda)^2}{p(1+\lambda)^2+1}(1-r)} + \sqrt{\frac{1-r}{(p(1+\lambda)^2+1)p}} - \lambda\sqrt{1-r} \\ &= \sqrt{1-r}\left(\sqrt{(1+\lambda)^2 + \frac{1}{p}} - \lambda\right). \end{aligned}$$

By setting $p = 1$ and $r = 0$, we obtain an actual choice from c^d and the corresponding utility of the pay-as-visit contract c^d.

$$a^d = \frac{(1+\lambda)^2}{(1+\lambda)^2+1}, \; e^d = \frac{1}{(1+\lambda)^2+1},$$

and

$$u(c^d|h) = \sqrt{(1+\lambda)^2+1} - \lambda.$$

For small λ, $u(c^m|h) > u(c^d|h)$, while for large λ, $u(c^m|h) < u(c^d|h)$.

Suppose that the agent chooses a higher level of exercise at the monthly contract, that is,

$$e^m = \frac{1-r}{(p(1+\lambda)^2+1)p} > e^d = \frac{1}{(1+\lambda)^2+1},$$

which is more likely to hold for small p and r.

3.2.2 Self-control in habit formation

In the subsequent analysis, $\beta(h)$ and $\alpha(h)$ in the PBT representation are assumed to be constant for simplicity. We consider the following recursive menu over time. Let

$$x^m = \{(c^m, x^m), (c^m, x^d)\}, \text{ and } x^d = \{(c^d, x^m), (c^d, x^d)\}.$$

The agent starts with x^d at an initial history. Suppose that at the initial history, $\lambda(e_0) = \lambda_0$ is sufficiently small. Since $u(c^m|e_0) > u(c^d|e_0)$, facing x^d, the agent's maximum temptation is given as

$$V((c^d, x^m)|e_0) = u(c^d|e_0) + \beta u(c^m|e_0) + \frac{\beta^2}{1-\beta} u(c^m|e_0),$$

which is achieved by choosing $(c^m, x^m) \in x^m$ all the time after the next period. If $(c^d, x^d) \in x^d$ is chosen,

$$V((c^d, x^d)|e_0) = u(c^d|e_0) + \beta u(c^d|e_0) + \frac{\beta^2}{1-\beta} u(c^m|e_0).$$

Hence, a self-control cost for resisting the temptation to ignore the habit formation is given by

$$\alpha\beta(u(c^d|e_0) - u(c^m|e_0)) < 0.$$

The utility of x^d is given by

$$\begin{aligned}
&W(x^d|e_0) \\
=&\max\left[u(c^d|e_0) + \beta W(x^m|e_0 e^d),\; u(c^d|e_0) + \beta W(x^d|e_0 e^d) + \alpha\beta(u(c^d|e_0) - u(c^m|e_0))\right] \\
=&u(c^d|e_0) + \beta \max[W(x^m|e_o e^d),\; W(x^d|e_0 e^d) + \alpha(u(c^d|e_0) - u(c^m|e_0))],
\end{aligned}$$

where e_0e^d is an updated history after choosing e^d in the menu c^d. Assume that e^d is not enough to develop a good habit for exercise and after the following history, the self-control issue has deteriorated. That is, $\lambda(e_0e^d)$ is sufficiently greater than λ_0 so that $u(c^d|e_0e^d) > u(c^m|e_0e^d)$. From the normative perspective, this effect of habit formation is considered. By the above inequality, we may well have $W(x^d|e_0e^d) > W(x^m|e_oe^d)$, or

$$\{(c^d, x^d)\} \succ_{e_0} \{(c^d, x^m)\}.$$

However, when α is sufficiently large, the agent tends to choose the monthly contract in the next period, that is, $(c^d, x^m) \in x^d$. This is because the agent is tempted to believe that even after the current level of low exercise, he will be able to maintain the ability of high self-control for exercise. Under this deceptive belief, x^m looks more attractive.

Although the agent is tempted to believe that the exercise level is given by $e^m(\lambda(e_0))$ of (12) in the next period after choosing $(c^d, x^m) \in x^d$, his actual choice of exercise at that time ends up with $e^m(\lambda(e_0e^d)) < e^m(\lambda(e_0))$ because of the bad habit formation.

3.3 Self-control fatigue and projection bias

People may fail to accumulate good habits due to the loss of self-control caused by fatigue. A puzzling feature of this observation is why people continue to exercise until they lose self-control and consequently fail to accumulate good habits. Assuming rational habit formation, if habit formation failure is foreseen, it is optimal not to start from the beginning, or to stop further exercise before burnout. One explanation is overconfidence or naiveté that prevents the agent from correctly anticipating his future behavior. Our PBT representation illustrates that even if the agent is sophisticated and correctly perceives the effects of habit formation, the projection bias will lead to excessive exercise until burnout.

In this subsection, we adopt the same setting of Sections 2.3.1 and 3.2.1. Let c^m and c^d denote the monthly and pay-per-visit contracts, respectively. From each contract, the agent chooses some pair (a, e) of a regular commodity and an exercise at the gym from the contract. A single period utility from a contract c, denoted by $u(c|h)$, depends on a history of exercise through a parameter $\lambda(h)$ representing the intensity of the temptation that comes from not exercising. As in Section 3.2.1, we use the following properties in the subsequent analysis.

- For small λ, $u(c^m|h) > u(c^d|h)$, while for large λ, $u(c^m|h) < u(c^d|h)$.

- At an optimum in the single period problem, $e^m > e^d$, that is, the agent exercises more in the monthly contact under the same λ.

We consider a recursive choice of contracts over time as follows:

$$x^m = \{(c^m, x^m), (c^m, x^d)\}, \text{ and } x^d = \{(c^d, x^m), (c^d, x^d)\}.$$

Consider the agent who has been exercising too much and is one step away from burnout. In the initial history h, $\lambda(h)$ is sufficiently low due to the effects of the exercise habit to better evaluate the exercise. That is, $u(c^m|h) > u(c^d|h)$ holds. However, if the agent continues to exercise further from this point, he will lose self-control due to fatigue, and λ will increase. More precisely, assume that $\lambda(h) = \lambda(he^m) < \lambda(he^m e^m)$, and $\lambda(he^m e^m)$ is sufficiently large and becomes constant afterward. Assume further that once the agent rests and chooses e^d after he^m, then good habits will be established and self-control will be maintained into the future. That is, $\lambda(he^m e^d)$ is sufficiently small and becomes constant afterward.

The agent starts with x^m at the initial history. In the subsequent analysis, $\beta(h)$ in the PBT representation is assumed to be constant for simplicity. By backward induction, consider the one-period ahead decision problem where the agent faces x^m at history he^m. The agent is tempted to believe that his future status of self-control is still $\lambda(he^m)$ with ignoring habits from the current contract c^m. Under the correct expectation, which takes into account the effects of habits, c^m is evaluated lower at history $he^m e^m$ due to the loss of self-control caused by fatigue. Under the deceptive belief, which overestimates future self-control for exercise, however, $(c^m, x^m) \in x^m$ is optimal at all times. The self-control cost to resist to this deceptive belief is given by $\alpha(he^m)\beta(u(c^m|he^m) - u(c^d|he^m)) > 0$. By the PBT representation,

$$\begin{aligned}
&W(x^m|he^m) \\
=&\max\{u(c^m|he^m) + \beta W(x^m|he^m e^m), \\
&\qquad u(c^m|he^m) + \beta W(x^d|he^m e^m) - \alpha(he^m)\beta(u(c^m|he^m) - u(c^d|he^m))\} \\
=&u(c^m|he^m) + \beta \max\left\{W(x^m|he^m e^m),\ W(x^d|he^m e^m) - \alpha(he^m)(u(c^m|he^m) - u(c^d|he^m))\right\}.
\end{aligned}$$

After history $he^m e^m$, since λ is assumed to be constant at the high value, $u(x^m|h') < u(x^d|h')$ holds for any history following $he^m e^m$, which implies $W(x^m|he^m e^m) < W(x^d|he^m e^m)$.

Nevertheless, if $\alpha(he^m)$ is high enough, $(c^m, x^m) \in x^m$ may be chosen at history he^m, in which case

$$W(x^m|he^m) = u(c^m|he^m) + \beta W(x^m|he^m e^m). \tag{13}$$

Suppose next that the agent faces x^d at history he^m, in which case, history $he^m e^d$ follows. Under this history, we assume that burnout is avoided and self-control is maintained due to the effects of light exercise. The agent is tempted to believe that his future status of self-control is still $\lambda(he^m)$. After history $he^m e^d$, since λ is assumed to be constant at the low value, $u(x^m|h') > u(x^d|h')$ holds for any history following $he^m e^d$, which implies $W(x^m|he^m e^d) > W(x^d|he^m e^d)$. The PBT representation evaluates x^d as

$$\begin{aligned}
&W(x^d|he^m)\\
=&\max\left\{u(c^d|he^m) + \beta W(x^m|he^m e^d),\right.\\
&\qquad\left. u(c^d|he^m) + \beta W(x^d|he^m e^d) - \alpha(he^m)\beta(u(c^m|he^m) - u(c^d|he^m))\right\}\\
=&u(c^d|he^m) + \beta \max\left\{W(x^m|he^m e^d),\ W(x^d|he^m e^d) - \alpha(he^m)(u(c^m|he^m) - u(c^d|he^m))\right\}\\
=&u(c^d|he^m) + \beta W(x^m|he^m e^d).
\end{aligned} \tag{14}$$

As a result, good habit formation is achieved with this history.

Now we consider how the agent evaluates x^m at history h. From the normative perspective, by (13) and (14), we may have $W(x^d|he^m) > W(x^m|he^m)$, or

$$\{(c^m, x^d)\} \succ_h \{(c^m, x^m)\}.$$

However, the agent is tempted to believe that his future status of self-control is still $\lambda(h)$. Under this deceptive belief, $(c^m, x^m) \in x^m$ is optimal at all times. The self-control cost to resist to this deceptive belief is given by $\alpha(h)\beta(u(c^m|h) - u(c^d|h)) > 0$. By the PBT representation,

$$\begin{aligned}
&W(x^m|h)\\
=&\max\left\{u(c^m|h) + \beta W(x^m|he^m), u(c^m|h) + \beta W(x^d|he^m) - \alpha(h)\beta(u(c^m|h) - u(c^d|h))\right\}\\
=&u(c^m|h) + \beta \max\left\{W(x^m|he^m),\ W(x^d|he^m) - \alpha(h)(u(c^m|h) - u(c^d|h))\right\}.
\end{aligned}$$

If $\alpha(h)$ is high enough, $(c^m, x^m) \in x^m$ may be chosen. Even though the agent is sophisticated and can foresee the future well enough, he gets tired from too much exercise, causing burnout.

A final remark is that the present observation can be further strengthened by introducing a history dependence of projection bias. It would be reasonable to assume that long-term persistence in an exercise habit will allow people to believe future self-control in exercise. Such a tendency corresponds to the assumption that $\alpha(h)$ is increasing in exercise history. Under this assumption, the projection bias is reinforced through time, and exercise tends to be the default option. This would lead to self-destructive burnout.

3.4 Bad habits and delaying commitment

If the agent anticipates a temptation at the choice from menu, he should exhibit a preference for commitment to avoid the temptation. More specifically, if drug addicts correctly anticipate their temptation from using drugs in the future, they should make some commitment, for example, checking in a rehabilitation center. Gul and Pesendorfer (2007, Section 5) illustrate an intertemporal choice pattern where a drug consumption increases over time up to some period, then jumps down to zero by checking in the rehabilitation center, and repeats the same cycle afterward. However, the tendency for drug addicts to delay such treatments has considerable evidence (see Cunningham et al. (1993)). By adopting the same setting in Section 2.3.1, we illustrate that if a self-control problem over the projection bias exists, checking in the rehabilitation tends to be delayed compared with the findings of Gul and Pesendorfer (2007).

3.4.1 Self-control on drug use

There are two commodities, a regular commodity a and a drug d. Let p denote the price of drug. A price of the regular commodity is normalized to be one. The corresponding budget is given by

$$c^o = \{(a, d) \in \mathbb{R}^2_+ \mid a + pd \leq 1\}.$$

The agent can take a rehabilitation in which case the drug consumption is forced to be zero with a fixed cost $r \in [0, 1]$. Thus, rehabilitation is viewed as a commitment option. It is formalized as a budget set (called "in-option" in contrast with "out-option" c^o)

$$c^i = \{(a, d) \in \mathbb{R}^2_+ \mid a = 1 - r,\ d = 0\} = \{(1 - r, 0)\}.$$

Assume that the agent cares only about regular consumption, but is tempted to consume a drug. For any menu c of pairs (a, d),

$$u(c|h) = \max_{(a,d)\in c} \left\{ \mathbf{u}(a) - \lambda(h) \left(\max_{(a',d')\in c} \mathbf{v}(d') - \mathbf{v}(d) \right) \right\}. \tag{15}$$

Assume also that λ depends on the history of the drug. More specifically, assume that λ depends only on the drug consumption in the last period. We assume that $\lambda(d)$ is increasing in d because bad habits of drug deteriorate the self-control problem.

From now on, assume that $\mathbf{u}(a) = \log a$ and $\mathbf{v}(d) = \log d$. Let us derive an actual choice from c^o and compute the corresponding utility of menu, $u(c^o|h)$. For simplicity, $\lambda(h)$ is denoted by λ. By applying the FOC, we can easily obtain

$$d^o = \frac{\lambda}{p(1+\lambda)}, \ a^o = \frac{1}{1+\lambda}.$$

By substituting these back into (15),

$$\begin{aligned} u(c^o|h) &= \log \frac{1}{1+\lambda} + \lambda \log \frac{\lambda}{p(1+\lambda)} - \lambda \log \frac{1}{p} \\ &= \log \left(\frac{\lambda}{1+\lambda} \right)^{\lambda} \frac{1}{1+\lambda}. \end{aligned}$$

Apparently, if c^i is chosen, a consumption is given by $a = 1 - r$ and $d = 0$, and hence,

$$u(c^i|h) = \log(1 - r).$$

Let

$$f(\lambda) = \left(\frac{\lambda}{1+\lambda} \right)^{\lambda} \frac{1}{1+\lambda}.$$

Note that f is strictly decreasing and satisfies $f(0) = 1$ and $\lim_{\lambda\to\infty} f(\lambda) = 0$. Thus, there exists a unique λ^* such that

$$\begin{aligned} u(c^o|h) > u(c^i|h) \quad &\text{if } \lambda < \lambda^* \\ u(c^o|h) \leq u(c^i|h) \quad &\text{if } \lambda \geq \lambda^*. \end{aligned} \tag{16}$$

3.4.2 Self-control over the projection bias

We consider the following recursive menu over time. Let

$$x^o = \{(c^o, x^o), (c^o, x^i)\}, \text{ and } x^i = \{(c^i, x^o), (c^i, x^i)\}.$$

The agent starts with x^o at an initial history. Every period, he chooses between the out-option c^o and the in-option c^i for the next period, while the current option (c^o in the case of x^o or c^i in the case of x^i) has been already determined in the last period.

We examine an intertemporal choice pattern of PBT model. For simplicity, assume $\beta(h) = \beta \in (0, 1)$ and $\alpha(h) = \alpha \geq 0$ for all h. Suppose also that at an initial history $d_0 = 0$, $\lambda(d_0) = \lambda_0 > 0$ is sufficiently small and hence the self-control problem is not so severe, in which case $u(c^o|0) > u(c^i|0)$ by (16).

First, as a benchmark, we consider the case of $\alpha = 0$, that is, the agent is not tempted from ignoring the effect of habit formation. Effectively, this case is equivalent to the analysis of Gul and Pesendorfer (2007, Section 5), where temptation from drug consumption and its effect on habits are investigated.

Since $\alpha = 0$, the utility of x^o at history d is given by

$$\begin{aligned} W(x^o|d) &= \max \left[u(c^o|d) + \beta W(x^o|d^o),\ u(c^o|d) + \beta W(x^i|d^o) \right] \\ &= u(c^o|d) + \beta \max \left[W(x^o|d^o),\ W(x^i|d^o) \right], \end{aligned} \tag{17}$$

where d^o is a drug consumption from the current c^o. The agent chooses an option from x^o by maximizing $W(\cdot|d^o)$. Note that $W(x^o|d)$ depends on d only through $u(c^o|d)$. Since $u(c^o|d)$ is decreasing in d, so is $W(x^o|d)$.

On the other hand, the utility of x^i at history d is given by

$$\begin{aligned} W(x^i|d) &= \max \left[u(c^i|d) + \beta W(x^o|d^i),\ u(c^i|d) + \beta W(x^i|d^i) \right] \\ &= \log(1 - r) + \beta \max \left[W(x^o|d^i),\ W(x^i|d^i) \right], \end{aligned} \tag{18}$$

where $d^i = 0$. Note that $W(x^i|d)$ is independent of d, and written as $W(x^i)$.

The following intertemporal choice is derived as in Proposition 7 of Gul and Pesendorfer (2007): From (17), as long as $W(x^o|d^o) \geq W(x^i)$, $(c^o, x^o) \in x^o$ is chosen; otherwise, $(c^o, x^i) \in x^o$ is chosen. In the latter case, the agent faces x^i in the next period. From (18), $(c^i, x^o) \in x^i$ is chosen because $W(x^o|0) > W(x^i)$. Then, the situation goes back to the initial position. In summary, there exists some k such that the drug consumption increases up to period k, and jumps down to zero; then it starts increasing again up to the next k periods, and so on. This choice pattern forms an intertemporal cycle.

Now, we consider the case of $\alpha > 0$, where the agent tends to believe that his future preference does not change in the next period (regarded as the projection bias) even after drug consumption and ignores the effect of (bad) habit formation. We illustrate that the agent is more likely to delay checking in the rehabilitation center.

Assume that d is sufficiently small. Facing x^o, the agent's maximum temptation is given as

$$V((c^o, x^o)|d) = \max_{(c_t)\in x^o} \sum_{t=0} \beta^t u(c_t|d) = u(c^o|d) + \beta u(c^o|d) + \frac{\beta^2}{1-\beta} u(c^o|d),$$

which is achieved by choosing $(c^o, x^o) \in x^o$ all the time. If $(c^o, x^i) \in x^o$ is chosen,

$$V((c^o, x^i)|d) = u(c^o|d) + \beta u(c^i|d) + \frac{\beta^2}{1-\beta} u(c^o|d).$$

Hence, a self-control cost for resisting the temptation to ignore the habit formation is given by

$$\alpha\beta(u(c^i|d) - u(c^o|d)) < 0.$$

Since $u(c^i|d) = \log(1-r)$, it is independent of d, and it is denoted by $u(c^i)$ from now on. The utility of x^o is given by

$$\begin{aligned} W(x^o|d) &= \max\left[u(c^o|d) + \beta W(x^o|d^o),\ u(c^o|d) + \beta W(x^i|d^o) + \alpha\beta(u(c^i) - u(c^o|d))\right] \\ &= u(c^o|d) + \beta \max[W(x^o|d^o),\ W(x^i|d^o) + \alpha(u(c^i) - u(c^o|d))]. \end{aligned}$$

Compared with (17), this equation shows that the agent is more reluctant to choose x^i because of the associated self-control cost $\alpha(u(c^i) - u(c^o|d)) < 0$. Hence, the agent tends to delay choosing x^i.

Note also that the projection bias may create a gap between the normative ranking and the actual choice. From the normative viewpoint, the agent correctly anticipates the effect of bad habits of drug consumption and hence may have a normative preference for commitment to rehabilitation in the next period. That is,

$$u(c^o|d) + \beta W(x^i|d^o) > u(c^o|d) + \beta W(x^o|d^o), \tag{19}$$

or $\{(c^o, x^i)\} \succ_d \{(c^o, x^o)\}$. Nevertheless, when α is sufficiently large, our agent may choose $(c^o, x^o) \in \{(c^o, x^o), (c^o, x^i)\}$ because of the projection bias. This pattern never happens in Gul and Pesendorfer (2007), where, together with (17) and (19), it is easy to see that $\{(c^o, x^i)\} \succ_d \{(c^o, x^o)\}$ if and only if $(c^o, x^i) \in \{(c^o, x^o), (c^o, x^i)\}$ is chosen at history d.

The observation in the previous paragraph is obtained from the general insight of Noor (2007), which first highlights the possibility of delaying commitment when the agent is tempted from the future options. Our model is more explicitly tied up with the projection bias story. In our model, the agent is tempted from the future option (i.e., drug in the next period) because he tends to ignore the effect of bad habits caused by the current drug consumption.

3.5 Time preference formation under self-control

This subsection reconsiders a model of endogenous determination of time preference. Consider a capital accumulation problem in which the agent can invest to increase the level of patience. In contrast to the model in which the agent has no self-control problem (Becker and Mulligan (1997), Shi and Epstein (1993), and Stern (2006)), we consider the agent who is tempted to act optimally according to the status quo level of patience.

3.5.1 Setting

There is one physical good which can be used either for consumption, investment in increasing the patience level, and for the reproduction of the same good through savings. Let $u : \mathbb{R}_+ \to \mathbb{R}$ denote the period-wise utility function over consumption, which is assumed to be invariant over time. That is, habit formation here is only about the level of patience. The function u is assumed to satisfy the standard properties such as strong monotonicity, strict concavity, continuity, differentiability, and the Inada condition. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ denote the production function, which is assumed to be strongly monotone and concave, continuous and differentiable.

Let $0 < \underline{b} < \bar{b} < 1$ denote the upper bound and lower bound for discount factor, and let $B = \{(b, b') \in [\underline{b}, \bar{b}]^2 : b' \geq b\}$. Let $q : B \to \mathbb{R}_+$ denote the cost function for changing discount factor, where $q(b, b')$ is the necessary cost of changing b into $b' \geq b$. The function q is assumed to satisfy the following properties.

(i) $q(b, b) = 0$ for all $b \in [\underline{b}, \bar{b}]$.

(ii) $q_1(b, b') < 0$ and $q_2(b, b') > 0$ for all $(b, b') \in intB$.

(iii) $\lim_{b' \to b} q_2(b, b') = 0$ and $\lim_{b' \to \bar{b}} q_2(b, b') = \infty$.

Given the current levels of capital k and discount factor b, the agent chooses (k', b'), where k' denotes the level of capital to carry over to the next period and b' denotes the level of patience in the next period. His consumption in the current period is given as the residual $f(k) - k' - q(b, b')$. On the other hand, by the projection bias, he is tempted to follow his status quo preference given the current level of b, where he simply solves the standard capital accumulation problem with the fixed discount factor b.

Thus the Bellman equations are given by

$$\begin{aligned} W(k,b) &= \max_{k',b'} \Big\{ u(f(k) - k' - q(b,b')) + bW(k',b') \\ &\quad + \alpha \left[u(f(k) - k' - q(b,b')) + bV(k',b) - \max_{\widetilde{k}'} \left\{ u(f(k) - \widetilde{k}') + bV(\widetilde{k}',b) \right\} \right] \Big\}, \\ V(k,b) &= \max_{\widetilde{k}'} \left\{ u(f(k) - \widetilde{k}') + bV(\widetilde{k}',b) \right\}. \end{aligned}$$

The first-order conditions are

$$\begin{aligned} (1+\alpha)u'(c) &= bW_1(k',b') + \alpha bV_1(k',b), \\ (1+\alpha)u'(c)q_2(b,b') &= bW_2(k',b'), \\ u'(\widetilde{c}) &= bV_1(\widetilde{k}',b). \end{aligned} \tag{20}$$

Denote the relevant policy functions by

$$\begin{aligned} &c = c(k,b),\ k' = s(k,b),\ b' = \beta(k,b), \\ &\widetilde{c} = \widetilde{c}(k,b),\ \widetilde{k}' = \widetilde{s}(k,b). \end{aligned}$$

The value functions take the form

$$\begin{aligned} W(k,b) &= (1+\alpha)u(f(k) - s(k,b) - q(b,\beta(k,b))) + bW(s(k,b),\beta(k,b)) \\ &\quad + \alpha bV(s(k,b),b) - \alpha u(f(k) - \widetilde{s}(k,b)) - \alpha bV(\widetilde{s}(k,b),b), \\ V(k,b) &= u(f(k) - \widetilde{s}(k,b)) + bV(\widetilde{s}(k,b),b). \end{aligned}$$

By taking derivatives of the above and arranging, we obtain the envelope conditions

$$\begin{aligned} W_1(k,b) &= f'(k)\left[(1+\alpha)u'(c(k,b)) - \alpha u'(\widetilde{c}(k,b))\right], \\ W_2(k,b) &= -(1+\alpha)u'(c(k,b))q_1(b,\beta(k,b)) + W(s(k,b),\beta(k,b)) \\ &\quad + \alpha V(s(k,b),b) + \alpha bV_2(s(k,b),b) \\ &\quad - \alpha V(\widetilde{s}(k,b),b) - \alpha bV_2(\widetilde{s}(k,b),b), \\ V_1(k,b) &= f'(k)u'(\widetilde{c}(k,b)). \end{aligned} \tag{21}$$

From the first-order conditions and the envelope conditions, we obtain the Euler equation for the evolution of consumption and savings as

$$\begin{aligned} & u'(c(k,b)) \\ = \ & bf'(s(k,b))\left\{u'(c(s(k,b),\beta(k,b))) - \frac{\alpha}{1+\alpha}\left(u'(\widetilde{c}(s(k,b),\beta(k,b))) - u'(\widetilde{c}(s(k,b),b))\right)\right\}. \end{aligned}$$

Note that when $\alpha = 0$, it reduces to the Euler equation with habit formation

$$u'(c(k,b)) \ = \ bf'(s(k,b))u'(c(s(k,b),\beta(k,b))).$$

It is not sensible to consider an interior steady state (k^*, b^*) with $\underline{b} < b^* < \bar{b}$, because at such steady state, the agent is not making any investment on increasing the patience level despite its arbitrarily large marginal return. This suggests that the self-control problem is really about transition dynamics, and will affect how fast the agent's patience level converges to the upper-bound or will lead him to perish in the long-run.

On the other hand, the Euler equation for the "tempting consumption path" is

$$u'(\widetilde{c}(k,b)) \ = \ bf'(\widetilde{s}(k,b))u'(\widetilde{c}(\widetilde{s}(k,b),b)),$$

which is the standard except that it describes only a hypothetical path in general because discount factor b is fixed.

There should be the Euler equation for the evolution of consumption and investment, which should be obtained by combining (20) and (21), but the implication is not immediate. The reason suggested by the envelope condition is that $W_2(k,b)$ depends on W itself unlike $W_1(k,b)$, that is, the effect of changing discount factor in the future is circulative.

3.5.2 Comparison with present bias

To obtain concrete observations, we adopt the numerical method. Theorem 2 in Section 4.2 guarantees the use of the value function iteration algorithm. We adopt the following parametrization:

$$\begin{aligned} u(c) &= c^{1-\rho}, \\ f(k) &= rk^a, \\ q(b,b') &= \gamma \tan^2 \frac{\pi(b'-b)}{2(\bar{b}-b)+\varepsilon}, \end{aligned}$$

where $0 < \rho < 1$, and ε is a sufficiently small number put to avoid a zero divisor securely.

We also adopt the following specification of the parameters:

α	ρ	r	a	$\underline{b}$	$\bar{b}$	γ	ε
2	0.5	1	0.5	0.01	0.5	0.01	0.0001

To compare, we also consider a solution following Gul and Pesendorfer (2004, 2007), in which the agent is tempted only by current consumption, in the form

$$\begin{aligned} W_{GP}(k,b) &= \max_{k',b'}\Big\{ u(f(k)-k'-q(b,b')) + bW_{GP}(k',b') \\ &\quad +\alpha\left[u(f(k)-k'-q(b,b')) - \max_{\widetilde{k}',\widetilde{b}'} u(f(k)-\widetilde{k}'-q(b,\widetilde{b}'))\right]\Big\}, \end{aligned}$$

where $\max_{\widetilde{k}',\widetilde{b}'} u(f(k)-\widetilde{k}'-q(b,\widetilde{b}'))$ is simply $u(f(k))$ in the current setting.

Below, PC refers to the perfect commitment solution, PBT refers to the projection bias temptation solution (our model), GP refers to the Gul-Pesendorfer solution, and SU refers to the stationary utility solution in which discount factor is assumed to be unchanged over time.

Figure 1a depicts the transition mappings for the discount factor, where the current capital level is fixed to be $k = 0.1$. Note that SU leads to the 45-degree line, since the discount factor is unchanged. Both PBT and GP lead to slower growth in patience level than PC, where GP predicts even slower growth. The reasons and meanings of slower growth are different, even with the same value of α. In GP, the slow growth comes from the fact that the agent is reluctant to spend resource for any purpose other than consumption. In PBT, the slower growth comes from the fact that the agent sees investment on patience as in PC is sub-optimal because of the projection bias.

Figure 1b presents the level of patience in the next period when the current discount factor is $b = 0.25$ and the amount of physical capital varies. Note that SU leads to the horizontal line. We see that the gap in the growth of patience level is consistent across capital levels.

Figure 2a depicts how savings vary across the current level of patience, where the current capital level is fixed to be $k = 0.1$. In contrast to the transition of patience level, PC, PBT and SU lead to similar graphs, whereas GP leads to one far below.

Given the same level of the current discount factor, the differences in temporary savings amount among PC, PBT and SU are small. However, this does not mean that the saving paths starting from a pair of initial capital and initial discount factor are similar, because PBT exhibit slower growth of patience level and SU allows no growth of patience.

We see that the GP model predicts significantly lower savings, which is because the agent is reluctant to spend resources for any purpose other than consumption. An asymmetry exists in that the agent in the GP model chooses to invest more on the patience level rather than on physical capital, provided that the total amount of investment is constant.

Figure 2b depicts how savings vary across current capital levels, where the current discount factor is fixed to be $b = 0.25$. Again, we see that the GP model predicts lower savings. Also, we see that *given* the current level of patience, the differences of temporary savings amount among PC, PBT and SU are small, which is a similar pattern to the comparison varying the current discount factors under a fixed capital level as in Figure 2a. The same remark applies.

4 Axiomatic foundation

4.1 Axioms

This section provides an axiomatic foundation for the PBT representation. First six axioms are fairly standard in the literature of dynamic decision making with menus of lotteries. See Gul and Pesendorfer (2004, 2007).

Axiom 1 (Order) For all $h \in H$, $\succsim_h$ is complete and transitive.

Axiom 2 (Continuity) For all $h \in H$, the set $\{(x, y) \in \mathcal{Z} \times \mathcal{Z} : x \succsim_h y\}$ is closed.

For all $x, y \in \mathcal{Z}$ and $\lambda \in [0, 1]$, define the mixture operation by

$$\lambda x + (1 - \lambda)y := \{\lambda l + (1 - \lambda)l' | l \in x, l' \in y\}.$$

Axiom 3 (Independence) For all $h \in H$, for all $x, y, z \in \mathcal{Z}$ and $\lambda \in (0, 1)$,

$$x \succ_h y \quad \Longrightarrow \quad \lambda x + (1 - \lambda)z \succ_h \lambda y + (1 - \lambda)z.$$

Axiom 4 (Set Betweeness) For all $h \in H$, for all $x, y \in \mathcal{Z}$,

$$x \succsim_h y \quad \Longrightarrow \quad x \succsim_h x \cup y \succsim_h y.$$

Axiom 5 (Timing Indifference) For all $h \in H$, for all $c \in C$, $x, y \in \mathcal{Z}$, and $\lambda \in [0, 1]$,

$$\{\lambda \circ (c, x) + (1 - \lambda) \circ (c, y)\} \sim_h \{(c, \lambda x + (1 - \lambda) y)\}.$$

In the statement of the axiom, $\lambda \circ (c, x) + (1 - \lambda) \circ (c, y)$ is a lottery which gives (c, x) or (c, y) with probabilities λ and $1 - \lambda$, respectively.

Axiom 6 (Dynamic Consistency) For all $h \in H$, for all $c \in C$ and $x, y \in \mathcal{Z}$,

$$\{(c, x)\} \succsim_h \{(c, y)\} \iff x \succsim_{hc} y.$$

From now on, we assume three axioms that impose some restrictions on the nature of temptation. The first two axioms are adapted from Noor (2007) with translation into a history-dependent model.

Let $\Delta_s(C \times \mathcal{Z}) \subset \Delta(C \times \mathcal{Z})$ be the set of probability measures with finite support. For any $\mu \in \Delta_s$, μ^1 and μ^2 denote the marginal distributions of μ on C and $\mathcal{Z}$, respectively. Take any $\nu \in \Delta(\mathcal{Z})$ with finite support. It can be written as $\nu = (\alpha_i, z_i)_{i=1}^n$, where $z_i \in \mathcal{Z}$ and $\alpha_i \in [0, 1]$ with $\sum_{i=1}^n \alpha_i = 1$. Define

$$\varphi(\nu) = \sum_{i=1}^n \alpha_i z_i \in \mathcal{Z}.$$

Notice that the difference between ν and $\varphi(\nu)$ is the timing of resolution of risk. In case of the former, after the resolution of lottery ν, the agent makes a choice from a realized menu, while in the latter, after he chooses from the menu $\varphi(\nu)$, an outcome is realized according to the chosen lottery. If the choice from the menu is made according to expected utility, the agent should be indifferent between ν and $\varphi(\nu)$.

Axiom 7 (Temptation Timing Indifference) For all $\mu \in \Delta(C \times \mathcal{Z})$ and $\eta, \nu \in \Delta_s(C \times \mathcal{Z})$, if $\{\mu\} \succ_h \{\mu, \eta\} \succ_h \{\eta\}$ and $\{\mu\} \succ_h \{\mu, \nu\} \succ_h \{\nu\}$, and if $\eta^1 = \nu^1$ and $\varphi(\eta^2) = \varphi(\nu^2)$, then $\{\mu, \eta\} \sim_h \{\mu, \nu\}$.

Notice that $\{\mu\} \succ_h \{\mu, \eta\} \succ_h \{\eta\}$ means that η is tempting than μ but the agent resists to this temptation. Thus, the evaluation of the menu $\{\mu, \eta\}$ reflects self-control costs at $\{\mu, \eta\}$. Temptation Timing Indifference imposes two restrictions. First, temptation preference satisfies separability across time, that is, temptation preference cares about only marginal distributions of a lottery on C and Z, in which case η and ν are equally tempting as long as $\eta^1 = \nu^1$ and $\eta^2 = \nu^2$. Second, temptation preference does not care about the timing of resolution of risk, in which case η^2 and ν^2 are indifferent to $\varphi(\eta^2)$ and $\varphi(\nu^2)$, respectively. Therefore, if $\eta^1 = \nu^1$ and $\varphi(\eta^2) = \varphi(\nu^2)$, then η and ν are equally tempting, and hence, $\{\mu, \eta\}$ should be indifferent to $\{\mu, \nu\}$.

To ensure that temptation preference admits a stationary recursive representation, Noor (2007) assumes the following axiom called Temptation Stationarity: For all $c \in C$ and $x, y \in \mathcal{Z}$,

$$x \succ x \cup y \iff \{(c, x)\} \succ \{(c, x), (c, y)\}. \tag{22}$$

This axiom states that preference for commitment does not change with one period delay after consumption c. Since Noor's model satisfies the stationarity axiom, $x \succ y$ if and only if $\{(c, x)\} \succ \{(c, y)\}$ for all x, y and c. Since this implies that preference over menus is stationary, Temptation Stationarity detects that the temptation preference is also stationary over time.

However, in the case of a history dependent model, consumption may change preference over menus from the next period on. Then, even if temptation preference is stationary, (22) may not hold. For example, suppose that $\{l\} \succ_h \{l, l'\}$. This ranking suggests that l is preferred to l' from the normative perspective but l' is tempting than l at history h. There may exist some consumption $c \in C$ such that at the updated history hc, the normative ranking between l and l' is reversed, and hence, together with Dynamic Consistency, we have $\{(c, \{l'\})\} \succ_h \{(c, \{l\})\}$. Then, Set Betweenness implies that $\{(c, \{l'\})\} \succsim_h \{(c, \{l\}), (c, \{l'\})\} \succsim_h \{(c, \{l\})\}$, which violates (22).

Notice that the above counterexample can be resolved if we apply Noor's Temptation Stationarity axiom only for all c that do not reverse the ranking between x and y across histories. The next axiom is motivated by such reasoning. To do so, we define one notion. For all $c \in C$ and $x, y \in \mathcal{Z}$, say that (c, y) is not tempted by (c, x) robustly if $\{(c, y)\} \succ_h \{(c, x)\}$ and for all $x^n \to x$ and $y^n \to y$, we have $\{(c, y^n)\} \sim_h \{(c, x^n), (c, y^n)\}$ for all sufficiently large n.

Axiom 8 (Temptation Stationarity) For all $h \in H$, for all $c \in C$ and $x, y \in \mathcal{Z}$ with $x \succ_h y$, if $\{(c, x)\} \succ_h \{(c, y)\}$, then

$$x \succ_h x \cup y \iff \{(c, x)\} \succ_h \{(c, x), (c, y)\},$$

and if $\{(c,y)\} \succ_h \{(c,x)\}$, then

$$x \succ_h x \cup y \quad \Longrightarrow \quad \{(c,y)\} \sim_h \{(c,x),(c,y)\}$$

and the converse also holds if (c,y) is not tempted by (c,x) robustly.

The first condition states that if c does not change the ranking between x and y, the preference for commitment does not change either before or after consumption c. The second condition states that if c reverses the ranking between x and y, the preference for commitment disappears after consumption c. If temptation preference is stationary, y is still tempting than x after consumption c. Since y is preferred to x both from normative and temptation preferences, internal conflict disappears and so does preference for commitment.

Notice that $\{(c,y)\} \sim_h \{(c,x),(c,y)\}$ suggests only that y is at least as tempting as x after consumption c, which does not imply $x \succ_h x \cup y$. If (c,y) is not tempted by (c,x) robustly, however, y should be strictly tempting than x, and hence the converse is also true.

The last axiom is concerned with the identification of preference over current consumption and time preference. For all $c \in C$, let $\boldsymbol{c}$ denote the consumption sequence $(c, \{(c, \{\cdots\})\})$.

Axiom 9 (Habit Free) There exist $\overline{c}, \underline{c} \in C$ such that for all $h \in H$, $\{(\overline{c}, \{\underline{\boldsymbol{c}}\})\} \succ_h \{\underline{\boldsymbol{c}}\}$, and

$$\{(\ell, \{\boldsymbol{c}\})\} \succsim_h \{(\ell', \{\boldsymbol{c}'\})\} \quad \Longrightarrow \quad \{(\ell, \{\boldsymbol{c}\})\} \sim_h \{(\ell, \{\boldsymbol{c}\}), (\ell', \{\boldsymbol{c}'\})\}$$

for all $\ell, \ell' \in \Delta(C)$ and $\boldsymbol{c}, \boldsymbol{c}' \in \{(\overline{c}, \{\underline{\boldsymbol{c}}\}), \underline{\boldsymbol{c}}\}$.

Notice that the only difference between $\{(\overline{c}, \{\underline{\boldsymbol{c}}\})\}$ and $\{\underline{\boldsymbol{c}}\}$ is whether current consumption is either $\overline{c}$ or $\underline{c}$. Thus, $\{(\overline{c}, \{\underline{\boldsymbol{c}}\})\} \succ_h \{\underline{\boldsymbol{c}}\}$ means that $\overline{c}$ and $\underline{c}$ are "numeraire" goods and their ranking is the same among all histories. Therefore, if the agent compares two options whose future menus are either $\{(\overline{c}, \{\underline{\boldsymbol{c}}\})\}$ or $\{\underline{\boldsymbol{c}}\}$, the current consumption does not affect their evaluations. In such a case, there is no internal conflict associated with habit formation; hence, the agent does not exhibit a preference for commitment.

For example, in the specification of $u(\cdot|h)$ as given in Section 2.3.1, C is assumed to be the set of menus, $C = \mathcal{K}(\Delta(A))$, and $u(\cdot|h)$ itself is assumed to be a self-control utility together with the assumption that the normative utility $\mathbf{u}(a)$ on A is history-independent. In such a setting, for any $\mathbf{u}(a) > \mathbf{u}(a')$, $\overline{c} := \{a\}$ and $\underline{c} := \{a'\}$ satisfy the Habit Free axiom. If C is assumed to be a compact interval of the real numbers as in Section 3.5, the maximum and the minimum consumption play the role for the numeraire in the Habit Free axiom.

4.2 The representation theorem

We consider the agent who suffers from self-control problems at any history.

Definition 2 $\{\succsim_h\}_{h\in H}$ is non-degenerate if for all $h \in H$, there exist $z, z' \in \mathcal{Z}$ such that $z \subset z'$ and $z \succ_h z'$.

We are ready to state our main representation theorem.

Theorem 1 Preference $\{\succsim_h\}$ is non-degenerate and satisfies Axioms 1-9 if and only if it admits a PBT representation.

Our representation result heavily relies on Noor (2007). Other than the history-dependent extension, we specify $u(\cdot|h) = v(\cdot|h)$ and $\beta(h) = \gamma(h)$ in Noor's model, which is a necessary restriction for our interpretation. The Habit Free axiom plays a significant role here.

To ensure the existence for the functional form $W : \mathcal{Z} \times H \to \mathbb{R}$ satisfying the functional equation (3), we apply the contraction mapping theorem. Since H is not compact, by technical reason, its closure, that is, $\overline{H} = C^\infty$, is taken here.

Theorem 2 For all continuous functions $u : C \times \overline{H} \to \mathbb{R}$, $\beta : \overline{H} \to (0, 1)$, $\alpha : \overline{H} \to \mathbb{R}_+$ satisfying $u(\underline{c}|h) = 0$ and $u(\overline{c}|h) = u(\overline{c}|h') > 0$ for all $h, h' \in H$, there is a unique continuous function $W : \mathcal{Z} \times \overline{H} \to \mathbb{R}$ that satisfies the functional equation (3).

We turn to the uniqueness of the representation. Let $\mathcal{L}$ be the set of perfect commitment menus where the agent is committed in every period. We identify a singleton menu with its only element. Then a perfect commitment menu can be viewed as a multistage lottery, considered by Epstein and Zin (1989), that is, $\mathcal{L}$ is a subdomain of $\mathcal{Z}$ satisfying $\mathcal{L} \simeq \Delta(C \times \mathcal{L})$.

If the agent does not exhibit self-control at any decision problem, neither self-control costs nor their parameter $\alpha(h)$ is materialized through behavior. We require that each $\succsim_h$ exhibits self-control at some menu consisting of perfect commitment lotteries. We say that $\{\succsim_h\}_{h\in H}$ satisfies Regularity if for all $h \in H$, there exist $l, l' \in \mathcal{L}$ such that $\{l\} \succ_h \{l, l'\} \succ_h \{l'\}$.

Theorem 3 If $\{\succsim_h\}_{h\in H}$ satisfies Regularity and is represented by two PBT representations $(u(\cdot|h),\beta(h),\alpha(h))_{h\in H}$ and $(u'(\cdot|h),\beta'(h),\alpha'(h))_{h\in H}$ with the common $\underline{c}\in C$, then there exists $\zeta>0$ such that for all h, $u'(\cdot|h)=\zeta u(\cdot|h)$, $\beta(h)=\beta'(h)$, and $\alpha(h)=\alpha'(h)$.

Since $u(\cdot|h)$ is an expected utility, it is unique up to positive linear transformation.[3] Moreover, the multiplier ζ must be constant across histories. The time preference or discount factor is uniquely identified. Regularity is used to show the uniqueness of the self-control parameter $\alpha(h)$.

4.3 Comparisons in preference parameters across histories

4.3.1 Comparisons of $\beta(h)$ across histories

To elicit the agent's degree of impatience or the magnitude of $\beta(h)$, it is enough to focus on the preference over the perfect commitment options and look at rankings between a sooner but smaller reward and a later but larger reward. It should be noted that when $u(\cdot|h)$ is history-dependent, the evaluation of present and future gains are not time-separable, and the current consumption will generally affect the future utility. Even if we observe the preferences of sooner-smaller and later-larger rewards, we cannot separate whether they reflect the magnitude of the discount factor or the change in future utility due to history dependence.

To avoid this problem, we design intertemporal trade-offs using only history-independent alternatives. The condition below states that the agent is said to be more patient at history h^1 than at h^2 if he prefers a later but larger reward at h^2 then he must prefer such reward at h^1 as well.

Definition 3 The agent is more patient at history h^1 than at h^2 if

$$\{(\mu\delta_{\overline{\mathbf{c}}}+(1-\mu)\delta_{\underline{\mathbf{c}}},\{(\overline{c},\{\underline{\mathbf{c}}\})\})\}\succsim_{h^2}\{(\nu\delta_{\overline{\mathbf{c}}}+(1-\nu)\delta_{\underline{\mathbf{c}}},\{\underline{\mathbf{c}}\})\}$$

implies

$$\{(\mu\delta_{\overline{\mathbf{c}}}+(1-\mu)\delta_{\underline{\mathbf{c}}},\{(\overline{c},\{\underline{\mathbf{c}}\})\})\}\succsim_{h^1}\{(\nu\delta_{\overline{\mathbf{c}}}+(1-\nu)\delta_{\underline{\mathbf{c}}},\{\underline{\mathbf{c}}\})\}$$

for all $\mu,\nu\in[0,1]$.

[3] Notice that the normalization of the representation requires $u(\underline{c}|h)=0$. Thus, adding a constant to $u(\cdot|h)$ is excluded.

The choice of $(\bar{c}, \{\underline{\mathbf{c}}\})$ as the later but larger reward is due to the reason discussed above. Because of habit formation, the discount rate at history h can only be the one between the current period and the next period, since the discount rate between a future period and a further future period is in general subject to a future consumption history until then that is not chosen yet. Also, whether a consumption bundle to be given in the next period deserves to be a "reward" may depend on consumption histories including the current consumption bundle, hence we choose the habit-free bundle $\bar{c}$.

Proposition 1 Assume that $\{\succsim_h\}_{h \in H}$ satisfies all the axioms of Theorem 1. Then the agent is more patient at history h^1 than at h^2 if and only if $\beta(h^1) \geq \beta(h^2)$.

Proof. Without loss of generality, let $u(\bar{c}|h) = \overline{u} > 0$ for all $h \in H$. Then, under Theorem 1, the condition is equivalent to

$$\mu\overline{u} + \beta(h^2)\overline{u} \geq \nu\overline{u} \implies \mu\overline{u} + \beta(h^1)\overline{u} \geq \nu\overline{u}$$

for all $\mu, \nu \in [0, 1]$, which is equivalent to $\beta(h^1) \geq \beta(h^2)$. ■

4.3.2 Comparisons of $\alpha(h)$ across histories

The magnitude of $\alpha(h)$ corresponds to the intensity of the projection bias, and the degree of this bias determines the extent to which the agent is willing to avoid temptation, and the extent to which he is able to exercise self-control over the bias. Thus, by comparing attitudes toward commitment across histories, we can compare the magnitude of $\alpha(h)$.

Again, notice that when the utility function and discount factor are history dependent, the current consumption will generally affect the future utility and future discount factor. Differences in these components between two histories can alter whether and to what extent self-control problems arise, even when faced with the same choice problem. In order to compare the magnitude of $\alpha(h)$, we need to focus on histories where these components are the same from those histories onward.

We consider a behavioral comparison about the attitude toward commitment. If the agent faces a more severe self-control problem at history h^1 than at history h^2, he is presumably more willing to commit to a specific plan at h^1.

Definition 4 The agent is *more willing to make a commitment at history* h^1 *than at* h^2 if for all $x \in \mathcal{Z}$ and $l \in \mathcal{L}$,

$$\{l\} \succsim_{h^2} x \implies \{l\} \succsim_{h^1} x.$$

This condition states that if the agent prefers a perfect commitment $\{l\}$ to a menu x at h^2, so does he at h^1.

For each $\overline{h} \in H$ and any $h = (c_1, \cdots, c_t) \in C^t$, let $\overline{h}h$ denote an updated history $(h, c_1, \cdots, c_t) \in H$. Let $C^0 = \emptyset$ with convention.

Proposition 2 Assume that $\{\succsim_h\}_{h \in H}$ satisfies all the axioms of Theorem 1 and Regularity. If the agent is more willing to make a commitment at history h^1 than at h^2, then there exists a PBT representation W with (u, β, α) such that for any finite history $h \in C^t$ and for any $t \geq 0$, (i) $u(\cdot|h^1 h) = u(\cdot|h^2 h)$, (ii) $\beta(h^1 h) = \beta(h^2 h)$, and (iii) $\alpha(h^1) \geq \alpha(h^2)$.

Appendices

A Proof of Theorem 1

A.1 Sufficiency

The proof of sufficiency takes the following three steps. First, we obtain a Gul-Pesendorder type representation for each history. To obtain continuity with respect to histories, we follow a construction of representations given by Gul and Pesendorfer (2007). Next, we show that the Gul-Pesendorfer type representation in each history is rewritten as the FT representation of Noor (2007). This step heavily relies on several lemmas proved in Noor (2007). We appropriately modify Noor's proof, where stationarity is assumed, to accommodate history dependence. Where the same proof holds, we omit the proof. Finally, we further impose necessary restrictions on the FT representation to obtain a PBT representation.

We use the following notation throughout. For all $l, l' \in \Delta(C \times \mathcal{Z})$ and $\lambda \in [0,1]$, $\lambda l + (1-\lambda) l'$ is simply denoted by $l \lambda l'$. Let $\mathbf{c}^* := (\overline{c}, \{\underline{\mathbf{c}}\})$.

First of all, the lemma below follows from the static Gul-Pesendorfer representation over $\mathcal{Z}$.

Lemma 1 For each $h \in H$, $\succsim_h$ allows a representation $W(\cdot|h) : \mathcal{Z} \to \mathbb{R}$ in the form

$$W(x|h) = \max_{l \in x} \left\{ U(l|h) + \left(V(l|h) - \max_{m \in x} V(m|h) \right) \right\},$$

where $U(\cdot|h), V(\cdot|h) : \Delta(C \times \mathcal{Z}) \to \mathbb{R}$ are mixture-linear and continuous. Moreover, W can be taken to satisfy $W(\{\mathbf{c}^*\}|h) = 1$ and $W(\{\underline{\mathbf{c}}\}|h) = 0$.

Recall the initial history $\overline{h}$. Let $(\widetilde{W}(z|\bar{h}), \widetilde{U}(l|\bar{h}), \widetilde{V}(l|\bar{h}))$ be the corresponding representation of $\succsim_{\bar{h}}$. Since every history $h \in H$ corresponds to a finite period of consumption from the initial history $\overline{h}$, the representation at h can be derived from $\widetilde{W}(z|\bar{h})$. A merit of this construction is that the joint continuity of representations in menus and histories follows from the continuity of $\widetilde{W}(z|\bar{h})$ in menus. Formally, for all $x \in \mathcal{Z}$ and $h = (\bar{h}c_n, \cdots, c_1) \in H$,

$$\widetilde{W}(\{(c_n, \{(c_{n-1}, \cdots, \{(c_1, x)\} \cdots)\})\}|\bar{h})$$

is simply denoted by $\widetilde{W}(c_n, c_{n-1}, \cdots, c_1, x|\bar{h})$. Define

$$W(x|h) := \theta(h)\widetilde{W}(c_n, c_{n-1}, \cdots, c_1, x|\bar{h}) + \zeta(h), \tag{23}$$

where

$$\theta(h) := \frac{1}{\widetilde{W}(c_n, c_{n-1}, \cdots, c_1, \{\mathbf{c}^*\}|\bar{h}) - \widetilde{W}(c_n, c_{n-1}, \cdots, c_1, \{\underline{\mathbf{c}}\}|\bar{h})},$$
$$\zeta(h) := -\theta(h)\widetilde{W}(c_n, c_{n-1}, \cdots, c_1, \{\underline{\mathbf{c}}\}|\bar{h}).$$

By Dynamic Consistency, $\widetilde{W}(c_n, c_{n-1}, \cdots, c_1, \{\mathbf{c}^*\}|\bar{h}) > \widetilde{W}(c_n, c_{n-1}, \cdots, c_1, \{\underline{\mathbf{c}}\}|\bar{h})$ because $\{\mathbf{c}^*\} \succ_h \{\underline{\mathbf{c}}\}$. Thus, $\theta(h) > 0$. Since $\widetilde{W}(\cdot|\bar{h})$ is continuous, $W(\cdot|\cdot) : \mathcal{Z} \times H \to \mathbb{R}$ is continuous. Moreover, by definition, $W(\{\mathbf{c}^*\}|h) = 1$ and $W(\{\underline{\mathbf{c}}\}|h) = 0$ for all h.

Lemma 2 The functional form $W(\cdot|h)$, defined as in (23), represents $\succsim_h$. Moreover, there exist continuous functions U and V such that

$$W(x|h) = \max_{l \in x} \left\{ U(l|h) + \Big(V(l|h) - \max_{m \in x} V(m|h) \Big) \right\}$$

and W, U, and V are mixture linear in their first argument.

Proof. We adopt the same argument as in Lemma A.1 of Gul and Pesendorfer (2007). By Dynamic Consistency, for all $h = (\bar{h}c_n, \cdots, c_1) \in H$, $x \succsim_h y$ if and only if

$$\{(c_n, \{(c_{n-1}, \cdots, \{(c_1, x)\} \cdots)\})\} \succsim_{\bar{h}} \{(c_n, \{(c_{n-1}, \cdots, \{(c_1, y)\} \cdots)\})\}.$$

Therefore, for all $h \in H$, $W(\cdot|h) : \mathcal{Z} \to \mathbb{R}$ represents $\succsim_h$.

By applying Timing-Indifference and Dynamic Consistency finitely many times,

$$\begin{aligned}&\{\lambda \circ (c_n, \{(c_{n-1}, \cdots, \{(c_1, x)\} \cdots)\}) + (1-\lambda) \circ (c_n, \{(c_{n-1}, \cdots, \{(c_1, y)\} \cdots)\})\} \\ &\sim_{\bar{h}} \{(c_n, \{(c_{n-1}, \cdots, \{(c_1, \lambda x + (1-\lambda)y)\} \cdots)\})\}.\end{aligned}$$

Since $\widetilde{W}(x|\bar{h})$ is mixture linear in x,

$$\begin{aligned}W(\lambda x + (1-\lambda)y|h) &= \theta(h)\widetilde{W}(c_n, \cdots, c_1, \lambda x + (1-\lambda)y|\bar{h}) + \zeta(h) \\ &= \theta(h)(\lambda\widetilde{W}(c_n, \cdots, c_1, x|\bar{h}) + (1-\lambda)\widetilde{W}(c_n, \cdots, c_1, y|\bar{h})) + \zeta(h) \\ &= \lambda W(x|h) + (1-\lambda)W(y|h),\end{aligned}$$

as desired.

For each $h \in H$, Lemma 1 ensures that there exists a representation $(\widetilde{W}(\cdot|h), \widetilde{U}(\cdot|h), \widetilde{V}(\cdot|h))$ for $\succsim_h$. Moreover, by Non-degeneracy and Habit Free, $\succsim_h$ is regular in the sense of Gul and Pesendorfer (2007, p.165). From the above observations, $W(\cdot|h)$ is cardinally equivalent to $\widetilde{W}(\cdot|h)$, that is, there exist $\gamma(h) > 0$ and $\eta(h) \in \mathbb{R}$ satisfying $W(x|h) = \gamma(h)\widetilde{W}(x|h) + \eta(h)$. Hence, by defining $U(x|h) = \gamma(h)\widetilde{U}(x|h) + \eta(h)$, $V(x|h) = \gamma(h)\widetilde{V}(x|h)$, W have the desired properties. ■

From now on, we show that $U(\cdot|h)$ and $V(\cdot|h)$ admit the form of the FT representation of Noor (2007). Define $\overline{V}(\cdot|h) : \mathcal{Z} \to \mathbb{R}$ and $\overline{(U+V)}(\cdot|h) : \mathcal{Z} \to \mathbb{R}$ by

$$\begin{aligned}\overline{V}(x|h) &:= \max_{l \in x} V(l|h), \\ \overline{(U+V)}(x|h) &:= \max_{l \in x}\{U(l|h) + V(l|h)\}.\end{aligned}$$

As shown by Lemmas A.7 and A.8 in Noor (2007), $\overline{V}(x|h)$ is mixture linear and continuous.

Lemma 3 For all $x, y \in \mathcal{Z}$,

(1) $x \succ_h x \cup y \iff \overline{V}(y|h) > \overline{V}(x|h)$ and $W(x|h) > W(y|h)$.

(2) $x \cup y \succ_h y \iff \overline{(U+V)}(x|h) > \overline{(U+V)}(y|h)$ and $W(x|h) > W(y|h)$.

(3) $x \succ_h x \cup y \succ_h y \iff \overline{(U+V)}(x|h) > \overline{(U+V)}(y|h)$ and $\overline{V}(y|h) > \overline{V}(x|h)$.

Proof. The proofs for (1) and (2) are exactly the same as in Noor (2007). If $\overline{(U+V)}(x|h) > \overline{(U+V)}(y|h)$ and $\overline{V}(y|h) > \overline{V}(x|h)$, we have

$$W(x|h) = \overline{(U+V)}(x|h) - \overline{V}(x|h) > \overline{(U+V)}(y|h) - \overline{V}(y|h) = W(y|h).$$

Thus, (3) also follows. ■

Lemma 4 $U(\cdot|h)$ has the form

$$U(l|h) = \int_{C\times\mathcal{Z}} \{u(c|h) + \beta(c|h)W(z|hc)\}\, \mathrm{d}l(c,z).$$

Proof. Since $U(\cdot|h)$ is an expected utility representation, it is written in the form

$$U(l|h) = \int_{C\times\mathcal{Z}} U((c,x)|h)\, \mathrm{d}l(c,z).$$

By Dynamic Consistency, both $U((c,\cdot)|h)$ and $W(\cdot|hc)$ represent the same ranking over $\mathcal{Z}$ conditional on h and c. Moreover, by Timing Indifference, $U((c,\cdot)|h)$ is mixture linear, and hence, there exist $\beta(c|h) > 0$ and $u(c|h) \in \mathbb{R}$ such that $U((c,z)|h) = u(c|h) + \beta(c|h)W(z|hc)$. Therefore,

$$U(l|h) = \int_{C\times\mathcal{Z}} \{u(c|h) + \beta(c|h)W(z|hc)\}\, \mathrm{d}l(c,z).$$

■

Lemma 5 For all $h \in H$, $u(\overline{c}|h) = 1$ and $u(\underline{c}|h) = 0$.

Proof. Since $0 = W(\{\underline{\mathbf{c}}\}|h) = u(\underline{c}|h) + \beta(\underline{c}|h) \times 0$, we have $u(\underline{c}|h) = 0$. Moreover, $1 = W(\{\mathbf{c}^*\}|h) = W(\{(\overline{c}, \underline{\mathbf{c}})\}|h) = u(\overline{c}|h) + \beta(\overline{c}|h) \times 0$. Thus, $u(\overline{c}|h) = 1$. ■

We want to show that $V(\cdot|h)$ has the form

$$V(l|h) = \int_{C\times\mathcal{Z}} \left\{v(c|h) + \gamma(h)\overline{V}(z|h)\right\} \mathrm{d}l(c,z),$$

where $\gamma(h) \in (0,1)$.

If $V(\cdot|h)$ is constant or $U(\cdot|h)$ is a positive affine transformation of $V(\cdot|h)$, it violates Non-degeneracy. Hence, suppose that $V(\cdot|h)$ is not constant and $U(\cdot|h)$ is not a positive affine transformation of $V(\cdot|h)$. It is impossible that $V(\cdot|h) = \theta U(\cdot|h) + \zeta$ for some $\theta \leq -1$. Indeed, then, for all $\{l\} \succ_h \{l'\}$, we have $\{l\} \succ_h \{l, l'\} \sim_h \{l'\}$, which violates the Habit Free axiom. Thus, either $\theta \in (-1, 0)$ or $U(\cdot|h)$ is not an affine transformation of $V(\cdot|h)$.

Let $\Delta_s \subset \Delta(C\times\mathcal{Z})$ be the set of probability measures with finite support. The following lemma is shown by the same argument as in Lemma A.2 of Noor (2007).

Lemma 6 There exist $\bar{l}, \underline{l} \in \Delta_s$ such that

$$\{\bar{l}\} \succ_h \{\bar{l}, \underline{l}\} \succ_h \{\underline{l}\}.$$

Moreover, for all finite $L \subset \Delta(C \times \mathcal{Z})$, there exists $\lambda \in (0, 1]$ such that for all $m \in L$

$$\{\bar{l}\} \succ_h \{\bar{l}, m\lambda\underline{l}\} \succ_h \{m\lambda\underline{l}\}.$$

The next two lemmas establish separability of $V(\cdot|h)$.

Lemma 7

$$V\left(\frac{1}{2}(c, z) + \frac{1}{2}(c', z')\middle| h\right) = V\left(\frac{1}{2}(c, z') + \frac{1}{2}(c', z)\middle| h\right).$$

Proof. Let $\nu^1 = \frac{1}{2}(c, z) + \frac{1}{2}(c', z')$ and $\nu^2 = \frac{1}{2}(c, z') + \frac{1}{2}(c', z)$. By Lemma 6, there exist $\bar{l}$ and $\underline{l}$ and $\lambda \in (0, 1]$ such that

$$\{\bar{l}\} \succ_h \{\bar{l}, \nu^1\lambda\underline{l}\} \succ_h \{\nu^1\lambda\underline{l}\} \text{ and } \{\bar{l}\} \succ_h \{\bar{l}, \nu^2\lambda\underline{l}\} \succ_h \{\nu^2\lambda\underline{l}\}.$$

Since $\nu^1\lambda\underline{l}$ and $\nu^2\lambda\underline{l}$ have the same marginals both on C and $\mathcal{Z}$, by Temptation Timing Indifference, $\{\bar{l}, \nu^1\lambda\underline{l}\} \sim_h \{\bar{l}, \nu^2\lambda\underline{l}\}$. By the representation,

$$\begin{aligned}
& U(\bar{l}|h) + V(\bar{l}|h) - V(\nu^1\lambda\underline{l}|h) = U(\bar{l}|h) + V(\bar{l}|h) - V(\nu^2\lambda\underline{l}|h) \\
\iff & V(\nu^1\lambda\underline{l}|h) = V(\nu^2\lambda\underline{l}|h) \\
\iff & \lambda V(\nu^1|h) + (1-\lambda)V(\underline{l}|h) = \lambda V(\nu^2|h) + (1-\lambda)V(\underline{l}|h) \\
\iff & V(\nu^1|h) = V(\nu^2|h),
\end{aligned}$$

as desired. ■

Lemma 8 There exist continuous functions $v(\cdot|h) : C \to \mathbb{R}$ and $\widehat{V}(\cdot|h) : \mathcal{Z} \to \mathbb{R}$ such that for all $\mu \in \Delta(C \times \mathcal{Z})$,

$$V(\mu|h) = \int_{C \times \mathcal{Z}} (v(c|h) + \widehat{V}(z|h))\, \mathrm{d}\mu.$$

Proof. The proof is exactly the same as Lemma A.4 in Noor (2007). ■

To show mixture linearity of $\widehat{V}(\cdot|h)$, we prepare the next lemma.

Lemma 9 For all $c \in C$, $z, z' \in \mathcal{Z}$, and $\lambda \in [0, 1]$,

$$V(\lambda(c, z) + (1-\lambda)(c, z')|h) = V((c, \lambda z + (1-\lambda)z')|h).$$

Proof. The proof is the same as in Lemma 7. Let $\nu^1 = \lambda(c,z) + (1-\lambda)(c,z')$ and $\nu^2 = (c, \lambda z + (1-\lambda)z')$. By Lemma 6, there exist $\bar{l}$ and $\underline{l}$ and $\lambda' \in (0,1]$ such that

$$\{\bar{l}\} \succ_h \{\bar{l}, \nu^1\lambda'\underline{l}\} \succ_h \{\nu^1\lambda'\underline{l}\} \text{ and } \{\bar{l}\} \succ_h \{\bar{l}, \nu^2\lambda'\underline{l}\} \succ_h \{\nu^2\lambda'\underline{l}\}.$$

Since $\nu^1\lambda'\underline{l}$ and $\nu^2\lambda'\underline{l}$ have the same marginals both on C and the same φ values on $\mathcal{Z}$, by Temptation Timing Indifference, $\{\bar{l}, \nu^1\lambda'\underline{l}\} \sim_h \{\bar{l}, \nu^2\lambda'\underline{l}\}$. By the representation,

$$\begin{aligned}
& U(\bar{l}|h) + V(\bar{l}|h) - V(\nu^1\lambda'\underline{l}|h) = U(\bar{l}|h) + V(\bar{l}|h) - V(\nu^2\lambda'\underline{l}|h) \\
\iff & V(\nu^1\lambda'\underline{l}|h) = V(\nu^2\lambda'\underline{l}|h) \\
\iff & \lambda' V(\nu^1|h) + (1-\lambda')V(\underline{l}|h) = \lambda' V(\nu^2|h) + (1-\lambda')V(\underline{l}|h) \\
\iff & V(\nu^1|h) = V(\nu^2|h),
\end{aligned}$$

as desired. ■

By Lemma 9, together with applying the same argument as in Lemma A.6 of Noor (2007), $\widehat{V}(\cdot|h)$ is mixture linear.

To obtain the PBT representation, we want to establish the cardinal equivalence between $v(\cdot|h)$ and $u(\cdot|h)$ and that between $\widehat{V}(\cdot|h)$ and $\overline{V}(\cdot|h)$. Moreover, $\beta(c|h)$ should be independent of c.

Lemma 10 For all compact subsets $x, y \subset \Delta(C) \times \{\mathbf{c}^*, \underline{\mathbf{c}}\}$,

$$x \succsim_h y \implies x \sim_h x \cup y.$$

Proof. First we show the lemma when x and y are finite. Since $\succsim_h$ satisfies Set Betweenness, by Lemma 2 of Gul and Pesendorfer (2001, p.1422), $W(x|h) = \min_{l \in x} \max_{l' \in x} W(\{l, l'\}|h)$ for all finite $x \in \mathcal{Z}$. Take any finite subset $x \subset \Delta(C) \times \{\mathbf{c}^*, \underline{\mathbf{c}}\}$. Fix any $l \in x$. By Habit Free, if $\{l'\} \succsim_h \{l\}$, $\{l'\} \sim_h \{l, l'\}$, and if $\{l\} \succ_h \{l'\}$, $\{l\} \sim_h \{l, l'\} \succ_h \{l'\}$. From the above observation,

$$W(x|h) = \min_{l \in x} \max_{l' \in \{l' \,|\, \{l'\} \succsim_h \{l\}\}.} W(\{l'\}|h). \tag{24}$$

Since $\{l' \,|\, \{l'\} \succsim_h \{\bar{l}\}\} \subset \{l' \,|\, \{l'\} \succsim_h \{l\}\}$ if $\{\bar{l}\} \succsim_h \{l\}$, (24) is minimized by a maximizer within x with respect to the commitment ranking. Thus, $W(x|h) = \max_{l \in x} U(l|h)$. Therefore, for all finite $x, y \subset \Delta(C) \times \{\mathbf{c}^*, \underline{\mathbf{c}}\}$, $x \succsim_h y$ implies that $W(x|h) = \max_{l \in x} U(l|h) = \max_{l \in x \cup y} U(l|h) = W(x \cup y|h)$.

Next consider all compact subsets $x, y \subset \Delta(C) \times \{\mathbf{c}^*, \underline{\mathbf{c}}\}$ with $x \succsim_h y$. If $x \sim_h y$, by Set Betweenness, $x \sim_h x \cup y \sim_h y$, so we are done. Suppose $x \succ_h y$. By Lemma 0 of Gul and

Pesendorfer (2001, p.1421), there exist sequences $x^n \to x$ and $y^n \to y$ such that x^n and y^n are finite subsets of x and y, respectively. By Continuity, $x^n \succ_h y^n$ for all sufficiently large n. By the above claim, $x^n \sim_h x^n \cup y^n$. Thus, we have $x \sim_h x \cup y$ as $n \to \infty$. ■

Lemma 11 For each $h \in H$, $v(\cdot|h)$ is cardinally equivalent to $u(\cdot|h)$, and $\beta(c|h)$ is independent of c.

Proof. Since $W(\{\underline{\mathbf{c}}\}|h) = 0$ for all h, for each menu $x_c \in \mathcal{K}(\Delta(C))$,

$$W((x_c, \{\underline{\mathbf{c}}\})|h) = \max_{\ell \in x_c} \{u(\ell|h) + v(\ell|h)\} - \max_{\ell \in x_c} v(\ell|h).$$

Since Lemma 10 implies that the agent does not exhibit preference for commitment on subsets of $\Delta(C) \times \{\mathbf{c}^*, \underline{\mathbf{c}}\}$, $v(\cdot|h)$ must be cardinally equivalent to $u(\cdot|h)$, and hence, there exists $\alpha(h) > 0$ and $\eta(h) \in \mathbb{R}$ such that

$$v(\cdot|h) = \alpha(h)u(\cdot|h) + \eta(h). \tag{25}$$

Next, since $W(\{\mathbf{c}^*\}|h) = 1$ for all h, together with (25), we have

$$W((x_c, \{\mathbf{c}^*\})|h) = \max_{\ell \in x_c} \{u(\ell|h) + \beta(\ell|h) + \alpha(h)u(\ell|h)\} - \max_{\ell \in x_c} \alpha(h)u(\ell|h),$$

where $\beta(\ell|h) = \int \beta(c|h)\, \mathrm{d}\ell(c)$. Again, Lemma 10 implies that the agent does not exhibit preference for commitment on subsets of $\Delta(C) \times \{\mathbf{c}^*, \underline{\mathbf{c}}\}$, $u(\cdot|h) + \beta(\cdot|h)$ must be cardinally equivalent to $u(\cdot|h)$. This is possible only when $\beta(c|h)$ is independent of c. ■

From now on, $\beta(c|h) = \beta(h)$. Next, we establish the cardinal equivalence between $\widehat{V}(\cdot|h)$ and $\overline{V}(\cdot|h)$.

Lemma 12 $\widehat{V}(\cdot|h)$ is not constant.

Proof. Seeking a contradiction, suppose that $\widehat{V}(\cdot|h)$ is constant, which is equal to $\kappa(h)$. By Lemmas 8 and 11, $V(l|h) = \alpha(h)u(l_c|h) + \eta(h) + \kappa(h)$, where l_c is the marginal of l on C. For all $\lambda \in (0, 1)$, define $l^\lambda = (\lambda \circ \overline{c} + (1 - \lambda) \circ \underline{c}, \{\mathbf{c}^*\})$. By Lemma 5, $W(\{l^\lambda\}|h) = \lambda u(\overline{c}|h) + (1 - \lambda)u(\underline{c}|h) + \beta(h) = \lambda + \beta(h)$. Since $\beta(h) > 0$, by continuity, $W(\{l^\lambda\}|h) > 1 = W(\{\mathbf{c}^*\}|h)$ as $\lambda \to 1$. On the other hand, by Lemma 5,

$$\begin{aligned} V(l^\lambda|h) &= \alpha(h)u(\lambda \circ \overline{c} + (1 - \lambda) \circ \underline{c}|h) + \eta(h) + \kappa(h) \\ &< \alpha(h) + \eta(h) + \kappa(h) = V(\mathbf{c}^*|h). \end{aligned}$$

By Lemma 3, $\{l^\lambda\} \succ_h \{l^\lambda, \mathbf{c}^*\}$, which contradicts to the Habit Free axiom. ■

As a preliminary step, Lemma A.9 in Noor (2007) shows that if $x \succ_h y$, then $\overline{V}(y|h) > \overline{V}(x|h)$ if and only if $\widehat{V}(y|h) > \widehat{V}(x|h)$. Since Noor's model satisfies Stationarity, $x \succ_h y$ is equivalent to $\{(c,x)\} \succ_h \{(c,y)\}$ for all c, which is not the case in our setting. We need the case-by-case argument.

Lemma 13 If $x \succ_h y$,

$$\overline{V}(y|h) > \overline{V}(x|h) \iff \widehat{V}(y|h) > \widehat{V}(x|h).$$

Proof. If $\{(c,x)\} \succ_h \{(c,y)\}$ for some $c \in C$, the same proof as in Lemma A.9 in Noor (2007) can be applied. Next, consider the case where $\{(c,y)\} \succsim_h \{(c,x)\}$ for some $c \in C$. By Dynamic Consistency, $y \succsim_{hc} x$. By Nondegeneracy, there exists $\tilde{x}, \tilde{y} \in \mathcal{Z}$ with $\tilde{y} \succ_{hc} \tilde{x}$. Let $x(\lambda) = \lambda x + (1-\lambda)\tilde{x}$ and $y(\lambda) = \lambda y + (1-\lambda)\tilde{y}$. By Independence, $y(\lambda) \succ_{hc} x(\lambda)$ for all $\lambda \in (0,1)$. Again, by Dynamic Consistency, $\{(c, y(\lambda))\} \succ_h \{(c, x(\lambda))\}$. Since $x \succ_h y$, Continuity implies that $x(\lambda) \succ_h y(\lambda)$ for all λ sufficiently close to one.

By the representation, $W(x(\lambda)|h) > W(y(\lambda)|h)$ and $W(\{(c,y(\lambda))\}|h) > W(\{(c,x(\lambda))\}|h)$. By Lemma 3,

$$x(\lambda) \succ_h x(\lambda) \cup y(\lambda) \iff \overline{V}(y(\lambda)|h) > \overline{V}(x(\lambda)|h),$$
$$\{(c,y(\lambda))\} \succ_h \{(c,x(\lambda)),(c,y(\lambda))\} \iff V((c,x(\lambda))|h) > V((c,y(\lambda))|h).$$

Moreover, by Lemma 8, $V((c,x(\lambda))|h) > V((c,y(\lambda))|h) \iff \widehat{V}(x(\lambda)|h) > \widehat{V}(y(\lambda)|h)$. By Temptation Stationarity, $x(\lambda) \succ_h x(\lambda) \cup y(\lambda)$ implies $\{(c,y(\lambda))\} \sim_h \{(c,x(\lambda)),(c,y(\lambda))\}$. Therefore, for all λ sufficiently close to one,

$$\overline{V}(y(\lambda)|h) > \overline{V}(x(\lambda)|h) \implies \widehat{V}(y(\lambda)|h) \geq \widehat{V}(x(\lambda)|h). \tag{26}$$

Step 1: $\overline{V}(y|h) > \overline{V}(x|h) \implies \widehat{V}(y|h) \geq \widehat{V}(x|h)$.

Assume $\overline{V}(y|h) > \overline{V}(x|h)$. By continuity of $\overline{V}(\cdot|h)$, for all λ sufficiently close to one, we have $\overline{V}(y(\lambda)|h) > \overline{V}(x(\lambda)|h)$. From (26), $\widehat{V}(y(\lambda)|h) \geq \widehat{V}(x(\lambda)|h)$. By continuity of $\widehat{V}(\cdot|h)$, $\widehat{V}(y|h) \geq \widehat{V}(x|h)$, as desired.

Step 2: $\widehat{V}(y|h) > \widehat{V}(x|h) \implies \overline{V}(y|h) \geq \overline{V}(x|h)$.

Assume $\widehat{V}(y|h) > \widehat{V}(x|h)$. For all λ sufficiently close to one, by continuity, $\widehat{V}(y(\lambda)|h) > \widehat{V}(x(\lambda)|h)$. Take any $x^n \to x(\lambda)$ and $y^n \to y(\lambda)$. By continuity, $\widehat{V}(y^n|h) > \widehat{V}(x^n|h)$. By Lemma 8, $V((c,y^n)|h) > V((c,x^n)|h)$. Moreover, for all such λ and n, Continuity implies that $\{(c, y^n)\} \succ_h \{(c, x^n)\}$, that is, $W(\{(c, y^n)\}|h) > W(\{(c, x^n)\}|h)$. By Lemma 3, $\{(c, y^n)\} \sim_h \{(c, x^n), (c, y^n)\}$. Since (c, y) is not tempted by (c, x) robustly, by Temptation Stationarity, $x(\lambda) \succ_h x(\lambda) \cup y(\lambda)$. By Lemma 3, $\overline{V}(y(\lambda)|h) > \overline{V}(x(\lambda)|h)$. By continuity, $\overline{V}(y|h) \geq \overline{V}(x|h)$ as $\lambda \to 1$.

Now we show the statement of the lemma. Suppose that $\overline{V}(y|h) > \overline{V}(x|h)$. We want to show that $\widehat{V}(y|h) > \widehat{V}(x|h)$. By Step 1, seeking a contradiction, suppose $\widehat{V}(y|h) = \widehat{V}(x|h)$. Since $\widehat{V}(\cdot|h)$ is not constant by Lemma 12, there exist $\tilde{x}$, $\tilde{y} \in \mathcal{Z}$ with $\widehat{V}(\tilde{x}|h) > \widehat{V}(\tilde{y}|h)$. By linearity, for all $\lambda \in (0,1)$, $\widehat{V}(\lambda x + (1-\lambda)\tilde{x}|h) > \widehat{V}(\lambda y + (1-\lambda)\tilde{y}|h)$. Moreover, by Continuity, for all λ sufficiently close to one, $\lambda x + (1-\lambda)\tilde{x} \succ_h \lambda y + (1-\lambda)\tilde{y}$. From the contraposition of Step 1, $\overline{V}(\lambda x + (1-\lambda)\tilde{x}|h) \geq \overline{V}(\lambda y + (1-\lambda)\tilde{y}|h)$. By continuity of $\overline{V}(\cdot|h)$, $\overline{V}(x|h) \geq \overline{V}(y|h)$ as $\lambda \to 1$, which is a contradiction.

To show the converse, note that $\overline{V}(\cdot|h)$ is not constant because $V(\cdot|h)$ is not constant. Thus, the symmetric argument goes through by replacing Step 1 with Step 2. ■

Lemma 14 For all x, y,

$$\overline{V}(y|h) > \overline{V}(x|h) \iff \widehat{V}(y|h) > \widehat{V}(x|h).$$

Proof. By Lemma 13, if $x \succ_h y$, $\overline{V}(y|h) > \overline{V}(x|h)$ if and only if $\widehat{V}(y|h) > \widehat{V}(x|h)$. The remaining proof is exactly the same as Lemma A.10 in Noor (2007). ■

Since $\overline{V}(y|h)$ and $\widehat{V}(y|h)$ are mixture linear and represent the same preference by Lemma 14, they are cardinally equivalent. There exist $\gamma(h) > 0$ and $\zeta(h) \in \mathbb{R}$ such that $\widehat{V}(x|h) = \gamma(h)\overline{V}(x|h) + \zeta(h)$. Moreover, $\gamma(h) \in (0,1)$. The proof of this part is exactly the same as Lemmas A.11 and A.12 in Noor (2007).

Because of normalization, we have $V(\underline{\mathbf{c}}|h) = \frac{v(\underline{c}|h)}{1-\gamma(h)} = 0$, and hence, $v(\underline{c}|h) = 0$.

Lemma 15 For all $h \in H$, $v(\cdot|h) = v(\overline{c}|h)u(\cdot|h)$ and $\beta(h) = \gamma(h)$.

Proof. By Lemma 11, there exist $\alpha(h) > 0$ and $\eta(h) \in \mathbb{R}$ such that

$$v(\cdot|h) = \alpha(h)u(\cdot|h) + \eta(h). \tag{27}$$

By Lemma 5, $u(\overline{c}|h) = 1$ and $u(\underline{c}|h) = 0$. By substituting these into (27), $0 = v(\underline{c}|h) = \alpha(h)u(\underline{c}|h) + \eta(h)$, that is, $\eta(h) = 0$. Moreover, $v(\overline{c}|h) = \alpha(h)u(\overline{c}|h) + \eta(h) = \alpha(h)$, that is, $v(\overline{c}|h) = \alpha(h)$. We have

$$v(\cdot|h) = v(\overline{c}|h)u(\cdot|h). \tag{28}$$

By Lemma 10, $U(\cdot|h)$ and $V(\cdot|h)$ are equivalent on the subdomain of $(\ell, \mathbf{c})$, where $\ell \in \Delta(C)$ and $\mathbf{c} \in \{\mathbf{c}^*, \underline{\mathbf{c}}\}$. Thus, $U((\ell, \{\mathbf{c}\})|h) = U((\ell', \{\mathbf{c}'\})|h)$ if and only if $V((\ell, \{\mathbf{c}\})|h) = V((\ell', \{\mathbf{c}'\})|h)$ for such alternatives. Let $\mathbf{c} = \mathbf{c}^*$ and $\mathbf{c}' = \underline{\mathbf{c}}$. From the representation, $u(\ell|h) + \beta(h) = u(\ell'|h)$ holds if and only if $v(\ell|h) + \gamma(h)v(\overline{c}|h) = v(\ell'|h)$. From (28), the latter is equivalent to $v(\overline{c}|h)u(\ell|h) + \gamma(h)v(\overline{c}|h) = v(\overline{c}|h)u(\ell'|h)$, or $u(\ell|h) + \gamma(h) = u(\ell'|h)$. Therefore, $\beta(h) = \gamma(h)$. ■

Recall $\alpha(h) = v(\overline{c}|h)$. Now redefine $V(\cdot|h)$ as $\frac{1}{\alpha(h)}V(\cdot|h)$. Then we obtain the PBT representation, as desired.

Lemma 16 $u : \Delta(C) \times H \to \mathbb{R}$, $\alpha : H \to \mathbb{R}_{++}$, and $\beta : H \to (0, 1)$ are continuous.

Proof. For all $\ell \in \Delta(C)$ and $h \in H$, $W((\ell, \underline{c})|h) = u(\ell|h)$. For all h, $W((\underline{c}, \mathbf{c}^*)|h) = \beta(h)$. Since W is continuous on $\mathcal{Z} \times H$, both $u(\ell|h)$ and $\beta(h)$ are continuous. By Lemma 2, $V(\ell|h)$ is continuous on $\Delta(C \times \mathcal{Z}) \times H$, and hence, $v(\ell|h)$ is also continuous. Thus, $\alpha(h) = v(\overline{c}|h)$ is continuous in h. ■

A.2 Necessity

The necessity of the first seven axioms comes from Gul and Pesendorfer (2007) and Noor (2007).

Temptation Stationarity: Suppose that $x \succ_h y$ and $\{(c, x)\} \succ_h \{(c, y)\}$. By Lemma 3, $x \succ_h x \cup y$ is equivalent to $\overline{V}(y|h) > \overline{V}(x|h)$, and $\{(c, x)\} \succ_h \{(c, x), (c, y)\}$ is equivalent to $V((c, y)|h) > V((c, x)|h)$, which is also equivalent to $\overline{V}(y|h) > \overline{V}(x|h)$ by the representation. Therefore, $x \succ_h x \cup y$ if and only if $\{(c, x)\} \succ_h \{(c, x), (c, y)\}$.

Next suppose $x \succ_h y$ and $\{(c, y)\} \succ_h \{(c, x)\}$. Suppose $x \succ_h x \cup y$. Since $x \succ_h x \cup y$ is equivalent to $\overline{V}(y|h) > \overline{V}(x|h)$, we have $V((c, y)|h) > V((c, x)|h)$. By the representation, $W(\{(c, y)\}|h) = W(\{(c, x), (c, y)\}|h)$, as desired. Conversely, suppose $\{(c, y)\} \sim_h \{(c, x), (c, y)\}$. By Lemma 3, $V((c, y)|h) \geq V((c, x)|h)$. If $V((c, y)|h) = V((c, x)|h)$, the non-constancy of $\overline{V}(\cdot|h)$ implies that there exist $x^n \to x$ and $y^n \to y$ such that $V((c, y^n)|h) < V((c, x^n)|h)$. Moreover, by continuity, $W(\{(c, y^n)\}|h) > W(\{(c, x^n)\}|h)$ for all sufficiently large n. Thus, by the representation, we have

$$W(\{(c, y^n)\}|h) > W(\{(c, x^n), (c, y^n)\}|h),$$

which contradicts the assumption that (c, y) is not tempted by (c, x) robustly. Therefore, we must have $V((c,y)|h) > V((c,x)|h)$, which is equivalent to $\overline{V}(y|h) > \overline{V}(x|h)$. By Lemma 3, we have $x \succ_h x \cup y$.

Habit Free: Let $\underline{\mathbf{c}} = (\underline{c}, \{(\underline{c}, \cdots)\})$ and $\mathbf{c}^* = (\overline{c}, \{\underline{\mathbf{c}}\})$. By the representation, $W(\{\underline{\mathbf{c}}\}|h) = 0$ and $W(\{\mathbf{c}^*\}|h) = u(\overline{c}|h)$. Since $u(\overline{c}|h)$ is independent of h, it is denoted by $\overline{u} \in \mathbb{R}$. For all ℓ, ℓ' $\Delta(C)$ and $\mathbf{c}, \mathbf{c}' \in \{\mathbf{c}^*, \underline{\mathbf{c}}\}$, suppose that $(\ell, \{\mathbf{c}\}) \succsim_h (\ell', \{\mathbf{c}'\})$. By the representation,

$$W(\{(\ell, \{\underline{\mathbf{c}}\})\}|h) = u(\ell|h) + \beta(h) \int W(\{\underline{\mathbf{c}}\}|hc)\, \mathrm{d}\ell(c) = u(\ell|h),$$

$$W(\{(\ell, \{\mathbf{c}^*\})\}|h) = u(\ell|h) + \beta(h) \int W(\{\mathbf{c}^*\}|hc)\, \mathrm{d}\ell(c) = u(\ell|h) + \beta(h)\overline{u},$$

$$V(\{(\ell, \{\underline{\mathbf{c}}\})\}|h) = u(\ell|h) + \beta(h) \int V(\{\underline{\mathbf{c}}\}|h)\, \mathrm{d}\ell(c) = u(\ell|h),$$

$$V(\{(\ell, \{\mathbf{c}^*\})\}|h) = u(\ell|h) + \beta(h) \int V(\{\mathbf{c}^*\}|h)\, \mathrm{d}\ell(c) = u(\ell|h) + \beta(h)\overline{u}.$$

That is, $W(\{(\ell, \{\underline{\mathbf{c}}\})\}|h) = V(\{(\ell, \{\underline{\mathbf{c}}\})\}|h)$ and $W(\{(\ell, \{\mathbf{c}^*\})\}|h) = V(\{(\ell, \{\mathbf{c}^*\})\}|h)$. Therefore, $W(\{(\ell, \{\mathbf{c}\})\}|h) \geq W(\{(\ell', \{\mathbf{c}'\})\}|h)$ if and only if $V(\{(\ell, \{\mathbf{c}\})\}|h) \geq V(\{(\ell', \{\mathbf{c}'\})\}|h)$. By the representation, $W(\{(\ell, \{\mathbf{c}\})\}|h) = W(\{(\ell, \{\mathbf{c}\}), (\ell', \{\mathbf{c}'\})\}|h)$, as desired.

B Proof of Theorem 2

We show that for all $(u(\cdot|h), \alpha(h), \beta(h))$, there exists $W(\cdot|h)$ satisfying the functional equation. Let $\mathcal{W}$ be the Banach space of all real-valued continuous functions on $\mathcal{Z} \times H$ with the sup-norm metric. For all $(x,h) \in \mathcal{Z} \times H$ and $W \in \mathcal{W}$, define

$$T(W)(x,h) \equiv \max_{l \in x} \left\{ \int (u(c|h) + \beta(h) W(z|hc))\, \mathrm{d}l(c,z) + \alpha(h) V(l|h) \right\} - \alpha(h) \max_{l \in z} V(l|h). \tag{29}$$

Since $u(\ell|h)$, $\alpha(h)$, and $\beta(h)$ are continuous in h, $T(W) \in \mathcal{W}$ for all $W \in \mathcal{W}$. Thus, the operator $T : \mathcal{W} \to \mathcal{W}$ is well-defined. To show that T is a contraction mapping, it suffices to verify that (i) T is monotonic, that is, $T(W^1) \geq T(W^2)$ whenever $W^1 \geq W^2$, and (ii) T satisfies the discounting property, that is, there exists $\overline{\beta} \in [0,1)$ such that for any W and $\delta \geq 0$, $T(W + \delta) \leq T(W) + \overline{\beta}\delta$ (see Aliprantis and Border (1994, p.97, Theorem 3.53)).

Step 1: T is monotonic.

Take any $W^1, W^2 \in \mathcal{W}$ with $W^1 \geq W^2$. Since

$$\int (u(c|h) + \beta(h) W^1(z|hc))\, \mathrm{d}l(c,z) \geq \int (u(c|h) + \beta(h) W^2(z|hc))\, \mathrm{d}l(c,z),$$

we have $T(W^1)(x,h) \geq T(W^2)(x,h)$.

Step 2: T satisfies the discounting property.

We first show that there exists $\overline{\beta} < 1$ such that $\overline{\beta} \geq \sup\{\beta(h) \,|\, h \in H\}$. Seeking a contradiction, suppose that there exists a sequence $\{h^n\}_{n=1}^{\infty} \subset H$ such that $\beta(h^n) \to 1$. Since H is compact, there exists a subsequence $\{h^m\}_{m=1}^{\infty}$ converging to some point $\bar{h} \in H$. By continuity of $\beta(h)$, we have $\beta(\bar{h}) = 1$, which is a contradiction.

Thus, for all $W \in \mathcal{W}$ and $\delta \geq 0$,

$$T(W+\delta)(x,h) = T(W)(x,h) + \beta(h)\delta \leq T(W)(x,h) + \overline{\beta}\delta.$$

By Steps 1 and 2, T is a contraction mapping. Thus, the fixed point theorem (see Gul and Pesendorfer (2007, Lemma 6)) ensures that there exists a unique $W \in \mathcal{W}$ satisfying $W = T(W)$. This W satisfies the equation (3).

C Proof of Theorem 3

For all h and $\ell \in \Delta(C)$, we have $W(\{(\ell, \{\underline{\boldsymbol{c}}\})\}|h) = u(\ell|h)$ and $W'(\{(\ell, \{\underline{\boldsymbol{c}}\})\}|h) = u'(\ell|h)$. Since both expected utilities represent the same preference over $\Delta(C)$, there exist $\zeta(h) > 0$ and $\gamma(h) \in \mathbb{R}$ such that $u'(\cdot|h) = \zeta(h)u(\cdot|h) + \gamma(h)$. Since $u(\underline{c}|h) = u'(\underline{c}|h) = 0$, $\gamma(h) = 0$.

For any fixed $c, c' \in C$, let $\ell(\lambda) = \lambda \circ c + (1-\lambda) \circ c'$. For all h, $\lambda \in [0,1]$, and $\ell \in \Delta(C)$,

$$\begin{aligned} W(\{(\ell(\lambda), \{(\ell, \{\underline{\boldsymbol{c}}\})\})\}|h) &= \lambda(u(c|h) + \beta(h)u(\ell|hc)) + (1-\lambda)(u(c'|h) + \beta(h)u(\ell|hc')) \\ &= u(\ell(\lambda)|h) + \beta(h)(\lambda u(\ell|hc) + (1-\lambda)u(\ell|hc')). \end{aligned}$$

Thus, $\{(\ell(\lambda), \{(\ell, \{\underline{\boldsymbol{c}}\})\})\} \succsim_h \{(\ell(\lambda), \{(\ell', \{\underline{\boldsymbol{c}}\})\})\}$ if and only if

$$\lambda u(\ell|hc) + (1-\lambda)u(\ell|hc') \geq \lambda u(\ell'|hc) + (1-\lambda)u(\ell'|hc'). \tag{30}$$

Similarly, from the other representation W', the same ranking holds if and only if

$$\lambda\zeta(hc)u(\ell|hc) + (1-\lambda)\zeta(hc')u(\ell|hc') \geq \lambda\zeta(hc)u(\ell'|hc) + (1-\lambda)\zeta(hc')u(\ell'|hc'),$$

or

$$\lambda u(\ell|hc) + (1-\lambda)\frac{\zeta(hc')}{\zeta(hc)}u(\ell|hc') \geq \lambda u(\ell'|hc) + (1-\lambda)\frac{\zeta(hc')}{\zeta(hc)}u(\ell'|hc'). \tag{31}$$

Lemma 17 For all $c, c' \in C$, $\zeta(hc) = \zeta(hc')$.

Proof. If $u(\cdot|hc)$ and $u(\cdot|hc')$ represent the same preference over $\Delta(C)$, we must have $u(\cdot|hc) = u(\cdot|hc')$ because of the normalization of PBT representation (u, β, α). Similarly, $u'(\cdot|hc) = u'(\cdot|hc')$. Thus, for the habit free outcome $\bar{c}$ for u,

$$\zeta(hc) = \zeta(hc)u(\bar{c}|hc) = u'(\bar{c}|hc) = u'(\bar{c}|hc') = \zeta(hc')u(\bar{c}|hc') = \zeta(hc').$$

Otherwise, there exist $\ell, \ell' \in \Delta(C)$ such that $u(\ell|hc) > u(\ell'|hc)$ and $u(\ell|hc') < u(\ell'|hc')$. In this case, if $\zeta(hc) \neq \zeta(hc')$, we can choose some λ such that (30) and (31) have the opposite inequalities, which contradicts that W and W' represent the same preference. Therefore, in any case, we have $\zeta(hc) = \zeta(hc')$. ■

By Lemma 17, for all $h \in H$, $c, c' \in C$, $\lambda \in [0, 1]$, and $\ell \in \Delta(C)$,

$$\begin{aligned} W'(\{(\ell(\lambda), \{(\ell, \{\underline{\boldsymbol{c}}\})\})\}|h) &= u'(\ell(\lambda)|h) + \beta'(h)(\lambda u'(\ell|hc) + (1-\lambda)u'(\ell|hc')) \\ &= \zeta(h)u(\ell(\lambda)|h) + \beta'(h)\zeta(hc)(\lambda u(\ell|hc) + (1-\lambda)u(\ell|hc')). \end{aligned}$$

If $\beta(h) \neq \beta'(h)\frac{\zeta(hc)}{\zeta(h)}$, we can find some $\lambda, \lambda' \in [0, 1]$ and $\ell, \ell' \in \Delta(C)$ such that W and W' induce the opposite rankings between $\{(\ell(\lambda), \{(\ell, \{\underline{\boldsymbol{c}}\})\})\}$ and $\{(\ell(\lambda'), \{(\ell', \{\underline{\boldsymbol{c}}\})\})\}$, which is a contradiction. Thus we have $\beta(h) = \beta'(h)\frac{\zeta(hc)}{\zeta(h)}$. Since this equality holds for all c, we must have $\zeta(hc) = \zeta(h)$, and hence, $\beta(h) = \beta'(h)$. Moreover, since $\zeta(hc) = \zeta(h)$ holds for all $h \in H$ and $c \in C$, $\zeta(h) = \zeta(h')$ for all $h, h' \in H$, as desired.

Since $u'(\cdot|h) = \zeta u(\cdot|h)$ and $\beta'(h) = \beta(h)$, we have $V'(l|h) = \zeta V(l|h)$ for all $l \in \Delta(C \times \mathcal{Z})$. Therefore, $(u(\cdot|h), \beta(h), \alpha'(h))_{h \in H}$ also represents the same preference. By Definition 4 about comparative attitude toward commitment, $\succsim_h$ is more willing to make a commitment than itself. Therefore, by Proposition 2, we have $\alpha'(h) = \alpha(h)$.

D Proof of Proposition 2

We first show the following preliminary result.

Lemma 18 If the agent is more willing to make a commitment at h^1 than at h^2, then for all $l, l' \in \mathcal{L}$,

$$\{l\} \succsim_{h^2} \{l'\} \iff \{l\} \succsim_{h_1} \{l'\}.$$

Proof. By definition, $\{l\} \succsim_{h^2} \{l'\}$ implies $\{l\} \succsim_{h^1} \{l'\}$. We want to show the converse. By the contraposition of Definition 4, we know that $\{l\} \succ_{h^1} \{l'\}$ implies $\{l\} \succ_{h^2} \{l'\}$.

Thus, it suffices to show that $\{l\} \sim_{h^1} \{l'\}$ implies $\{l\} \succsim_{h^2} \{l'\}$. Since linearity implies $W(\{\lambda l + (1-\lambda)\overline{\boldsymbol{c}}\}|h^1) > W(\{\lambda l' + (1-\lambda)\underline{\boldsymbol{c}}\}|h^1)$ for all $\lambda \in (0,1)$, by assumption, we have $W(\{\lambda l + (1-\lambda)\overline{\boldsymbol{c}}\}|h^2) > W(\{\lambda l' + (1-\lambda)\underline{\boldsymbol{c}}\}|h^2)$. By continuity, $\{l\} \succsim_{h^2} \{l'\}$ as $\lambda \to 1$. ■

Lemma 19 The agent is more willing to make a commitment at h^1 than at h^2 if and only if there exists a PBT representations W with (u, β, α) such that for any finite history $h \in C^t$ and for any t, (i) $u(\cdot|h^1 h) = u(\cdot|h^2 h)$, (ii) $\beta(h^1 h) = \beta(h^2 h)$, and (iii) $W(x|h^2) \geq W(x|h^1)$ for all $x \in \mathcal{Z}$.

Proof. (If part) For all $x \in \mathcal{Z}$ and $l \in \mathcal{L}$, suppose $\{l\} \succsim_{h^2} x$. By assumption, $W(\{l\}|h^1) = W(\{l\}|h^2) \geq W(x|h^2) \geq W(x|h^1)$. Thus, $\{l\} \succsim_{h^1} x$, as desired.

(only-if part) By Lemma 18, $\succsim_{h^1}$ and $\succsim_{h^2}$ are equivalent on $\mathcal{L}$. On this subdomain, the preference is represented by a recursive utility

$$W(\{l\}|h^i) = \int (u(c|h^i) + \beta(h^i) W(\{l'\}|h^i c)) \, \mathrm{d}l(c, l')$$

for $i = 1, 2$. Thus, $u(\cdot|h^i h)$ and $\beta(h^i h)$ can be taken to be identical for any finite history $h \in C^t$. Note that for all $x \in \mathcal{Z}$, there exists $l \in \mathcal{L}$ such that $\{l\} \sim_{h^2} x$. By Definition 4, $\{l\} \sim_{h^2} x$ implies $\{l\} \succsim_{h^1} x$. Therefore, $W(x|h^2) = W(\{l\}|h^2) = W(\{l\}|h^1) \geq W(x|h^1)$. ■

By Regularity, suppose that there exist $l, l' \in \mathcal{L}$ such that $\{l\} \succ_{h^1} \{l, l'\} \succ_{h^1} \{l'\}$. Since $U(\cdot|h^1) = U(\cdot|h^2)$ and $V(\cdot|h^1) = V(\cdot|h^2)$ by Lemma 19, we have $\{l\} \succ_{h^2} \{l, l'\}$. Moreover, since the agent is more willing to make a commitment at h^1 than at h^2, $\{l, l'\} \succ_{h^1} \{l'\}$ implies $\{l, l'\} \succ_{h^2} \{l'\}$. Hence, in any case, we have $\{l\} \succ_{h^2} \{l, l'\} \succ_{h^2} \{l'\}$. Lemma 19 and the representation imply that

$$\begin{aligned} U(l|h^2) - \alpha(h^2)(V(l'|h^2) - V(l|h^2)) &= W(\{l, l'\}|h^2) \geq W(\{l, l'\}|h^1) \\ &\geq U(l|h^1) - \alpha(h^1)(V(l'|h^1) - V(l|h^1)). \end{aligned}$$

Since $U(l|h^1) = U(l|h^2)$ and $V(l'|h^1) - V(l|h^1) = V(l'|h^2) - V(l|h^2)$, $\alpha(h^1) \geq \alpha(h^2)$.

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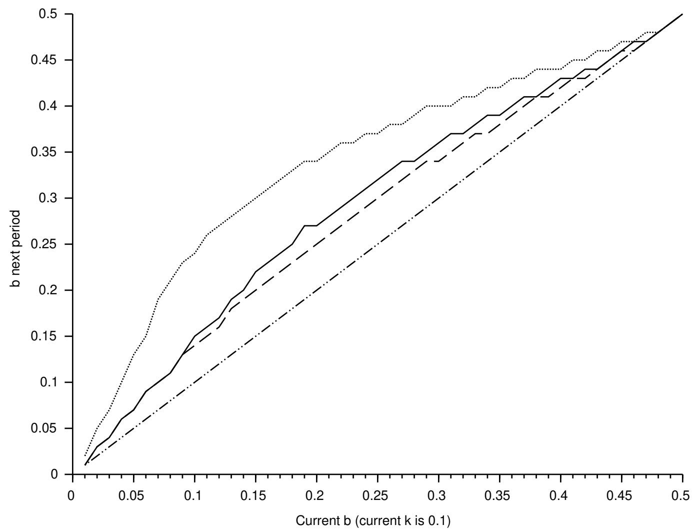
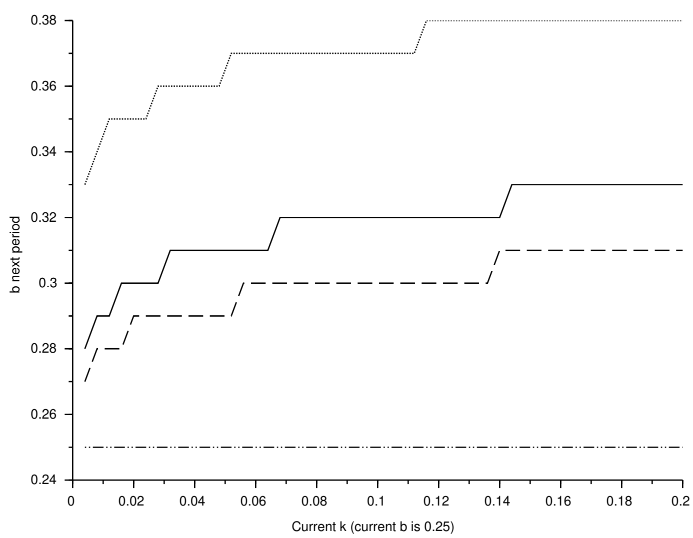
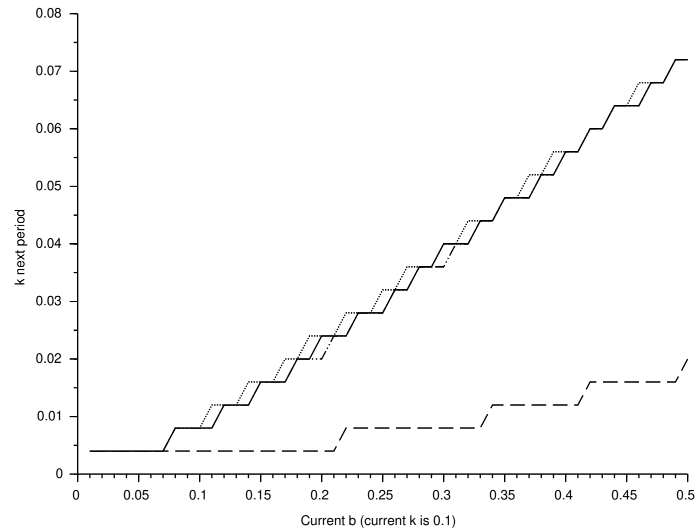


(a) Transition of patience level when $k = 0.1$. From above: PC (dotted), PBT (solid), GP (dashed), SU (dash-dot, the 45-degree line)

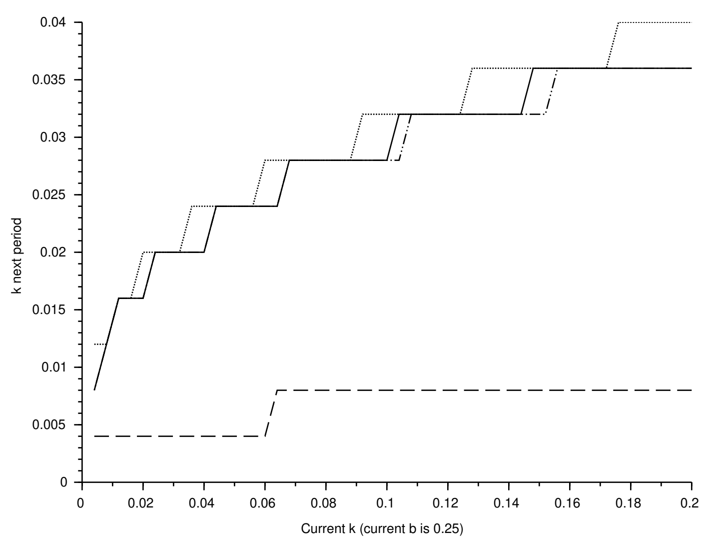


(b) Transition of patience level along capital level when $b = 0.25$. From above: PC (dotted), PBT (solid), GP(dashed), SU (dash-dot, the horizontal line)

Figure 1:



(a) Savings along patience level when $k = 0.1$. From above: PC (dotted), PBT (solid), SU (dash-dot), GP(dashed)



(b) Savings along capital level when $b = 0.25$. From above: PC (dotted), PBT (solid), SU (dash-dot), GP(dashed)

Figure 2: