Online Appendix to "Self-Control Games"

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In Section 3 of "Self-Control Games" by Norio Takeoka and Takashi Ui, peer effects affect only the strength of temptation through $\kappa_i(m_{-i}) \geq 0$. The purpose of this online appendix is to provide an axiomatic foundation for a special class of self-control preferences in which a normative payoff function is independent of the opponents' actions and a temptation payoff function is multiplicatively separable from the opponents' actions. That is, player i's normative and temptation payoff functions are of the form

$$u_i(m) = f_i(m_i),\ v_i(m) = \kappa_i(m_{-i})g_i(m_i) \text{ for all } m \in \Delta(A), \tag{1}$$

where $f_i : \Delta(A_i) \to \mathbb{R}$, $g_i : \Delta(A_i) \to \mathbb{R}$, and $\kappa_i : \prod_{j\neq i} \Delta(A_j) \to \mathbb{R}_+$ are continuous linear functions. Then, his payoff of choosing $M_i \in \mathcal{M}_i$ is

$$U_i(M_i, m_{-i}) = \max_{m_i \in M_i} \left(f_i(m_i) - \kappa_i(m_{-i}) \left(\max_{m_i' \in M_i} g_i(m_i') - g_i(m_i) \right) \right) \tag{2}$$

and his choice correspondence is

$$C_i(M_i, m_{-i}) = \arg\max_{m_i \in M_i} \left(f_i(m_i) + \kappa_i(m_{-i})g_i(m_i) \right).$$

Let $\succsim_i$ be player i's preference relation on $\mathcal{K}(\Delta(A))$. In addition to GP's axioms, we use the following axioms.

Axiom 1 (Indifferent Commitment)**.** For all $m_{-i}, m'_{-i} \in \prod_{j\neq i} \Delta(A_j)$, there exist $m_i, m_i' \in \Delta(A_i)$ such that

$$\{(m_i, m_{-i})\} \sim_i \{(m_i, m'_{-i})\} \nsim_i \{(m_i', m_{-i})\} \sim_i \{(m_i', m'_{-i})\}.$$

This axiom requires that, for any pair of the opponents' action profiles, there exist at least two actions to which full commitment yields the same payoffs.

For each $m_{-i} \in \prod_{j \neq i} \Delta(A_j)$, let $\succsim_{i,m_{-i}}$ denote a conditional preference relation on $\mathcal{K}(\Delta(A_i))$ defined by

$$M_i \succsim_{i,m_{-i}} M'_i \;\Leftrightarrow\; M_i \times \{m_{-i}\} \succsim_i M'_i \times \{m_{-i}\} \;\Leftrightarrow\; U_i(M_i, m_{-i}) \geq U_i(M'_i, m_{-i}),$$

where $M_i \times \{m_{-i}\} = \{x \in \prod_{j \in \mathcal{I}} \Delta(A_j) \mid x_i \in M_i, x_j = m_j \text{ for } j \neq i\} \in \mathcal{K}(\Delta(A))$. The opponents' action m_{-i} is said to be *nondegenerate* (with respect to the induced conditional preference relation) if there exist $m_i, m'_i \in \Delta(A_i)$ with $\{m_i\} \succ_{i,m_{-i}} \{m_i, m'_i\}$; that is, m'_i is normatively less preferred and more tempting than m_i. Note that the nondegeneracy of m_{-i} requires the existence of a tempting action.

Axiom 2 (Identical Tempting Actions)**.** For all nondegenerate $m_{-i}, m'_{-i} \in \prod_{j \neq i} \Delta(A_j)$ and for all $m_i, m'_i \in \Delta(A_i)$, if $\{m_i\} \succ_{i,m_{-i}} \{m_i, m'_i\}$ then $\{m_i\} \succ_{i,m'_{-i}} \{m_i, m'_i\}$.

This axiom requires that if an action is tempting given m_{-i}, then it is also given any m'_{-i}.

The next axiom requires that any pair of the opponents' actions can be ordered by a player's attitude toward commitment according to Definition 1.

Definition 1. For all $m_{-i}, m'_{-i} \in \prod_{j \neq i} \Delta(A_j)$, player i is *more willing to make a commitment at m'_{-i} than at m_{-i}* if $\{m_i\} \succsim_{i,m_{-i}} M_i$ implies $\{m_i\} \succsim_{i,m'_{-i}} M_i$ for all $M_i \in \mathcal{K}(\Delta(A_i))$ and $m_i \in \Delta(A_i)$. We write $m'_{-i} \geq_i^C m_{-i}$ if this condition holds.

Axiom 3 (Comparative Commitment Attitude)**.** The binary relation $\geq_i^C$ is complete; that is, for all $m_{-i}, m'_{-i} \in \prod_{j \neq i} \Delta(A_j)$, either $m'_{-i} \geq_i^C m_{-i}$ or $m_{-i} \geq_i^C m'_{-i}$ holds.

When $m'_{-i} \geq_i^C m_{-i}$, if player i prefers to make a commitment to m_i rather than to have a menu M_i given m_{-i}, he also does given m'_{-i}. Presumably, this is because player i anticipates a stronger peer effect toward temptation under m'_{-i}. The above axiom requires that any pair $m_{-i}, m'_{-i} \in \prod_{j \neq i} \Delta(A_j)$ can be ordered by their strength of peer effects.

The following is our characterization result.

Theorem 1. *Let $\succsim_i$ be player i's self-control preference relation on $\mathcal{K}(\Delta(A))$ represented by self-control utility*

$$U_i(M_i, m_{-i}) = \max_{m_i \in M_i} \left(u_i(m_i, m_{-i}) - \left(\max_{m'_i \in M_i} v_i(m'_i, m_{-i}) - v_i(m_i, m_{-i}) \right) \right). \tag{3}$$

The following statements are equivalent.

(i) $\succsim_i$ *satisfies Indifferent Commitment, Identical Tempting Actions, and Comparative Commitment Attitude.*

(ii) *For all* $m_{-i} \in \prod_{j \neq i} \Delta(A_j)$*, the conditional preference relation* $\succsim_{i,m_{-i}}$ *is represented by* $U_i(M_i, m_{-i})$ *of the form* (2) *such that* $\kappa_i(m'_{-i}) \geq \kappa_i(m_{-i})$ *if and only if* $m'_{-i} \geq_i^C m_{-i}$*, where a strict inequality holds if* $\succsim_{i,m_{-i}} \neq \succsim_{i,m'_{-i}}$.

The above theorem guarantees that the strength of temptation $\kappa_i(m_{-i})$ can be captured by the order $\geq_i^C$.

In a different context, Gul and Pesendorfer (2007) axiomatize a self-control representation similar to (2). They consider an infinite horizon extension of the GP model with habit formation in consumption and study a model of addiction. As a special case, they axiomatize a recursive self-control representation in which past consumption affects only the strength of temptation. The role of past consumption in their representation corresponds to that of peer effects in our representation (2). Gul and Pesendorfer (2007) consider the difference between a normative choice derived from normative utility and an actual choice in the second stage derived from a compromise between normative and temptation utilities, and show that the larger difference implies stronger temptation. On the other hand, we define Comparative Commitment Attitude to measure the strength of temptation.

In the remainder of this online appendix, we give a proof of Theorem 1. It is straightforward to check that (ii) implies (i). We show that (i) implies (ii). We first show that a normative payoff function is independent of the opponents' actions.

Lemma 1. *If* $\succsim_i$ *satisfies Indifferent Commitment and Comparative Commitment Attitude, then there exists* $f_i : \Delta(A_i) \to \mathbb{R}$ *such that* $u_i(m_i, m_{-i}) = f_i(m_i)$ *for all* $m_i \in \Delta(A_i)$ *and* $m_{-i} \in \prod_{j \neq i} \Delta(A_j)$.

Proof. Take arbitrary $m_{-i}, m'_{-i} \in \prod_{j \neq i} \Delta(A_j)$. By Comparative Commitment Attitude, we can assume $m_{-i} \geq_i^C m'_{-i}$ without loss of generality. If $\{x_i\} \succsim_{i,m_{-i}} \{y_i\}$ then $\{x_i\} \succsim_{i,m'_{-i}} \{y_i\}$, which implies that if $\{x_i\} \sim_{i,m_{-i}} \{y_i\}$ then $\{x_i\} \sim_{i,m'_{-i}} \{y_i\}$.

We show that if $\{x_i\} \succ_{i,m_{-i}} \{y_i\}$ then $\{x_i\} \succ_{i,m'_{-i}} \{y_i\}$. Seeking a contradiction, suppose that $\{x_i\} \sim_{i,m'_{-i}} \{y_i\}$. Let $\overline{x}_i, \underline{x}_i \in \Delta(A_i)$ be such that $\{\overline{x}_i\} \succ_{i,m'_{-i}} \{\underline{x}_i\}$, which exist by Indifferent Commitment. Then, $\{\lambda y_i + (1-\lambda)\overline{x}_i\} \succ_{i,m'_{-i}} \{\lambda x_i + (1-\lambda)\underline{x}_i\}$ for all $\lambda \in (0,1)$ by linearity of u_i, but $\{\lambda x_i + (1-\lambda)\underline{x}_i\} \succ_{i,m_{-i}} \{\lambda y_i + (1-\lambda)\overline{x}_i\}$ for sufficiently large $\lambda \in (0,1)$ by continuity of u_i, which implies $\{\lambda x_i + (1-\lambda)\underline{x}_i\} \succsim_{i,m'_{-i}} \{\lambda y_i + (1-\lambda)\overline{x}_i\}$ because $m_{-i} \geq_i^C m'_{-i}$, a contradiction.

We have shown that $\{x_i\} \succsim_{i,m_{-i}} \{y_i\}$ if and only if $\{x_i\} \succsim_{i,m'_{-i}} \{y_i\}$; that is, $\succsim_{i,m_{-i}}$ and $\succsim_{i,m'_{-i}}$ restricted to singletons are identical. Since the two identical preference relations on $\Delta(A_i)$ are represented by $u_i(\cdot, m_{-i})$ and $u_i(\cdot, m'_{-i})$ respectively, we must have $u_i(\cdot, m_{-i}) = \alpha u_i(\cdot, m'_{-i}) + \beta$ with $\alpha > 0$ and $\beta \in \mathbb{R}$. By Indifferent Commitment, there exist $m_i, m'_i \in \Delta(A_i)$ such that $\{(m_i, m_{-i})\} \sim_i \{(m_i, m'_{-i})\} \nsim_i \{(m'_i, m_{-i})\} \sim_i \{(m'_i, m'_{-i})\}$, or equivalently,

$u_i(m_i, m_{-i}) = u_i(m_i, m'_{-i}) \neq u_i(m'_i, m_{-i}) = u_i(m'_i, m'_{-i})$. This implies that $\alpha = 1$ and $\beta = 0$. Thus, $u_i(\cdot, m_{-i}) = u_i(\cdot, m'_{-i})$ for all $m_{-i}, m'_{-i} \in \prod_{j\neq i} \Delta(A_j)$, and we can choose $f_i(\cdot) = u_i(\cdot, m_{-i})$. □

To show the remaining part of the theorem, we use the following lemmas. Lemma 2 is taken from Lemma 0 of Gul and Pesendorfer (2001).

Lemma 2. *For each $M_i \in \mathcal{K}(\Delta(A_i))$, there exists a sequence of finite subsets $\{M_i^k \subseteq M_i\}_k^\infty$ such that $M_i^k \to M_i$ as $k \to \infty$ in the Hausdorff metric.*

Lemma 3. *Suppose that $m_{-i} \in \prod_{j\neq i} \Delta(A_j)$ is nondegenerate; that is, there exist $x_i, y_i \in \Delta(A_i)$ satisfying $\{x_i\} \succ_{i,m_{-i}} \{x_i, y_i\}$. Then, for $x_i, y_i \in \Delta(A_i)$ with $\{x_i\} \succ_{i,m_{-i}} \{y_i\}$, $v_i(y_i, m_{-i}) > v_i(x_i, m_{-i})$ if and only if $\{x_i\} \succ_{i,m_{-i}} \{x_i, y_i\}$.*

Proof. Suppose that $\{x_i\} \succ_{i,m_{-i}} \{y_i\}$, i.e., $u_i(x_i, m_{-i}) > u_i(y_i, m_{-i})$. Then, $\{x_i\} \succ_{i,m_{-i}} \{x_i, y_i\}$ if and only if

$$\begin{aligned} U_i(\{x_i\}, m_{-i}) &= u_i(x_i, m_{-i}) \\ &> U_i(\{x_i, y_i\}, m_{-i}) \\ &= \max_{\{x_i, y_i\}} (u_i(\cdot, m_{-i}) + v_i(\cdot, m_{-i})) - \max_{\{x_i, y_i\}} v_i(\cdot, m_{-i}). \end{aligned}$$

If $v_i(x_i, m_{-i}) \geq v_i(y_i, m_{-i})$, then $U_i(\{x_i, y_i\}, m_{-i}) = u_i(x_i, m_{-i})$, which is a contradiction. Thus, we must have $v_i(y_i, m_{-i}) > v_i(x_i, m_{-i})$. Conversely, if $v_i(y_i, m_{-i}) > v_i(x_i, m_{-i})$, then

$$\begin{aligned} U_i(\{x_i, y_i\}, m_{-i}) &= \max\{u_i(x_i, m_{-i}) + v_i(x_i, m_{-i}) - v_i(y_i, m_{-i}), u_i(y_i, m_{-i})\} \\ &< u_i(x_i, m_{-i}) = U_i(\{x_i\}, m_{-i}), \end{aligned}$$

which implies that $\{x_i\} \succ_{i,m_{-i}} \{x_i, y_i\}$. □

We are ready to establish the separability of a temptation payoff function.

Lemma 4. *If $\succsim_i$ satisfies Indifferent Commitment, Identical Tempting Actions, and Comparative Commitment Attitude, then there exist $\kappa_i : \prod_{j\neq i} \Delta(A_j) \to \mathbb{R}_+$ and $g_i : \Delta(A_i) \to \mathbb{R}$ such that $v_i(m_i, m_{-i}) = \kappa_i(m_{-i}) g_i(m_i)$ for all $m_i \in \Delta(A_i)$ and $m_{-i} \in \prod_{j\neq i} \Delta(A_j)$.*

Proof. By Lemma 1, there exists $f_i : \Delta(A_i) \to \mathbb{R}$ such that $u_i(\cdot, m_{-i}) = f_i(\cdot)$ for all m_{-i}. Note that $f_i(m_i) \geq f_i(m'_i)$ if and only if $\{m_i\} \succsim_{i,m_{-i}} \{m'_i\}$ for all m_{-i}.

Consider degenerate $m_{-i} \in \prod_{j\neq i} \Delta(A_j)$ such that $\{x_i\} \sim_{i,m_{-i}} \{x_i, y_i\}$ for all $x_i, y_i \in \Delta(A_i)$ with $f_i(x_i) \geq f_i(y_i)$. For such $x_i, y_i \in \Delta(A_i)$,

$$U_i(\{x_i, y_i\}, m_{-i}) = U_i(\{x_i\}, m_{-i}) = f_i(x_i).$$

Then, we can show that

$$U_i(M_i, m_{-i}) = \max_{M_i} f_i(\cdot) \tag{4}$$

for all $M_i \in \mathcal{K}(\Delta(A_i))$. This is true if $|M_i| = 2$. Suppose that (4) holds if $|M| = k$ with $2 \leq k \leq n$. For M_i with $|M_i| = n + 1$, let $x_i^* \in \arg\max_{M_i} f_i(\cdot)$. Then, for any $x_i \in M_i \setminus \{x_i^*\}$, Set Betweenness implies $\{x_i^*, x_i\} \succsim_{i,m_{-i}} M_i \succsim_{i,m_{-i}} M_i \setminus \{x_i\}$ or $M_i \setminus \{x_i\} \succsim_{i,m_{-i}} M_i \succsim_{i,m_{-i}} \{x_i^*, x_i\}$, but $\{x_i^*\} \sim_{i,m_{-i}} \{x_i^*, x_i\} \sim_{i,m_{-i}} M_i \setminus \{x_i\}$ by the induction hypothesis. Hence, we must have $\{x_i^*\} \sim_{i,m_{-i}} M_i$. By induction, (4) holds for any finite M_i, and by Lemma 2 and continuity of U_i, (4) holds for all $M_i \in \mathcal{K}(\Delta(A_i))$. Therefore, if $\kappa_i(m_{-i}) = 0$ then we have the representation (2) for any g_i.

Let $m_{-i}, m'_{-i} \in \prod_{j \neq i} \Delta(A_j)$ be nondegenerate. By Identical Tempting Actions, $\{x_i\} \succ_{i,m_{-i}} \{x_i, y_i\}$ and $\{x_i\} \succ_{i,m'_{-i}} \{x_i, y_i\}$ are equivalent, and such x_i and y_i exist by nondegeneracy. Thus, by Lemma 3, for $x_i, y_i \in \Delta(A_i)$ with $f_i(x_i) > f_i(y_i)$,

$$v_i(y_i, m_{-i}) > v_i(x_i, m_{-i}) \;\Leftrightarrow\; v_i(y_i, m'_{-i}) > v_i(x_i, m'_{-i}). \tag{5}$$

Furthermore, as we show below, (5) holds for all $x_i, y_i \in \Delta(A_i)$. Accordingly, there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $v_i(\cdot, m'_{-i}) = \alpha v_i(\cdot, m_{-i}) + \beta$, where we can choose $\beta = 0$ since only the difference of temptation payoffs matter. By setting $g_i(\cdot) = v_i(\cdot, m_{-i})$ for some nondegenerate m_{-i}, we have $v_i(\cdot, m'_{-i}) = \kappa_i(m'_{-i}) g_i(\cdot)$ for all m'_{-i}.

We prove that (5) is true for all $x_i, y_i \in \Delta(A_i)$. Note that (5) is true for all $x_i, y_i \in \Delta(A_i)$ with $f_i(x_i) > f_i(y_i)$. Seeking a contradiction, suppose otherwise. Then, there exist $x_i, y_i \in \Delta(A_i)$ such that $f_i(x_i) = f_i(y_i)$, $v_i(x_i, m_{-i}) \geq v_i(y_i, m_{-i})$, and $v_i(x_i, m'_{-i}) < v_i(y_i, m'_{-i})$. Let $x'_i, y'_i \in \Delta(A_i)$ be such that $\{y'_i\} \succ_{i,m_{-i}} \{x'_i, y'_i\}$. Then, $f_i(y'_i) > f_i(x'_i)$, $v_i(x'_i, m_{-i}) > v_i(y'_i, m_{-i})$, and $v_i(x'_i, m'_{-i}) > v_i(y'_i, m'_{-i})$ by Lemma 3. Thus, for sufficiently small $\lambda \in (0, 1)$,

$$v_i(\lambda x'_i + (1 - \lambda) x_i, m_{-i}) > v_i(\lambda y'_i + (1 - \lambda) y_i, m_{-i}), \tag{6}$$

$$v_i(\lambda x'_i + (1 - \lambda) x_i, m'_{-i}) < v_i(\lambda y'_i + (1 - \lambda) y_i, m'_{-i}). \tag{7}$$

Since $f_i(y_i) = f_i(x_i)$, we have $f_i(\lambda y'_i + (1 - \lambda) y_i) > f_i(\lambda x'_i + (1 - \lambda) x_i)$ for all $\lambda \in (0, 1)$. Then, by (6) and Lemma 3, $\{\lambda y'_i + (1 - \lambda) y_i\} \succ_{i,m_{-i}} \{\lambda x'_i + (1 - \lambda) x_i, \lambda y'_i + (1 - \lambda) y_i\}$ for all $\lambda \in (0, 1)$. By Identical Tempting Actions, $\{\lambda y'_i + (1 - \lambda) y_i\} \succ_{i,m'_{-i}} \{\lambda x'_i + (1 - \lambda) x_i, \lambda y'_i + (1 - \lambda) y_i\}$, and $v_i(\lambda x'_i + (1 - \lambda) x_i, m'_{-i}) > v_i(\lambda y'_i + (1 - \lambda) y_i, m'_{-i})$ for all $\lambda \in (0, 1)$ by Lemma 3. This contradicts to (7) for sufficiently small $\lambda \in (0, 1)$. □

To complete the proof, we show that $\kappa_i(m'_{-i}) \geq \kappa_i(m_{-i})$ if and only if $m'_{-i} \geq_i^C m_{-i}$. The next lemma establishes the "only if" part.

Lemma 5. *Suppose that U_i is given by (2). If $\kappa_i(m'_{-i}) \geq \kappa_i(m_{-i})$, then $m'_{-i} \geq_i^C m_{-i}$.*

Proof. Suppose that $\{x_i\} \succsim_{i,m_{-i}} M_i$. If $\kappa_i(m'_{-i}) \geq \kappa_i(m_{-i})$, then

$$\begin{aligned} U_i(\{x_i\}, m_{-i}) &= f_i(x_i) \\ &\geq \max_{m_i \in M_i} \left(f_i(m_i) - \kappa_i(m_{-i}) \left(\max_{m'_i \in M_i} g_i(m'_i) - g_i(m_i) \right) \right) \\ &\geq \max_{m_i \in M_i} \left(f_i(m_i) - \kappa_i(m'_{-i}) \left(\max_{m'_i \in M_i} g_i(m'_i) - g_i(m_i) \right) \right). \end{aligned}$$

Since $U_i(\{x_i\}, m'_{-i}) = f_i(x_i)$, we have $\{x_i\} \succsim_{i,m'_{-i}} M_i$, which implies $m'_{-i} \geq_i^C m_{-i}$. □

If m_{-i} is not nondegenerate, then $\kappa_i(m_{-i})$ is characterized as follows.

Lemma 6. *Suppose that U_i is given by* (2). *Let $m_{-i} \in \prod_{j \neq i} \Delta(A_j)$ be not nondegenerate. Then, we can set $\kappa_i(m_{-i}) = 0$. Furthermore, $\kappa_i(m'_{-i}) \geq \kappa_i(m_{-i})$ and $m'_{-i} \geq_i^C m_{-i}$ for all $m'_{-i} \in \Delta(m_{-i})$.*

Proof. By the argument in the proof of Lemma 4, we can set $\kappa_i(m_{-i}) = 0$. This implies that $\kappa_i(m'_{-i}) \geq \kappa_i(m_{-i})$ and $m'_{-i} \geq_i^C m_{-i}$ for all $m'_{-i} \in \prod_{j \neq i} \Delta(A_j)$ by Lemma 5. □

We distinguish the following two types of nondegeneracy. The opponents' action profile $m_{-i} \in \prod_{j \neq i} \Delta(A_j)$ is said to be *regular* if it is nondegenerate and there exist $x_i, y_i \in \Delta(A_i)$ such that $\{x_i\} \succ_{i,m_{-i}} \{x_i, y_i\} \succ_{i,m_{-i}} \{y_i\}$; m_{-i} is said to be *nonregular* if it is nondegenerate and $\{x_i, y_i\} \sim_{i,m_{-i}} \{y_i\}$ for all $x_i, y_i \in \Delta(A_i)$ with $\{x_i\} \succ_{i,m_{-i}} \{x_i, y_i\}$. Nonregularity is equivalent to the following stronger condition.

Lemma 7. *The opponents' action profile $m_{-i} \in \prod_{j \neq i} \Delta(A_j)$ is nonregular if and only if it is nondegenerate and $\{x_i, y_i\} \sim_{i,m_{-i}} \{y_i\}$ for all $x_i, y_i \in \Delta(A_i)$ with $\{x_i\} \succ_{i,m_{-i}} \{y_i\}$.*

Proof. It is enough to show the "only if" part. Let m_{-i} be nonregular. There exist $x_i, y_i \in \Delta(A_i)$ such that $\{x_i\} \succ_{i,m_{-i}} \{x_i, y_i\} \sim_{i,m_{-i}} \{y_i\}$. Seeking a contradiction, suppose that the lemma does not hold. Then, there exist $x'_i, y'_i \in \Delta(A_i)$ such that $\{x'_i\} \sim_{i,m_{-i}} \{x'_i, y'_i\} \succ_{i,m_{-i}} \{y'_i\}$. Then, $\{\lambda x'_i + (1-\lambda)x_i\} \succ_{i,m_{-i}} \{\lambda y'_i + (1-\lambda)y_i\}$ for all $\lambda \in [0, 1]$. Since m_{-i} is nonregular, either of the following holds for all $\lambda \in [0, 1]$:

$$\begin{aligned} \{\lambda x'_i + (1-\lambda)x_i\} &\succ_{i,m_{-i}} \{\lambda x'_i + (1-\lambda)x_i, \lambda y'_i + (1-\lambda)y_i\} \sim_{i,m_{-i}} \{\lambda y'_i + (1-\lambda)y_i\}, \\ \{\lambda x'_i + (1-\lambda)x_i\} &\sim_{i,m_{-i}} \{\lambda x'_i + (1-\lambda)x_i, \lambda y'_i + (1-\lambda)y_i\} \succ_{i,m_{-i}} \{\lambda y'_i + (1-\lambda)y_i\}. \end{aligned}$$

The following sets are nonempty and closed by continuity of U_i and disjoint since $\{\lambda x'_i + (1-\lambda)x_i\} \succ_{i,m_{-i}} \{\lambda y'_i + (1-\lambda)y_i\}$:

$$\begin{aligned} I_1 &\equiv \{\lambda \in [0, 1] \mid \{\lambda x'_i + (1-\lambda)x_i\} \sim_{i,m_{-i}} \{\lambda x'_i + (1-\lambda)x_i, \lambda y'_i + (1-\lambda)y_i\}\}, \\ I_2 &\equiv \{\lambda \in [0, 1] \mid \{\lambda x'_i + (1-\lambda)x_i, \lambda y'_i + (1-\lambda)y_i\} \sim_{i,m_{-i}} \{\lambda y'_i + (1-\lambda)y_i\}\}. \end{aligned}$$

Since $[0,1]$ is a connected set, there exists $\lambda \in [0,1] \setminus (I_1 \cup I_2)$, which satisfies

$$\{\lambda x_i' + (1-\lambda)x_i\} \succ_{i,m_{-i}} \{\lambda x_i' + (1-\lambda)x_i, \lambda y_i' + (1-\lambda)y_i\} \succ_{i,m_{-i}} \{\lambda y_i' + (1-\lambda)y_i\}.$$

This implies that m_{-i} is regular, a contradiction. □

If m_{-i} is nonregular, then $\kappa_i(m_{-i})$ is characterized as follows.

Lemma 8. *Suppose that U_i is given by* (2). *Let $m_{-i} \in \prod_{j\neq i} \Delta(A_j)$ be nonregular. Then, we can set $\kappa_i(m_{-i}) = 1$. Furthermore, $\kappa_i(m_{-i}) \geq \kappa_i(m'_{-i})$ and $m_{-i} \geq_i^C m'_{-i}$ for all $m'_{-i} \in \prod_{j\neq i} \Delta(A_j)$.*

Proof. By Lemma 7, $\{x_i, y_i\} \sim_{i,m_{-i}} \{y_i\}$ for all $x_i, y_i \in \Delta(A_i)$ with $\{x_i\} \succ_{i,m_{-i}} \{y_i\}$. By Set Betweenness, $\{x_i\} \succ_{i,m_{-i}} \{y_i\}$ if and only if $\{x_i, y_i\} \succ_{i,m_{-i}} \{y_i\}$, which is equivalent to $v_i(y_i, m_{-i}) > v_i(x_i, m_{-i})$ by Lemma 3. Thus, $f_i(x_i) > f_i(y_i)$ if and only if $v_i(y_i, m_{-i}) > v_i(x_i, m_{-i})$. Since $v_i(\cdot, m_{-i})$ and $-f_i(\cdot)$ represent the same ranking over $\Delta(A_i)$, there exists $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $v_i(\cdot, m_{-i}) = -\alpha f_i(\cdot) + \beta$, where we can choose $\beta = 0$ since only the difference of temptation payoffs matter. Then, by (2),

$$U_i(M_i, m_{-i}) = \max_{M_i} (1-\alpha) f_i(\cdot) + \alpha \min_{M_i} f_i(\cdot).$$

Thus, if $\{x_i\} \succ_{i,m_{-i}} \{y_i\}$ (i.e. $f_i(x_i) > f_i(y_i)$), then

$$U_i(\{x_i, y_i\}, m_{-i}) = \max_{\{x_i,y_i\}} (1-\alpha) f_i(\cdot) + \alpha \min_{\{x_i,y_i\}} f_i(\cdot) = f_i(y_i)$$

since $\{x_i, y_i\} \sim_{i,m_{-i}} \{y_i\}$. Therefore, $\alpha \geq 1$ and $U_i(M_i, m_{-i}) = \min_{M_i} f_i(\cdot)$, whereby we can set $\kappa_i(m_{-i}) = 1$ and $g_i(\cdot) = -f_i(\cdot)$ for all nonregular $m_{-i} \in \prod_{j\neq i} \Delta(A_j)$. For $m'_{-i} \neq m_{-i}$, we must have $v_i(\cdot, m'_{-i}) = \kappa_i(m'_{-i}) g_i(\cdot) = -\kappa_i(m'_{-i}) f_i(\cdot)$, and by (2),

$$U_i(M_i, m_{-i}) = \max_{M_i} (1 - \kappa_i(m'_{-i})) f_i(\cdot) + \kappa_i(m'_{-i}) \min_{M_i} f_i(\cdot).$$

If m'_{-i} is nonregular, we can set $\kappa_i(m'_{-i}) = 1$; otherwise, we must have $\kappa_i(m'_{-i}) < 1$ because $U_i(M_i, m_{-i}) = \min_{M_i} f_i(\cdot)$ implies nonregularity. Accordingly, $1 = \kappa_i(m_{-i}) \geq \kappa_i(m'_{-i})$ for all $m'_{-i} \in \prod_{j\neq i} \Delta(A_j)$. Then, Lemma 5 implies that $m_{-i} \geq_i^C m'_{-i}$ for all $m'_{-i} \in \prod_{j\neq i} \Delta(A_j)$. □

To characterize $\kappa_i(m_{-i})$ with regular m_{-i}, we use the following lemma.

Lemma 9. *Suppose that U_i is given by* (2). *If $m'_{-i} \geq_i^C m_{-i}$ then $U_i(M_i, m_{-i}) \geq U_i(M_i, m'_{-i})$ for all $M_i \in \mathcal{K}(\Delta(A_i))$.*

Proof. By (2), there exist $\overline{x}_i, \underline{x}_i \in \Delta(A_i)$ such that $f_i(\overline{x}_i) \geq U_i(M_i, m_{-i}) \geq f_i(\underline{x}_i)$. Since f_i is continuous, there exists $x_i \in \Delta(A_i)$ such that $f_i(x_i) = U_i(M_i, m_{-i})$, i.e., $\{x_i\} \sim_{i,m_i} M_i$. Since $m'_{-i} \geq_i^C m_{-i}$, we have $\{x_i\} \succsim_{i,m'_i} M_i$, which implies that $U_i(M_i, m_{-i}) = f_i(x_i) = U_i(\{x_i\}, m'_{-i}) \geq U_i(M, m'_{-i})$. □

If m_{-i} is regular, then $\kappa_i(m_{-i})$ is characterized as follows.

Lemma 10. *Suppose that U_i is given by* (2)*. Let $m_{-i}, m_{-i} \in \prod_{j\neq i} \Delta(A_j)$ be regular. If $m'_{-i} \geq_i^C m_{-i}$, then $\kappa_i(m'_{-i}) \geq \kappa_i(m_{-i})$.*

Proof. Suppose that $m'_{-i} \geq_i^C m_{-i}$. Since m_{-i} is regular, there exist $x_i, y_i \in \Delta(A_i)$ such that $\{x_i\} \succ_{i,m_{-i}} \{x_i, y_i\} \succ_{i,m_{-i}} \{y_i\}$. Thus, $f_i(x_i) = U_i(\{x_i\}, m_{-i}) > U_i(\{x_i, y_i\}, m_{-i}) > U_i(\{y_i\}, m_{-i}) = f_i(y_i)$ and $g_i(y_i) > g_i(x_i)$ by Lemma 3. Since

$$\begin{aligned} U_i(\{x_i, y_i\}, m_{-i}) &= \max_{\{x_i,y_i\}} \left(f_i(\cdot) + \kappa_i(m_{-i})g_i(\cdot)\right) - \max_{\{x_i,y_i\}} \kappa_i(m_{-i})g_i(\cdot) \\ &= \max\{f_i(x_i) - \kappa_i(m_{-i})(g_i(y_i) - g_i(x_i)), f_i(y_i)\} \\ &> f_i(y_i), \end{aligned}$$

we have $U_i(\{x_i, y_i\}, m_{-i}) = f_i(x_i) - \kappa_i(m_{-i})(g_i(y_i) - g_i(x_i))$. By Lemma 9,

$$\begin{aligned} f_i(x_i) - \kappa_i(m'_{-i})(g_i(y_i) - g_i(x_i)) &\leq \max\{f_i(x_i) - \kappa_i(m'_{-i})(g_i(y_i) - g_i(x_i)), f_i(y_i)\} \\ &= U_i(\{x_i, y_i\}, m'_{-i}) \\ &\leq U_i(\{x_i, y_i\}, m_{-i}) \\ &= f_i(x_i) - \kappa_i(m_{-i})(g_i(y_i) - g_i(x_i)). \end{aligned}$$

The above is reduced to $(\kappa_i(m'_{-i}) - \kappa_i(m_{-i}))(g_i(y_i) - g_i(x_i)) \geq 0$, and thus $\kappa_i(m'_{-i}) - \kappa_i(m_{-i}) \geq 0$ since $g_i(y_i) - g_i(x_i) > 0$. □

We are ready to establish the "if" part.

Lemma 11. *Suppose that U_i is given by* (2)*. If $m'_{-i} \geq_i^C m_{-i}$, then $\kappa_i(m'_{-i}) \geq \kappa_i(m_{-i})$. A strict inequality holds if $\succsim_{i,m_{-i}} \neq \succsim_{i,m'_{-i}}$.*

Proof. By Lemma 6, if m_{-i} is not nondegenerate, then $m'_{-i} \geq_i^C m_{-i}$ for all $m'_{-i} \in \prod_{j\neq i} \Delta(A_j)$ and $\kappa_i(m'_{-i}) \geq \kappa_i(m_{-i}) = 0$.

By Lemma 8, if m'_{-i} is nonregular, then $m'_{-i} \geq_i^C m_{-i}$ for all $m_{-i} \in \prod_{j\neq i} \Delta(A_j)$ and $1 = \kappa_i(m'_{-i}) \geq \kappa_i(m_{-i})$.

In the other case, both m_{-i} and m'_{-i} are regular. By Lemma 10, if $m'_{-i} \geq_i^C m_{-i}$, then $\kappa_i(m'_{-i}) \geq \kappa_i(m_{-i})$. It should be noted that if there is no nonregular opponents' action profile, then $\kappa_i(m_{-i}) \leq 1$ is not necessarily true.

Finally, by (2), if $\kappa_i(m'_{-i}) = \kappa_i(m_{-i})$ then $\succsim_{i,m_{-i}} = \succsim_{i,m'_{-i}}$, which implies that a strict inequality holds if $\succsim_{i,m_{-i}} \neq \succsim_{i,m'_{-i}}$. □

References

Gul, F., Pesendorfer, W., 2001. Temptation and self-control. Econometrica 69, 1403–1436.

Gul, F., Pesendorfer, W., 2007. Harmful addiction. Rev. Econ. Stud. 74, 147–172.