Optimal Discounting with Negative Consumption Utility *

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Abstract

Noor and Takeoka [3] propose a generalization of the discounted utility, called the costly empathy representation, where a current self empathizes future selves at cognitive costs and optimally chooses a discount function for each consumption stream. This model accommodates several anormalies of the discounted utility including the magnitude effect, the common difference effect, preference for increasing sequences, and so on. In the present study, we extend the costly empathy representation to an intertemporal choice environment with negative consumption utilities. This extension allows for distinction between gains and losses relative to a reference point and explaining anormalies of the discounted utility such as the sign effect.

1 Introduction

Noor and Takeoka [3] extend the metaphor of multiple selves by incorporating costly empathy into it. The key behavioral implication of costly empathy is that impatience reduces with the magnitude of rewards. The model unifies several experimental and empirical findings in the intertemporal choice literature including the magnitude effect, the common difference effect, preference for increasing sequences, and so on.

Noor and Takeoka [3] consider the following model, called the Costly Empathy (CE) representation. There are (finite) discrete time periods $t = 0, 1, \cdots, T$. Let $C = \mathbb{R}_+$ be the set of positive consumption and $X = C^{T+1}$ be the set of consumption streams. For any instantaneous utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ with $u(0) = 0$ and cognitive cost functions $\{\varphi_t(d)\}_{t=1}^{T}$, where $\varphi_t : [0, 1] \to \mathbb{R}_+ \cup \{\infty\}$, with certain properties, we say that preference $\succsim$ over X admits a CE representation if it is represented by the utility function $U : X \to \mathbb{R}_+$ defined by

$$U(x) = u(x_0) + \sum_{t \geq 1} D_{u(x_t)}(t)u(x_t), \qquad x \in X, \tag{1}$$

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$$\text{where } D_{u(x_t)}(t) = \arg\max_{D(t)\in[0,1]} \{D(t)u(x_t) - \varphi_t(D(t))\}. \tag{2}$$

The expression (1) looks like the discounted utility, but the discount function $D_{u(x_t)}(t)$ depends on the utility of x at time t, which is in stark contrast with the standard discounted utility model. The expression (2) shows that $D_{u(x_t)}$ is derived as a solution to the cognitive optimization problem for allocating empathy to future selves. Higher empathy to self t corresponds to greater discount function $D(t) \in [0, 1]$. The agent, interpreted as self 0, can choose $D(t)$ at a cognitive cost $\varphi_t(D(t))$.

The present study extends the CE model so that consumption 0 can be interpreted as a reference point and negative consumption is permitted. Now a consumption space is assumed to be $C = \mathbb{R}$ and allows for simple lotteries over C, denoted by Δ. Accordingly, $u : C \to \mathbb{R}$ with $u(0) = 0$ is interpreted as a VNM function. A Costly Empathy (CE) representation is a tuple $(u, \{\varphi_t\})$ such that $\succsim$ is represented by the function $U : X \to \mathbb{R}$ defined by

$$U(x) = u(x_0) + \sum_{t\geq 1} D_{u(x_t)}(t)u(x_t), \qquad x \in X = \Delta^{T+1},$$

$$\text{where } D_{u(x_t)}(t) = \arg\max_{D(t)\in[0,1]} \{D(t)|u(x_t)| - \varphi_t(D(t))\}.$$

The key hypothesis in the model is in the cognitive optimization problem, where the agent considers the discounted sum of absolute value of utility of consumption of future selves. Consequently, the worst off selves are not ignored in the allocation of empathy. If loss aversion ($|u(-c)| > u(c)$) is assumed, then the model gives rise to the *sign effect*, that is, greater patience when dealing with a loss of \$$c$ than a gain of \$$c$ (see Hardistry et al [2] for a recent reference).

2 The Model

2.1 Primitives

There are $T + 1 < \infty$ periods, starting with period 0. The space C of outcomes is assumed to be $C = \mathbb{R}$. The element 0 is interpreted as a reference point or an initial endowment. Positive and negative consumption should be understood to be gains and losses from the reference point. Let Δ denote the set of simple lotteries over C, with generic elements $p, q, ...$ We will refer to p as consumption. Consider the space of consumption streams $X = \Delta^{T+1}$, endowed with the product topology. A typical element in X is denoted by $x = (x_0, x_1, \cdots, x_T)$. The primitive of our model is a preference $\succsim$ over X.

Let $\Delta_0 \subset X$ denote the set of streams $x = (p, 0, \cdots, 0)$ that offer consumption p immediately and 0 in every subsequent period. Abusing notation, we often use p to denote both a lottery $p \in \Delta$ and a stream $(p, 0, \cdots, 0) \in \Delta_0$. Thus, 0 also denotes the stream $(0, \cdots, 0)$. An element of Δ that is a mixture between two consumption alternatives $p, q \in \Delta$ is denoted $\alpha \circ p + (1 - \alpha) \circ q$ for any $\alpha \in [0, 1]$. The same mixture is also regarded as $\alpha \circ p + (1 - \alpha) \circ q \in \Delta_0$. In particular, the mixture between p and 0 is denoted by $\alpha \circ p$.

Denote by p^t the stream that pays $p \in \Delta$ at time t and 0 in all other periods. Such a stream is called a *dated reward*.

Say that a stream x is *positive* if $x_t \succsim 0$ for all t, and it is *negative* if $x_t \precsim 0$ for all t. Let X_+ denote the set of positive streams, that is,

$$X_+ := \{x \in X \mid x_t \succsim 0,\ \forall t\}.$$

Note that via the identification between p and $(p, 0, .., 0)$, it is meaningful to say that $p \in \Delta$ is positive or negative. The choice domain of Noor and Takeoka [3] is $X = \Delta(\mathbb{R}_+)^{T+1}$ and effectively identified with X_+ in the current set up.

2.2 Functional Form

We extend the costly empathy representation of Noor and Takeoka [3] allowing for negative consumption utilities.

Consider an instantaneous utility function $u : \Delta \to \mathbb{R}$, and a cost function $\varphi_t : [0, 1] \to \mathbb{R}_+ \cup \{\infty\}$ for each $t > 0$. Say that a tuple $(u, \{\varphi_t\}_{t=1}^T)$ is *basic* if

(i) u is mixture linear with a vNM utility index $u : C \to \mathbb{R}$ satisfying continuity, strictly increasingness, and $u(0) = 0$.
(ii) for each $t > 0$, the cost function φ_t is represented by the Riemann integral

$$\varphi_t(D(t)) = \int_0^{D(t)} \varphi_t'(\delta)\, \mathrm{d}\delta, \qquad D(t) \in [0, 1],$$

of a marginal cost function φ_t' for which there exist parameters $0 \le \underline{d}_t \le \overline{d}_t \le 1$ such that $\varphi_t' : [0, 1] \to \mathbb{R}_+ \cup \{\infty\}$ is left-continuous, continuous at 0 in particular, takes the value 0 on $[0, \underline{d}_t]$, is strictly increasing on $(\underline{d}_t, \overline{d}_t]$, takes the value ∞ on $(\overline{d}_t, 1]$, and is weakly increasing in t.

The conditions of the cost function provided in part (ii) is the same as in Noor and Takeoka [3]. In part (i), the VNM function is extended to negative consumption (losses relative to the reference point 0).

The following representation is an extension of the CE representation of Noor and Takeoka [3] allowing for negative consumption utilities.

Definition 1 (CE Representation) *A Costly Empathy (CE) representation is a basic tuple $(u, \{\varphi_t\})$ such that $\succsim$ is represented by the function $U : X \to \mathbb{R}$ defined by*

$$U(x) = u(x_0) + \sum_{t \ge 1} D_{u(x_t)}(t) u(x_t), \qquad x \in X,$$

$$\text{s.t. } D_{u(x_t)}(t) = \arg \max_{D(t) \in [0,1]} \{D(t)\, |u(x_t)| - \varphi_t(D(t))\}.$$

An interpretation has been provided in the Introduction. The axiomatization of Noor and Takeoka [3] can be regarded as a characterization of the above fuctional on X_+.

As in Noor and Takeoka [3], we can consider the following tractable special case of the CE representation.

Definition 2 (Homogeneous CE) *A homogeneous CE representation* $(u, m, \overline{d}_t, a_t)$ *is a CE representation* $(u, \{\varphi_t\})$ *such that for all* t*,*

$$\varphi_t(d) = \begin{cases} a_t d^m & \text{if } d \in [0, \overline{d}_t], \\ \infty & \text{if } d \in (\overline{d}_t, 1], \end{cases}$$

where (i) $m > 1$ *and (ii)* $a_t > 0$ *is increasing in* t*.*

Since the cost function is differentiable on $[0, \overline{d}_t]$ the cognitive optimization problem can be solved in the usual way by including the constraint $d \leq \overline{d}_t$.

3 Reduced From

3.1 Characterization

In order to develop an intuition for its structure, we first establish that the CE model can be nested within a class of representations that maintains the DU model's additive separability across time but permits magnitude-dependent discounting:

Definition 3 (General Discounted Utility Representation) *A General Discounted Utility (GDU) representation for a preference* $\succsim$ *over* X *is a tuple* (u, D) *where* $u : \Delta \to \mathbb{R}$ *is an expected utility such that its VNM function* $u : C \to \mathbb{R}$ *is continuous and strictly increasing with* $u(0) = 0$ *and for each* $c \in C$*,* $D_{|u(c)|} : \{1, \cdots, T\} \to [0, 1]$ *is an absolute-value magnitude-dependent discounting function which is weakly decreasing in* t *and continuous for all* $u(c) > 0$*, such that* $\succsim$ *is represented by a strictly increasing function* $U : X \to \mathbb{R}$ *defined by*

$$U(x) = u(x_0) + \sum_{t \geq 1} D_{|u(x_t)|}(t) u(x_t), \qquad x \in X.$$

A GDU representation (u, D) *is unbounded if* $u(C) = \mathbb{R}$*.*

The GDU model permits $D_r(t)$ to increase or decreasing with $r > 0$. The condition that U is strictly increasing requires that $D_r(t)r$ must be strictly increasing, thereby placing a limit on how negative the slope of $D_r(t)$ can be. That said, our study will lead us into the following GDU subclass:

Definition 4 (Magnitude-Decreasing Impatience) *A GDU representation* (u, D) *exhibits magnitude-decreasing impatience (MDI) if* $D_r(t)$ *is weakly increasing in* $r > 0$ *for all* t*.*

Since $D_r(t)$ depends only on positive payoffs r, the following results are obtained as a corollary of Theorem 1 of Noor and Takeoka [3].

Corollary 1 *A preference $\succsim$ over X admits a CE representation if and only if it admits a GDU representation that exhibits magnitude-decreasing impatience.*

Corollary 2 *The following statements are equivalent for any preference $\succsim$ over X, and any tuple $(u, m, \overline{d}_t, a_t)$ as in the Definition 2.*
(a) $\succsim$ admits a homogeneous CE representation $(u, m, \overline{d}_t, a_t)$.
(b) $\succsim$ admits a GDU representation (u, D) such that for all t,

$$D_r(t) = \begin{cases} \kappa_t r^{\frac{1}{m-1}} & \text{if } 0 < r \leq \overline{r}_t \\ \overline{d}_t & \text{if } r > \overline{r}_t \end{cases},$$

where $\kappa_t = (ma_t)^{-\frac{1}{m-1}}$ and $\overline{r}_t = ma_t\overline{d}_t^{m-1}$.

Corollary 1 suggests that it is enough to characterize the GDU representation with MDI for axiomatizing the CE representation. By Corollary 2, we have to ensure that $D_r(t)$ is homogeneous up to some threshold and is constant thereafter to axiomatize the homogeneous CE representation.

3.2 Implications

As shown by Noor and Takeoka [3], the CE model can accommodate several anormalies of the DU model including the magnitude effect and the common difference effect on the domain of positive streams. In addition, the model extended to negative utilities can accommodate the following experimental and empirical evidence.

3.2.1 Sign Effect

The *sign effect* refers to the finding that people are more patient when dealing with a loss of \$$c$ than a gain of \$$c$. For example, Thaler [5] report that a median subject tends to be indifferent between a gain of \$15 now and a gain of \$60 in a year, while the subject is indifferent between a loss of \$15 now and a loss of \$20 in a year. See Hardistry et al [2] for a recent reference. Consider the reduced form of the homogeneous CE model (Corollary 2). If $|u(-c)| > u(c)$ then we obtain

$$D_{-c}(t) = \kappa_t \, |u(-c)|^{\frac{1}{m-1}} > \kappa_t u(c)^{\frac{1}{m-1}} = D_c(t).$$

3.2.2 Magnitude Hypothesis in Consumption Smoothing

There is considerable evidence that consumption tends to respond to anticipated income increases more than what is implied by standard models of consumption smoothing. Moreover, this response is inversely correlated with the size or magnitude of anticipated income increases, that is, for small income changes, consumption tends to overreact to them, while consumption pattern tends to be consistent with consumption smoothing for medium or large income changes. This evidence is known as the *magnitude hypothesis in consumption smoothing* (Browning and Collado [1], Scholnick [4]), which has been attributed to bounded rationality or costs associated with the mental processing of small anticipated income changes.

The CE representation may predict a similar behavioral pattern where $\succsim$ exhibits preference for concentration in streams with small payoffs, whereas it exhibits preference for consumption smoothing otherwise. Consider the reduced form of the homogeneous CE model (Corollary 2). A stream x is interpreted as a stream of gains from the reference point. Note that $D_{u(x_t)}(t)u(x_t)$ has a flatter curvature up to $x_t \leq u^{-1}(\bar{r}_t)$ because of the convex transformation over u, and gets more concave beyond the threshold. Thus, the agent is more likely to choose a skewed consumption stream over a smoothed consumption stream when these streams are small.

4 Behavioral Foundation

Consider a binary relation $\succsim$ over the space of consumption streams $X = \Delta^{T+1}$ as defined in Section 2.1.

4.1 CE Representation

Axiom 1 (Regularity) *(a) (Order). $\succsim$ is complete and transitive.*
(b) (Continuity). For all $x \in X$, $\{y \in X : y \succsim x\}$ and $\{y \in X : x \succsim y\}$ are closed.
(c) (Impatience). For any positive $p \in \Delta$ and $t < t'$,

$$(p)^t \succsim (p)^{t'}.$$

(d) (C-Monotonicity): for all $c, c' \in C$,

$$c \geq c' \Longleftrightarrow c \succsim c'.$$

(e) (Monotonicity) For any $x, y \in X$,

$$(x_t, 0, .., 0) \succsim (y_t, 0, .., 0) \text{ for all } t \Longrightarrow x \succsim y.$$

Moreover, if $(x_t, 0, .., 0) \succ (y_t, 0, .., 0)$ for some t, then $x \succ y$.
(f) (Risk Preference). For any $p, p', p'' \in \Delta$ and $\alpha \in (0, 1]$,

$$p \succ p' \Longrightarrow \alpha \circ p + (1 - \alpha) \circ p'' \succ \alpha \circ p' + (1 - \alpha) \circ p''.$$

(g) (Present Equivalents). For any stream x there exist $c, c' \in C$ s.t.

$$c \succsim x \succsim c'.$$

Order and Continuity are standard. Impatience requires that positive outcomes are weakly preferred sooner rather than later. C-Monotonicity states that more consumption is better than less. While C-Monotonicity applies only to immediate consumption, Monotonicity is a property on arbitrary streams: it requires that point-wise preferred streams are preferred. Risk Preference imposes vNM Independence only on immediate consumption. Present Equivalents states that for any stream, there are immediate consumption levels that are better and worse than x. Given Order and Continuity, this ensures that each stream x has a present equivalent $c_x \in C$. Notably, each x has a unique present equivalent c_x (by C-Monotonicity, $x \sim c_x > c_y \sim y$ implies $c_x \succ c_y$ and therefore $x \succ y$).

For notational convenience, for all streams x, $y \in X$ and all t, let xty denote the stream that pays according to x at t and according to y otherwise.

Axiom 2 (Separability) *For all $x \in X$ and all t,*

$$\frac{1}{2} \circ c_{xt0} + \frac{1}{2} \circ c_{0tx} \sim \frac{1}{2} \circ c_x + \frac{1}{2} \circ c_0.$$

For any stream x and $\alpha \in [0, 1]$, $\alpha \circ x$ denotes the stream $(\alpha \circ x_0, \cdots, \alpha \circ x_T)$, and is interpreted as a scaling down of the stream by lottery α. As argued by Noor and Takeoka [3], the magnitude effect can be identified via violation of Homotheticity: For any positive streams $x \in X_+$ and any $\alpha \in (0, 1)$,

$$c_x \sim x \Longrightarrow \alpha \circ c_x \sim \alpha \circ x.$$

Axiom 3 (Weak Homotheticity) *For any $x \in X_+$ and any $\alpha \in (0, 1)$,*

$$c_x \sim x \Longrightarrow \alpha \circ c_x \succsim \alpha \circ x.$$

We extend the model to accommodate negative outcomes. We say that $p^* \in \Delta$ is an *absolute value* of $p \in \Delta$ if:[1] $p^* \sim p$ when p is positive , or p^* satisfies

$$\frac{1}{2} \circ p + \frac{1}{2} \circ p^* \sim 0$$

when p is negative. That is, the 50-50 lottery over p and p^* is as good as receiving 0.

For any stream x, define x^* by the stream that replaces each outcome x_t with an absolute value $(x_t)^*$, that is,

$$x_t^* = (x_t)^* \text{ for all } t.$$

Note that the absolute value p^* is not unique, since anything in its indifference class will also be an absolute value for p.

We impose a Symmetry axiom so that axioms on positive streams translate into restrictions on negative ones.

[1] For simplicity, we use this terminology rather than the more accurate one that p^* *has the same absolute value as* p. This terminology anticipates the fact that in terms of the representation p, p^* will satisfy $u(p^*) = |u(p)|$.

Axiom 4 (Symmetry) *If x is a negative stream then*

$$(c_x)^* = c_{x^*}.$$

Consider a negative stream x and its present equivalent c_x. The axiom states that the absolute value $(c_x)^*$ of this present equivalent is the same as the present equivalent c_{x^*} of the stream's absolute value x^*.

As noted earlier, present equivalents carry information about the agent's assessment of the outcomes and his impatience towards them. So the axiom suggests that the agent's impatience towards two streams is identical when the streams give outcomes that have identical absolute values. This suggests that if impatience is not constant, it can only change with the absolute value of outcomes.

Theorem 1 *A non-degenerate preference $\succsim$ on X satisfies Regularity, Separability, Weak Homotheticity, and Symmetry if and only if it admits a CE representation.*

If there are two CE representations $(u^i, \{\varphi_t^i\})$, $i = 1, 2$ of the same preference $\succsim$, then there exists $\lambda > 0$ such that (i) $u^2 = \lambda u^1$, (ii) $\varphi_t^2 = \lambda \varphi_t^1$ for each t.

A proof sketch is as follows. By Lemma 1 in the Appendix, there exists an expected utility representation u on lotteries. There exists a representation U on X which is an extension of u. Since $\succsim$ satisfies Separability, Lemma 2 ensures that $U : X \to \mathbb{R}$ admits an additively separable form where each component function U_t is defined on Δ. For any negative dated reward, Lemma 3 shows that the absolute value of its present equivalent is equal to the dated reward of its absolute value. Together with Lemma 3, in Lemma 4, we show that $U : X \to \mathbb{R}$ admits a GDU representation such that the discount function depends only on the absolute value of payoffs. By Weak Homotheticity, the discount function is weakly increasing in positive payoffs. The result obtains by invoking Corollary 1.

4.2 Homogeneous CE Representation

Say that a stream $x \in X_+$ is *ℓ-Magnitude Sensitive* if the agent's impatience strictly reduces whenever the stream is made less desirable.

Definition 5 (ℓ-Magnitude-Sensitivity) *A stream $x \in X_+$ is ℓ-Magnitude Sensitive if*

$$c_x \sim x \Longrightarrow \alpha \circ c_x \succ \alpha \circ x \text{ for all } \alpha \in (0, 1).$$

The set of all ℓ-Magnitude Sensitive streams is denoted by $X_\ell \subset X_+$.

The set X_ℓ can be identified with the set of magnitude sensitive streams considered in Noor and Takeoka [3] in particular by interpreting the operation $\circ$ as the mixture operation over lotteries. We consider two axioms on X_ℓ, called X^*-Regularity and X^*-Homogeneity, introduced by Noor and Takeoka [3]. Details are omitted.

Theorem 2 *A non-degenerate preference $\succsim$ on X satisfies Regularity, Separability, Weak Homotheticity, Symmetry, X^*-Regularity, and X^*-Homogeneity if and only if it admits a homogeneous CE representation.*

Moreover, if there are two homogeneous CE representations $(u^i, m^i, \overline{d}_t^i, a_t^i)$, $i = 1, 2$ of the same preference $\succsim$, then there exists $\lambda > 0$ such that (i) $u^2 = \lambda u^1$, (ii) $\overline{d}_t^2 = \overline{d}_t^1$, $a_t^2 = \lambda a_t^1$, and $m^2 = m^1$ for each t.

Since $\succsim$ satisfies Regularity, Separability, Weak Homotheticity, and Symmetry, $\succsim$ admits a GDU representation. Since the discount function depends only on the absolute value of utility of consumption, axioms on positive streams are enough to impose particular structures on the discount function. As shown by Noor and Takeoka [3, Theorem 7], X^*-Regularity and X^*-Homogeneity on positive streams imply that there exists $\overline{r}_t > 0$ such that $D_r(t)$ is homogeneous on $(0, \overline{r}_t]$ and $D_r(t)$ is constant on $(\overline{r}_t, \infty)$. Theorem 2 obtains by invoking Corollary 2.

A Appendix: Characterization of GDU Representations

Theorem 3 *A preference $\succsim$ over $X = \Delta^{T+1}$ satisfies Regularity and Separability if and only if it admits an unbounded GDU representation.*

Moreover, two GDU representations (u^i, D^i), $i = 1, 2$ represent the same preference $\succsim$ if and only if there exists a scalar $\lambda > 0$ s.t. (i) $u^2 = \lambda u^1$, and (ii) for all $p \in \Delta$ and $t > 0$,

$$D^1_{|u^1(p)|}(t) = D^2_{|u^2(p)|}(t).$$

Proof. Necessity is obvious. We show sufficiency.

Lemma 1 *The preference $\succsim|_{\Delta_0}$ is represented by a utility function $u : \Delta \to \mathbb{R}$ with $u(0) = 0$ which is continuous, mixture linear, and homogeneous (that is, $u(\alpha p) = \alpha u(p)$ for all $\alpha \geq 0$.) Moreover, the preference $\succsim$ on X is represented by a continuous utility function $U : X \to \mathbb{R}$ such that $U(p) = u(p)$ for all $p \in \Delta_0$.*

Proof. By Weak Regularity, $\succsim|_{\Delta_0}$ satisfies the vNM axioms. There exists a continuous mixture linear function $u : \Delta \to \mathbb{R}$ which represents $\succsim|_{\Delta_0}$ and which can be chosen so that $u(0) = 0$.

Establish homogeneity of u next. If $\alpha \in [0, 1]$, by mixture linearity of u, together with identifying αp with $\alpha p + (1 - \alpha)0$,

$$u(\alpha p) = u(\alpha p + (1 - \alpha)0) = \alpha u(p) + (1 - \alpha)u(0) = \alpha u(p).$$

If $\alpha > 1$, we identify αp with $p' \in \Delta$ satisfying $p = \frac{1}{\alpha}p' + \frac{\alpha-1}{\alpha}0$. Then, mixture linearity of u implies that $u(p) = \frac{1}{\alpha}u(p')$, that is, $u(\alpha p) = u(p') = \alpha u(p)$, as desired.

For any $x \in X$, the Present Equivalents axiom ensures that there exists $c_x \in C$ such that $c_x \sim x$. Define $U(x) = u(c_x)$. By construction, U represents $\succsim$. Moreover, for all $p \in \Delta$, $U(p) = u(p)$. In particular, we have $U(0) = u(0) = 0$.

To show the continuity of U, take any sequence $x^n \to \widehat{x}$. There exists a corresponding present equivalent $c_{x^n} \sim x^n$. Since $U(x^n) = u(c_{x^n})$ and u is continuous, we want to show that $c_{x^n} \to c_{\widehat{x}}$.

Claim 1 *The present equivalent is continuous, that is, if $x^n \to x$, then $c_{x^n} \to c_{\widehat{x}}$.*

Proof. Take any $\overline{c}$ and $\underline{c}$ such that $\overline{c} > c_{\widehat{x}} > \underline{c}$. Let $W = \{x \in X \,|\, \overline{c} \succ x \succ \underline{c}\}$. Since $x^n \to \widehat{x} \sim c_{\widehat{x}}$, by Continuity, we can assume $x^n \in W$ for all n without loss of generality.

Seeking a contradiction, suppose $c_{x^n} \not\to c_{\widehat{x}}$. Then, there exists a neighborhood of $c_{\widehat{x}}$, denoted by $B(c_{\widehat{x}})$, such that $c_{x^m} \notin B(c_{\widehat{x}})$ for infinitely many m. Let $\{x^m\}$ denote the corresponding subsequence of $\{x^n\}$. Since $x^n \to \widehat{x}$, $\{x^m\}$ also converges to $\widehat{x}$. Without loss of generality, we can assume $x^m \in W$, that is, $\overline{c} \succ x^m \sim c_{x^m} \succ \underline{c}$. By C-Monotonicity, $\overline{c} > c_{x^m} > \underline{c}$. Thus, $\{c_{x^m}\}$ belongs to a compact interval $[\underline{c}, \overline{c}]$, and hence, there exists a convergent subsequence $\{c_{x^\ell}\}$ with a limit $\widetilde{c} \neq c_{\widehat{x}}$. On the other hand, since $x^\ell \to \widehat{x}$ and $x^\ell \sim c_{x^\ell}$, Continuity implies $\widehat{x} \sim \widetilde{c}$. Since $c_{\widehat{x}}$ is unique, $c_{\widehat{x}} = \widetilde{c}$, which is a contradiction. ■

The symmetric argument can be applied for the case that $0 \succsim \overline{x}, x^n$ for all n. Finally, suppose that $\overline{x} \sim 0$. If $x^n \sim 0$ for some n, $U(x^n) = 0 = U(\overline{x})$ for such n. Thus, we can assume without loss of generality that $x^n \not\sim 0$ for all n. For the subsequence $\{x^m\}$ of $\{x^n\}$ satisfying $x^m \succ 0$, we have $x^m \to \overline{x}$. By the above argument, $U(x^m) \to U(\overline{x})$. Similarly, for the subsequence $\{x^m\}$ of $\{x^n\}$ satisfying $0 \succ x^m$, we have $U(x^m) \to U(\overline{x})$. Therefore, $U(x^n) \to U(\overline{x})$, as desired. ■

Lemma 2 *U can be written as an additively separable utility form, i.e. $U : X \to \mathbb{R}$ s.t. for all $x \in X$,*

$$U(x) = u(x_0) + \sum_{t \geq 1} U_t(x_t),$$

where u is given as in Lemma 1 and $U_t : \Delta \to \mathbb{R}$ is continuous with $U_t(0) = 0$ for each t. Moreover, u is unbounded from above.

Proof. Take any $x = (x_0, x_1, \cdots, x_T) \in X$ s.t. $x \not\sim 0$. By Monotonicity, there exists some $t > 0$ with $x_t \succ 0$. We start with the case where there are two $x_t, x_s \not\sim 0$. By notational convenience, denote such a stream by $(x_t, x_s, 0, \cdots, 0)$. By Separability,

$$\frac{1}{2} \circ c_{(0,x_s,0,\cdots,0)} + \frac{1}{2} \circ c_{(x_t,0,\cdots,0)} \sim \frac{1}{2} \circ c_{(x_t,x_s,0,\cdots,0)} + \frac{1}{2} \circ 0.$$

Since u is mixture linear,

$$\begin{aligned} & u(c_{(0,x_s,0,\cdots,0)}) + u(c_{(x_t,0,\cdots,0)}) = u(c_{(x_t,x_s,0,\cdots,0)}) + u(0) \\ \iff & U(0, x_s, 0, \cdots, 0) + U(x_t, 0, \cdots, 0) = U(x_t, x_s, 0, \cdots, 0). \end{aligned}$$

Define $U_t(x_t) = U(x_t, 0, \cdots, 0)$ and $U_s(x_s) = U(0, x_s, 0, \cdots, 0)$. Then, we have

$$U(x_t, x_s, 0, \cdots, 0) = U_t(x_t) + U_s(x_s). \tag{3}$$

In particular, if $t = 0$, $U_t(x_t) = u(x_t)$.

If a stream has three outcomes $x_t, x_s, x_r \not\sim 0$, denote it by $(x_t, x_s, x_r, 0, \cdots, 0)$. From the above argument, we have (3). By Separability,

$$\frac{1}{2} \circ c_{(0,0,x_r,0,\cdots,0)} + \frac{1}{2} \circ c_{(x_t,x_s,0,\cdots,0)} \sim \frac{1}{2} \circ c_{(x_t,x_s,x_r,0,\cdots,0)} + \frac{1}{2} \circ 0.$$

Since u is mixture linear,

$$\begin{aligned} & u(c_{(0,0,x_r,0,\cdots,0)}) + u(c_{(x_t,x_s,0,\cdots,0)}) = u(c_{(x_t,x_s,x_r,0,\cdots,0)}) + u(0) \\ \Longleftrightarrow\ & U(0, 0, x_r, 0 \cdots, 0) + U(x_t, x_s, 0, \cdots, 0) = U(x_t, x_s, x_r, 0, \cdots, 0). \end{aligned}$$

Define $U_r(x_r) = U(0, 0, x_r, 0, \cdots, 0)$. Then, we have

$$U(x_t, x_s, x_r, 0, \cdots, 0) = U_r(x_r) + U(x_t, x_s, 0, \cdots, 0) = U_t(x_t) + U_s(x_s) + U_r(x_r).$$

By repeating the same argument finitely many times, we have

$$U(x) = \sum_{t \geq 0} U_t(x_t),$$

where $U_t(x_t)$ is defined as $U_t(x_t) = U(0, \cdots, 0, x_t, 0, \cdots, 0)$. By definition, $U_t(0) = 0$. Since U is continuous, U_t is also continuous. Moreover, $U_0(x_0) = U(x_0, 0, \cdots, 0) = u(x_0)$.

Finally, we show that u must be unbounded. First, we show that u is unbounded above. By seeking a contradiction, suppose otherwise. Then, the range of u is nonempty and has an upper bound. There exists a supremum $\overline{v}$ of the range of u. Since U_t is non-constant by Monotonicity, there exists some $\tilde{p} \in \Delta$ with $U_t(\tilde{p}) > 0$. Take a lottery $\overline{p} \in \Delta$ such that $\overline{v} - u(\overline{p}) < U_t(\tilde{p})$. Consider the stream $\overline{x}$ which pays $\overline{p}$ in period 0, $\tilde{p}$ in period t, and zero otherwise. By the representation,

$$U(\overline{x}) = u(\overline{p}) + U_t(\tilde{p}) > \overline{v}.$$

Since $\overline{v}$ is the supremum of $u(\Delta)$, the above inequality contradicts to the Present Equivalents axiom. By the symmetric argument, we can show that u is unbounded below. ■

We show properties of an absolute value of streams.

Lemma 3 *(1) For all negative outcomes $p \in \Delta$, $u(p) = -u(p^*)$.*
(2) For any dated reward p^t with a negative outcome, $(c_{p^t})^* \sim (p^*)^t$.

Proof. (1) If $0 \succ p$, by definition, its absolute value $p^* \in \Delta$ satisfies

$$\frac{1}{2} \circ p + \frac{1}{2} \circ p^* \sim 0.$$

Since u is mixture linear, $u(p) = -u(p^*)$.

(2) Since the dated reward p^t with this negative outcome is a negative stream, by Symmetry, $(c_{p^t})^* = c_{(p^t)^*}$. Since $(p^t)^* = (p^*)^t$ by definition, we have a desired result. ■

Lemma 4 *The function $U : X \to \mathbb{R}$ defined as in Lemma 2 can be written as follows:*

$$U(x) = u(x_0) + \sum_{t \geq 1} D_{|u(x_t)|}(t) u(x_t),$$

where for all $t \geq 1$, $D_{|u(p)|}(t) \in [0, 1]$ and $D_{|u(p)|}(t)$ is continuous in $|u(p)|$.

Proof. Taking the additive representation from Lemma 2, by Monotonicity, we have that $U_t(x_t)$ can be written as an increasing transformation of $u(x_t)$. So we can write $U_t(x_t)$ as $U_t(u(x_t))$. Define D_x by $D_{u(x_t)}(t) = \frac{U_t(u(x_t))}{u(x_t)} > 0$ for any $x_t \in \Delta$ with $x_t \not\sim 0$. Then

$$U(x) = u(x_0) + \sum_{t \geq 1} D_{u(x_t)}(t) u(x_t).$$

By Lemma 3, the representation implies

$$\begin{aligned} D_{u(p)}(t)u(p) &= U(p^t) = u(c_{p^t}) = -u((c_{p^t})^*) = -u(c_{(p^t)^*}) \\ &= -U((p^t)^*) = -D_{u(p^*)}(t)u(p^*) = D_{|u(p)|}(t)u(p), \end{aligned}$$

and hence, $D_{u(p)}(t) = D_{|u(p)|}(t)$, as desired.

Since u and U_t are continuous, so is $D_{u(p)}(t)$ in $u(p)$ on the domain of $u(p) \neq 0$. Moreover, since $|u(p)|$ is continuous, so is $D_{|u(p)|}(t)$ for all $|u(p)| \neq 0$.

By Impatience, for all positive p and $t \geq 1$, $u(p) = U(p^0) \geq U(p^t) = D_{|u(p)|}(t)u(p)$, which implies $D_{|u(p)|}(t) \leq 1$. ■

Finally we establish uniqueness.

Lemma 5 *If GDU (u^i, D^i) for $i = 1, 2$ both represent the same preference $\succsim$, then there exists a scalar $\lambda > 0$ s.t. for all $p \in \Delta$ and $t > 0$,*

$$u^1(p) = \lambda u^2(p) \text{ and } D^1_{|u^1(p)|}(t) = D^2_{|u^2(p)|}(t).$$

Proof. By considering the restriction $\succsim|_{\Delta_0}$ and applying the Mixture Space Theorem, we obtain $\lambda > 0$ and γ s.t. $u^1 = \lambda u^2 + \gamma$. Due to the normalization $u(0) = 0$ in the representation, we have $\gamma = 0$. Thus $u^1 = \lambda u^2$.

Next, observe that there exists a present equivalent c_{p^t} for each dated reward p^t, the representation implies that $u^i(c_{p^t}) = D^i_{|u^i(p)|}(t)u^i(p)$ for any $p \in \Delta$ and $t > 0$. Since $u^1 = \lambda u^2$, we therefore see that

$$D^1_{|u^1(p)|}(t) = \frac{u^1(c_{p^t})}{u^1(p)} = \frac{u^2(c_{p^t})}{u^2(p)} = D^2_{|u^2(p)|}(t),$$

as desired. ■ ■

B Proof of Theorem 1

By Regularity and Separability, Theorem 3 ensures that $\succsim$ admits a GDU representation.

Lemma 6 *$\succsim$ satisfies Weak Homotheticity if and only if $D_r(t)$ is weakly increasing in $r > 0$.*

Proof. By the GDU representation, for any $p \in \Delta$ and $t > 0$, it must be that $U(p^t) = D_{u(p)}(t)u(p)$. Take any $\alpha \in (0, 1]$. Since u is mixture linear, $\alpha u(q) = u(\alpha \circ q)$ for any $q \in \Delta$. For the dated reward p^t,

$$\begin{aligned} \alpha D_{u(p)}(t)u(p) &= \alpha U(p^t) = \alpha u(c_{p^t}) = u(\alpha \circ c_{p^t}) \\ &\geq U(\alpha \circ x) = D_{u(\alpha \circ p)}(t)u(\alpha \circ p) = \alpha D_{\alpha u(p)}(t)u(p), \end{aligned}$$

where the inequality holds since Weak Homotheticity requires $\alpha \circ c_x \succsim \alpha \circ x$. Conclude that $D_{u(p)}(t) \geq D_{\alpha u(p)}(t)$, that is, D is weakly increasing in utility of magnitude, as desired. The converse directions of this claim is straightforward to establish. ■

Together with this lemma, the sufficiency of Theorem 1 obtains by invoking Corollary 1.

For the necessity, it is enough to verify the Symmetry axiom. Assume that $\succsim$ is represented by a CE representation. Take any negative stream x and its absolute value x^*. Note that the absolute value of the utility stream of x^* is written as $(|u(x_t)|)_{t=0}^T$, which is denoted by $|u(x)|$. By representation,

$$\begin{aligned} U(x^*) &= |u(x_0)| + \sum_{t \geq 1} D_{|u(x)|}|u(x_t)|, \text{ and} \\ U(x) &= u(x_0) + \sum_{t \geq 1} D_{|u(x)|}u(x_t). \end{aligned}$$

Therefore,

$$u(c_{x^*}) = U(x^*) = -U(x) = -u(c_x) = u((c_x)^*),$$

as desired.

Uniqueness

Consider two CE representations $(u^i, \{\varphi_t^i\})$, $i = 1, 2$, that represent the same preference. Their reduced forms are denoted by $U^i(x) = u^i(x_0) + \sum_{t \geq 1} D^i_{|u^i(x_t)|}(t)u^i(x_t)$, where $D^i_{|u^i(x)|}$ is an optimal discount function. Since these are GDU representations that represent the same preference, by Theorem 3, there exists $\lambda > 0$ such that (i) $u^2 = \lambda u^1$, and (ii) for all $c \in C$ and t,

$$D^1_{|u^1(c)|}(t) = D^2_{|u^2(c)|}(t). \tag{4}$$

Since $D_r^i(t)$ depends only on positive payoffs, the uniqueness of the CE representations extended to negative payoffs is equivalent to that of the CE representation on the positive streams. The latter uniqueness is established by Noor and Takeoka [3, Theorem 6].

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