

Subjective Prior over Subjective States, Stochastic Choice, and Updating *

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Abstract

In a dynamic model with uncertainty, a decision tree and preference at each decision node have been taken as primitives. This assumption requires the analyst to know all the uncertainties a decision maker (DM) perceives. Kreps (1979) and Dekel, Lipman and Rustichini (2001) show that the subjective states can be identified with the set of uncertain future preferences and be derived from preference over menus.

However, it is not immediate how we can use their models for dynamic choice because the realized subjective state is not observable for the analyst. We propose introducing some objective states into their model. There are two added benefits of this modification. First, we can derive a meaningful probability over the subjective state space, which makes stochastic prediction of subsequent choice possible. Second, in our model, subjective states have correlation with objective states. Hence, the analyst can infer the ex post probability over the subjective states from the realized objective information. We derive a unique updating rule of the subjective probability over the subjective state space in response to the objective information.

Keywords: subjective state space, stochastic choice, subjective probability, Bayesian updating, dynamic consistency.

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1 Introduction

1.1 Motivation and Outline

In order to model intertemporal choice under uncertainty, a decision tree and preference at each decision node have been taken as primitives in a standard dynamic setting. This modeling requires the analyst to know all the uncertainties a decision maker (DM) perceives. However, the DM may have some subjective contingencies in her mind, which are different from the given objective states. Moreover, by the nature of subjective states, those contingencies are not directly observable for the analyst.

The answer suggested by Kreps [8, 9] and Dekel, Lipman and Rustichini [1] (Henceforth DLR) is that the analyst can derive the subjective uncertainties from preference over opportunity sets (or menus) of alternatives. When considering menus as choice objects, we have in mind the following timing of decisions: (i) choose a menu x ; and (ii) choose an alternative $a \in x$ in the next period, though such an ex post stage is not in the formal model.

The following example illustrates the basic idea of Kreps [8]: The DM will choose this evening between chicken dinner and fish dinner at a restaurant. If the DM has to make a reservation for dinner, she is indifferent between chicken and fish. Nevertheless, the DM may strictly prefer not to make a reservation because she is reluctant to commit herself right now to decide between chicken and fish. That is, the ranking

$$\{\text{chicken, fish}\} \succ \{\text{chicken}\} \sim \{\text{fish}\} \quad (1)$$

is appealing in terms of flexibility. This ranking reveals that the DM perceives two subjective states. She expects that, at one state, chicken is strictly preferred to fish, while this ranking is reversed at the other state. Ranking (1) reflects the DM's awareness of uncertainty about her future preferences. In Kreps's model, the subjective state space is identified with a set of ex post preferences over alternatives.

Can we use their models for dynamic choice? More precisely, when observing the DM choose a menu, can we predict choice out of the menu in the next period? The answer is not immediate. Even if the analyst can collect information on the rankings over menus and derive the DM's subjective states, the realized subjective state is unobservable for the analyst. Hence, we can not predict subsequent choice based on the ex post preference.

Now a question is how we can make subjective states observable. The answer we suggest here is to introduce some objective states, say Ω , into the Kreps's model. The idea is simple. If there exist objective states, the DM's subjective states may be correlated with those objective states. If a joint distribution over the objective states and the subjective states is known, the analyst can infer the ex post probability over the subjective states from the realized objective information. As long as the distribution has correlation, this procedure provides some information about the subjective states.

There are two added benefits of this modification. First, unlike Kreps [8, 9] and DLR, we can pin down a meaningful probability measure over a subjective state space. Why is

this possible? Unlike a menu of some abstract alternatives, we can now consider a menu of *acts* as a choice object. An act means a function from the objective state space into a given outcome space. In this setup, we can identify the subjective state space with a set of subjective expected utility (SEU) representations over acts. An SEU representation has two components: taste over outcomes and belief over Ω . In our model, subjective uncertainties concern only beliefs over Ω , and do not affect preference over outcomes. This state-independence of outcome preference is the reason why we can pin down a subjective probability over the subjective state space.

Our model has implications for dynamic choice behavior. Once a subjective probability is uniquely derived, it is possible to predict subsequent choice stochastically. We provide a condition under which a subjective probability over the subjective state space generates a unique stochastic choice in the ex post stage.

As mentioned above, in our model, subjective states concern beliefs about Ω . In other words, subjective states have correlation with objective states. This is the second benefit of introducing the objective states. Imagine the situation where, before choice out of the predetermined menu takes place, the DM receives an additional information telling that $A \subset \Omega$ is the “true” event. Presumably, she updates the initial subjective probability somehow in response to this objective information. This updating rule is relevant for the analyst because the updated probability provides more accurate stochastic prediction of the subsequent choice. We derive a unique updating rule of the subjective probability over the subjective state space in response to the objective information.

1.2 Domain and Functional Form

We consider the following choice objects: Let Ω be a finite objective state space. Let Z be a compact metric outcome space and $\Delta(Z)$ be the set of lotteries over Z . Let $\mathcal{H}$ be the set of functions, $h : \Omega \rightarrow \Delta(Z)$, called Anscombe-Aumann acts. Let $\mathcal{P}(\mathcal{H})$ be the set of non-empty subsets of $\mathcal{H}$. Preference $\succeq$ is defined on the domain $\mathcal{D} \equiv \mathcal{P}(\mathcal{H})$.

Our hypothesis is that, when choosing a menu, the DM has in mind the “ex post” stage where choice out of the menu takes place. If so, preference on menus should reflect the DM’s dynamic perspective.

We axiomatize preference $\succeq$ on $\mathcal{D}$ admitting the following representation: there exist a state space $S \subset \Delta(\Omega)$, a countably additive Borel probability measure μ over S , and a non-constant mixture linear function $u : \Delta(Z) \rightarrow \mathbb{R}$ such that $U : \mathcal{D} \rightarrow \mathbb{R}$, defined by

$$U(x) \equiv \int_S \sup_{h \in x} \left(\sum_{\omega \in \Omega} u(h(\omega)) p(\omega) \right) d\mu(p), \quad (2)$$

represents $\succeq$. This functional form justifies our hypothesis, that is, the DM behaves as if she anticipates the ex post stage where choice out of the menu takes place. She expects some subjective signal, p , to arrive in the next period according to the probability measure μ , and is aware that the ex post choice takes place so as to maximize the signal-dependent SEU representation over $\mathcal{H}$.

In (2), the subjective state space S is identified with a subset of beliefs over Ω . This special nature allows us to pin down a unique subjective probability μ .

1.3 An Example: Real Option

As an example, consider a DM facing the decision whether she purchases a real estate. The profit from the real estate depends on objective states, ω_1 and ω_2 . If ω_1 happens, it generates a net gain \$1000, whereas it causes a net loss \$1000 otherwise. Hence, the real estate is regarded as an act from $\{\omega_1, \omega_2\}$ into $\mathbb{R}$. On the other hand, if the DM does not purchase it, she receives nothing no matter what objective state is realized.

Now suppose

$$\left\{ \begin{bmatrix} \$1000 & \omega_1 \\ -\$1000 & \omega_2 \end{bmatrix} \right\} \succeq \left\{ \begin{bmatrix} 0 & \omega_1 \\ 0 & \omega_2 \end{bmatrix} \right\}. \quad (3)$$

This ranking says that the DM wants to buy the real estate. Presumably, she has a prior over $\{\omega_1, \omega_2\}$ evaluating that ω_1 is more likely to happen.

Nevertheless, the DM may strictly prefer delaying a decision. Consider the real option making procrastination possible with some (opportunity) cost $c > 0$. For c small enough, the ranking

$$\left\{ \begin{bmatrix} \$1000 - c & \omega_1 \\ -\$1000 - c & \omega_2 \end{bmatrix}, \begin{bmatrix} -c & \omega_1 \\ -c & \omega_2 \end{bmatrix} \right\} \succ \left\{ \begin{bmatrix} \$1000 & \omega_1 \\ -\$1000 & \omega_2 \end{bmatrix} \right\} \quad (4)$$

is appealing. Ranking (4) reveals that the DM anticipates two subjective signals to arrive in the next period. One signal suggests that ω_1 is more likely, while the other signal conversely tells that ω_2 is more likely. The left hand side of (4), that is, the real option, allows the DM to choose between purchasing the real estate and not purchasing it, depending on subjective signals. On the other hand, if she chooses the right hand side, she has to commit herself to purchase the real estate no matter what signal arrives. Thus, the DM prefers the real option even though she has to pay a positive cost. Our resulting model (2) is consistent with ranking (3) and (4).

Under the ranking (3), the “standard” model implies the ranking

$$\left\{ \begin{bmatrix} \$1000 & \omega_1 \\ -\$1000 & \omega_2 \end{bmatrix} \right\} \succeq \left\{ \begin{bmatrix} \$1000 - c & \omega_1 \\ -\$1000 - c & \omega_2 \end{bmatrix}, \begin{bmatrix} -c & \omega_1 \\ -c & \omega_2 \end{bmatrix} \right\}, \quad (5)$$

for any $c \geq 0$. Thus, the real option has no value.

1.4 Updating of the Initial Prior

To capture how μ is updated in response to objective information, we consider the set of conditional preferences $\{\succeq_A\}_{A \subset \Omega}$ over $\mathcal{P}(\mathcal{H})$. Each $\succeq_A$ is interpreted as preference over menus when the DM is told that $A \subset \Omega$ is a “true” event. We impose all the axioms ensuring result (2) on the ex ante preference $\succeq_\Omega$, that is, preference under no additional

information. Imposing, in addition, “dynamic consistency” on the conditional preferences $\{\succeq_A\}_{A \subset \Omega}$, we show that each $\succeq_A$ admits representation (2) with components (μ_A, u) , where μ_A is a probability measure over $\Delta(\Omega)$ and u is a risk preference independent of A .

Our question is what is the relation between the initial prior μ_Ω and μ_A , that is, how to update μ_Ω in response to the objective information $A \subset \Omega$. Our axioms pin down the following unique updating rule:

Step 1: Adjust μ to μ^* so as to reflect the “reliance” of each $p \in \Delta(\Omega)$ in terms of the additional information A .

Step 2: Update each $p \in \Delta(\Omega)$ by Bayes’ Rule, and derive the non-negative measure ν_A over $\Delta(A)$ as the distribution of μ^* induced by Bayes’ Rule.

Step 3: Normalize ν_A to obtain a probability measure.

The probability over $\Delta(\Omega)$ derived from these steps exactly coincides with μ_A .

To illustrate the above updating rule, assume that Ω consists of three states, that is, a red ball R , a blue ball, B , and a green ball, G . The DM satisfying our axioms has a subjective probability μ over $\Delta(\Omega)$. Since Ω consists of three states, $\Delta(\Omega)$ can be identified with the probability triangle such as Figure 1. For simplicity, assume μ has the finite support $\{p_1, \dots, p_4\} \subset \Delta(\Omega)$.

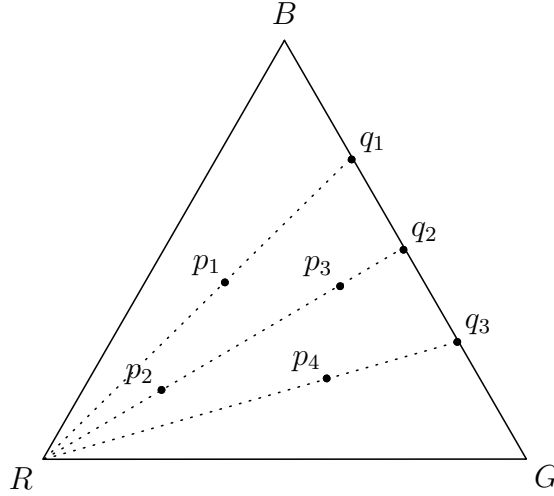


Figure 1: probability triangle and updating rule

Now suppose that the DM receives the objective information $A \equiv \{B, G\}$. First, she reevaluates p_i by taking into account its reliance in terms of the information A . For example, though p_2 and p_3 induces the same conditional probability $q_2 = p_2(\cdot|A) = p_3(\cdot|A)$, $p_2(A)$ is less than $p_3(A)$. Thus, p_2 is less reliable than p_3 in terms of the event A . The DM takes this into account. Precisely, the new evaluation of p_i is $\mu^*(p_i) \equiv \mu(p_i)p_i(A)$. Second, she updates each p_i by Bayes’ Rule and obtains the set $\{q_1, q_2, q_3\} \subset \Delta(A)$, where $q_1 = p_1(\cdot|A)$,

$q_2 = p_2(\cdot|A) = p_3(\cdot|A)$, and $q_3 = p_4(\cdot|A)$. Third, she induces the distribution ν_A over $\{q_1, q_2, q_3\}$ by

$$\nu_A(q_1) \equiv \mu^*(p_1), \nu_A(q_2) \equiv \mu^*(p_2) + \mu^*(p_3), \nu_A(q_3) \equiv \mu^*(p_4).$$

This non-negative measure ν_A over $\Delta(A)$ may not be a probability measure. Finally, take a normalization $\nu_A/\nu_A(\Delta(A))$, which must coincide with μ_A representing conditional preference $\succeq_A$. This is a unique updating rule consistent with dynamic consistency.

1.5 Related Literature

Kreps [8, 9] provides an axiomatic foundation of a subjective state space. Dekel, Lipman and Rustichini [1] show uniqueness of the subjective state space. DLR take as the domain the set of non-empty subsets of lotteries over alternatives. Though they have several models, we focus the additive representation with a non-negative measure, that is,

$$U(x) = \int_S \sup_{l \in x} u(l, s) d\mu(s), \quad (6)$$

where S is a state space, μ is a non-negative measure over S , and $u(\cdot, s)$ is a state-dependent expected utility function over lotteries.

DLR fail to provide a unique probability over the subjective state space because of the state-dependence of the ex-post utility functions. Hence, μ in (6) cannot be interpreted as the DM's belief about subjective uncertainties.

Domains consisting of menus with objective states are not new. Epstein [2] introduces the domain $\mathcal{P}(\mathcal{H})$ and provides non-Bayesian updating models. Takeoka [13] derives not only a subjective probability but also a subjective decision tree by taking into account the domain $\mathcal{P}(\mathcal{P}(\mathcal{H}))$. Hyogo [7] takes the product domain $A \times \mathcal{P}(\mathcal{H})$, where A is the set of actions, and provides an axiomatic foundation of a subjective experimentation.

Nehring [10], Ghirardato [4] and Ozdenoren [11] are other literature considering menus with objective states. These authors take, as the domain, the set of set-valued Savage acts or of set-valued Anscombe-Aumann acts.

In the updating context within the Savage model, Epstein and Le Breton [3] show that dynamic consistency implies that the DM is probabilistically sophisticated and updates the prior by Bayes' Rule. Ghirardato [5] shows the similar result under the subjective expected utility setting.

2 Subjective Probability over a Subjective State Space

2.1 Domain

Let Ω be a finite objective state space with $\#\Omega = n$. Let Z be a compact metric outcome space and $\Delta(Z)$ be the set of all Borel probability measures over Z with the

weak convergence topology. Let $\mathcal{H}$ be the set of all Anscombe-Aumann acts, that is, $\mathcal{H} \equiv \{h|h : \Omega \rightarrow \Delta(Z)\}$. Since $\Delta(Z)$ is a compact metric space, so is $\mathcal{H}$ under the product topology.

To capture the subjective states of the DM, we adapt the modeling as in Kreps [8] and DLR. Let $\mathcal{P}(\mathcal{H})$ be the set of all non-empty subsets of $\mathcal{H}$ with the Hausdorff topology.¹ Generic elements are denoted by $x, y, \dots$, and interpreted as menus or opportunity sets of acts. Preference $\succeq$ is defined on $\mathcal{D} \equiv \mathcal{P}(\mathcal{H})$.

What we have in mind is the following timing of decisions:

Period 0: choose a menu x

Period 1⁻: receive a subjective signal s

Period 1: choose an act $h \in x$

Period 1⁺: An objective state is realized and the DM receives the lottery prescribed by h .

Notice that this time line, except period 0, and a subjective signal s are not parts of the formal model. If the DM has in mind the above timing of decisions and expects to receive a subjective signal, preference in period 0 should reflect the DM's perception of those subjective signals. Thus, preference over $\mathcal{P}(\mathcal{H})$ is relevant for deriving the subjective states.

2.2 Axioms

The first five axioms on $\succeq$ are the same as in DLR. However, we impose the same axioms on preference on $\mathcal{P}(\mathcal{H})$ rather than on $\mathcal{P}(\Delta(Z))$.

AXIOM 2.1 (Order): $\succeq$ is complete and transitive.

AXIOM 2.2 (Continuity): For all $x \in \mathcal{D}$, $\{y \in \mathcal{D} | x \succeq y\}$ and $\{y \in \mathcal{D} | y \succeq x\}$ are closed.

AXIOM 2.3 (Strong Nondegeneracy): There exist $l, l' \in \Delta(Z)$ such that $\{l\} \succ \{l'\}$.

For any acts $h, h' \in \mathcal{H}$ and $\lambda \in [0, 1]$, we can define the mixture act $\lambda h + (1 - \lambda)h' \in \mathcal{H}$ by mixing lotteries $h(\omega)$ and $h'(\omega)$ state by state. For all menus $x, x' \in \mathcal{D}$ and $\lambda \in [0, 1]$, define the mixture by

$$\lambda x + (1 - \lambda)x' \equiv \{\lambda h + (1 - \lambda)h' | h \in x, h' \in x'\}.$$

AXIOM 2.4 (Independence): For all $x, y, z \in \mathcal{D}$ and for all $\lambda \in (0, 1]$,

$$x \succ y \Rightarrow \lambda x + (1 - \lambda)z \succ \lambda y + (1 - \lambda)z.$$

¹Details are relegated to Appendix A.

As in DLR, Independence is justified by the following two steps: For any $x, z \in \mathcal{P}(\mathcal{H})$ and $\lambda \in [0, 1]$, consider the lottery $\lambda \circ x + (1 - \lambda) \circ z$, which assigns x with probability λ and z with probability $(1 - \lambda)$. This lottery is an informal object because it is not in the domain. vNM independence axiom implies that, for any $\lambda \in (0, 1]$, if x is strictly preferred to y , then $\lambda \circ x + (1 - \lambda) \circ z$ is strictly preferred to $\lambda \circ y + (1 - \lambda) \circ z$.

As the second step, we argue that the DM is indifferent between $\lambda \circ x + (1 - \lambda) \circ z$ and $\lambda x + (1 - \lambda)z$. The difference between these two objects is the timing of resolution of the randomization $(\lambda, 1 - \lambda)$. For the former, the randomization has been realized before the DM chooses an act out of the menu she receives, while, for the latter, the randomization is still unresolved when she chooses an act out of the menu $\lambda x + (1 - \lambda)z$. Thus, indifference between these objects means that the DM does not care in which order the randomization $(\lambda, 1 - \lambda)$ is realized. This is appealing if the DM believes that her future preference over $\mathcal{H}$ surely satisfies mixture linearity because she can expect the same outcome out of the above two objects.

The next axiom says that a bigger menu is always weakly preferred.

AXIOM 2.5 (Monotonicity): For all $x, x' \in \mathcal{D}$, $x' \supset x \Rightarrow x' \succeq x$.

A bigger menu is preferable because it allows the DM to leave more options until the next period. Thus, Monotonicity is consistent with preference for flexibility.

The next axiom has no counterpart in DLR. The axiom is meaningful only when there are some objective states. For $h \in \mathcal{H}$, let

$$O(h) \equiv \{h' \in \mathcal{H} \mid \{h(\omega)\} \succeq \{h'(\omega)\} \text{ for all } \omega\}.$$

In terms of commitment preference $\{l\} \succeq \{l'\}$, any act in $O(h)$ is dominated by h state by state. Hence, $O(h)$ is the set of all dominated acts by h .

This dominance notion can be adapted to menus. Let

$$O(x) \equiv \cup_{h \in x} O(h).$$

Notice that $O(x)$ is also a menu and that $O(x)$ is bigger than x when $\succeq$ satisfies Order. Any act in $O(x)$ is dominated by some act in x in the above sense.

AXIOM 2.6 (Risk Preference Certainty): For all $x \in \mathcal{D}$, $x \sim O(x)$.

This axiom can be justified as follows: suppose that the DM surely knows her future risk preference, that is, the rankings over $\Delta(Z)$. Then the future risk preference coincides with the commitment preference $\{l\} \succeq \{l'\}$ though these two preferences are conceptually different. Now $O(x)$ can be reinterpreted as the set of all acts dominated by some act in x in terms of the future risk preference. Even though $O(x)$ is bigger than x , the additional part $O(x) \setminus x$ is surely valueless because the DM never chooses a dominated act in the future. Since $O(x)$ has no additional value in terms of flexibility, x and $O(x)$ should be indifferent.

2.3 Additive SEU Representation

In this section, we introduce a functional form with a subjective state space and a subjective probability, and provide the main representation theorem.

First of all, the set of subjective signals can be effectively identified with the set of preferences over $\mathcal{H}$. As suggested in the Introduction, subjective signals themselves do not matter for the DM. Subjective signals are relevant only because they convey some information about the future preferences.

The second remark is that preference over $\mathcal{H}$ can have a special representation, that is, a subjective expected utility (SEU) representation. An SEU representation has two components: a risk preference $u : \Delta(Z) \rightarrow \mathbb{R}$ and a belief p over Ω . Thus, it is possible that subjective signals have no effect on u , but affect beliefs over Ω . Then the set of subjective signals can be identified with $\Delta(\Omega)$.

The above argument leads to the functional form $U : \mathcal{P}(\mathcal{H}) \rightarrow \mathbb{R}$, defined by

$$U(x) \equiv \int_{\Delta(\Omega)} \sup_{h \in x} U_1(h, p) d\mu(p), \quad (7)$$

where

$$U_1(h, p) \equiv \sum_{\omega \in \Omega} u(h(\omega))p(\omega),$$

μ is a countably additive Borel probability measure over $\Delta(\Omega)$ and $u : \Delta(Z) \rightarrow \mathbb{R}$ is a non-constant mixture linear function.

The following is the analogue of the additive EU representation provided by DLR for menus of acts:

Definition 2.1. *Preference $\succeq$ on $\mathcal{P}(\mathcal{H})$ admits an additive SEU representation if there exists the functional form (7) with components (μ, u) representing $\succeq$.*

This representation can be interpreted as follows: the DM behaves as if she has in mind the timing of decisions described in Section 2.1 and anticipates a subjective signal to arrive before choosing an act out of the menu. The DM is certain about her future risk preference. Thus, subjective signals exclusively concern beliefs about Ω . She faces, in period 0, the uncertainty about the subjective signals and have a belief μ over these signals.

The following is the main theorem in this section. A proof appears in Appendix B.1.

Theorem 2.1. *The following statements are equivalent:*

- (a) *Preference $\succeq$ on $\mathcal{D}$ satisfies Order, Continuity, Strong Nondegeneracy, Independence, Monotonicity and Risk Preference Certainty.*
- (b) *Preference $\succeq$ on $\mathcal{D}$ admits an additive SEU representation.*

Three remarks are in order regarding additive SEU representations. Suppose that $\succsim$ on $\mathcal{P}(\mathcal{H})$ admits an additive SEU representation. Notice that the commitment preference over $\mathcal{H}$ is represented by the SEU form

$$U(\{h\}) = \sum_{\omega} u(h(\omega)) \bar{p}(\omega),$$

where $\bar{p} \in \Delta(\Omega)$ is the mean belief in terms of the second-order probability $\mu \in \Delta(\Delta(\Omega))$. Since the commitment preference reflects the ex ante perspective, the belief $\bar{p}$ is interpreted as the ex ante belief or initial prior over Ω . Each belief p expected to arrive in period 1 can be interpreted as the ex post belief over Ω . Thus, it depends on subjective signals how the DM updates the initial prior $\bar{p}$.

If the subjective states are interpreted as primitives, the functional form (7) is consistent with the standard Bayesian model. Let S be the subjective state space. Then, $S \times \Omega$ can be regarded as the “full” state space. The DM has a marginal distribution μ over S and a conditional probability system $q : S \rightarrow \Delta(\Omega)$ defined by $q(p) = p$ for all $p \in S$. Equivalently, the DM has a single belief over $S \times \Omega$ and updates the belief by Bayes’ Rule when a signal $s \in S$ arrives.

A final remark concerns the relation to DLR. An additive SEU representation is not an extension of their additive representation to the setting of acts. The direct counterpart of their representation is the functional form

$$U(x) \equiv \int_S \sup_{h \in x} U_1(h, s) d\mu(s), \quad (8)$$

where S is a state space, μ is a non-negative measure over S , and $U_1(\cdot, s)$ is a state-dependent mixture linear function over $\mathcal{H}$. This representation is consistent with Axiom 2.1-2.5, but may violate Risk Preference Certainty. An additive SEU representation is a special case of the functional form (8), and Risk Preference Certainty plays a key role to determine the special structure. As shown in the next subsection, the added benefit of this axiom is to pin down a unique belief over the subjective state space.

2.4 Uniqueness

Suppose there exist two additive SEU representations (μ, u) and (μ', u') representing the same preference. If $\mu \neq \mu'$, we cannot interpret μ as a belief of the DM about subjective states. The following theorem ensures that preference on $\mathcal{D}$ admits a unique additive SEU representation. A proof is relegated to Appendix B.2.

Theorem 2.2. *If two additive SEU representations (μ, u) and (μ', u') represent the same preference, then:*

- (i) *u and u' are cardinally equivalent, that is, there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u' = \alpha u + \beta$; and*

(ii) $\mu = \mu'$.

Unlike DLR, we can pin down a meaningful probability measure μ over a subjective state space. The reason comes from a special nature of an additive SEU representation (μ, u) , that is, subjective uncertainties exclusively concern beliefs over Ω , and do not affect risk preference. This state-independence of risk preference makes our result possible.

2.5 Generated Stochastic Choice

An additive SEU representation

$$U(x) \equiv \int_{\Delta(\Omega)} \sup_{h \in x} U_1(h, p) d\mu(p), \quad (9)$$

where

$$U_1(h, p) \equiv \sum_{\omega \in \Omega} u(h(\omega))p(\omega),$$

has an implication of choice in the ex post stage. The basic idea is as follows: suppose that the DM has preference over menus represented by functional form (9) and that we observe her choose a menu x in period 0. In period 1, the DM has an SEU over $\mathcal{H}$ depending on the realized subjective signal p , which comes about according to μ . Thus, μ can be interpreted as a random subjective expected utility. If the DM receives p in period 1⁻, (9) suggests that she will choose an act $h \in x$ so as to maximize the ex post SEU $U_1(\cdot, p)$. We can expect the random SEU μ to deliver a stochastic choice over x , which is a stochastic prediction of choice in the ex post stage.

Prior to proceeding to the formal argument, we point out some remarks. The probability μ is a part of the representation. That is, it is a subjective belief over signals. Hence, μ may have nothing to do with the “true” probability over those signals. The argument about the ex post choice as outlined above relies on the hypothesis that subjective belief μ over signals coincides with the objective probability.² For example, imagine the situation where the DM knows the true probability over the signals, but the analyst does not. Then, μ in (9) can be interpreted as the private information of the DM, which is elicited from the DM’s observable behavior.

Second, uniqueness of additive SEU representations, shown in Section 2.4, is crucial for deriving a meaningful stochastic prediction of the ex post choice. Without uniqueness of μ , the stochastic choice generated by the representation is not unique, either.

To model choice in the ex post stage, we adapt the argument in Gul and Pesendorfer [6].³ We focus on menus consisting of finite elements, that is,

$$\mathcal{D}_F \equiv \{x \in \mathcal{D} | \#x < \infty\}.$$

²This hypothesis is well-known as the rational expectations hypothesis.

³Gul and Pesendorfer [6] consider as the primitive a stochastic choice among a finite menu of lotteries and characterize the condition under which there exists a “random expected utility” rationalizing the stochastic choice. We consider acts rather than lotteries as choice objects, and ask under what conditions a “random subjective expected utility” μ induces a stochastic choice.

A *stochastic choice* is a function $\rho : \mathcal{D}_F \rightarrow \Delta(\mathcal{H})$ such that $\rho^x(x) = 1$ for any $x \in \mathcal{D}_F$.

Let

$$M(x, p) \equiv \{h \in x \mid U_1(h, p) \geq U_1(h', p), \text{ for any } h' \in x\}.$$

When the DM has a menu x and p arrives, she chooses $h \in M(x, p)$ in period 1. However, since $M(x, p)$ may contain more than two elements, we need an arbitrarily specified tie-breaking rule to derive a stochastic choice. To avoid this arbitrariness, we pay attention to random SEUs admitting, almost surely, a unique maximizer for any x . For any $x \in \mathcal{D}_F$, let

$$N(x, h) \equiv \{p \in \Delta(\Omega) \mid U_1(h, p) \geq U_1(h', p), \text{ for all } h' \in x\},$$

and

$$N^+(x, h) \equiv \{p \in \Delta(\Omega) \mid U_1(h, p) > U_1(h', p), \text{ for all } h' \in x \text{ with } h' \neq h\}.$$

That is, $N(x, h)$ is the set of p supporting h as a maximizer among x in terms of $U_1(\cdot, p)$. Similarly, $N^+(x, h)$ is the set of p such that h is a unique maximizer with respect to $U_1(\cdot, p)$. For $h \notin x$, let $N(x, h) = N^+(x, h) = \emptyset$.

For some $x \in \mathcal{D}_F$, $N^+(x, h)$ is empty for all $h \in x$. For example, take h and h' with $h \neq h'$ satisfying $u(h(\omega)) = u(h'(\omega))$ for all ω . Then, for any $p \in \Delta(\Omega)$, $U_1(h, p) = U_1(h', p)$. Therefore, $N^+(\{h, h'\}, h) = N^+(\{h, h'\}, h') = \emptyset$. To ensure $N^+(x, h) \neq \emptyset$ for some $h \in x$, we focus on the domain

$$\mathcal{D}_F^+ \equiv \{x \in \mathcal{D}_F \mid \forall h, h' \in x, \exists \omega, u(h(\omega)) \neq u(h'(\omega))\}.$$

Say that a random SEU (μ, u) is *regular* if, for any $x \in \mathcal{D}_F^+$,

$$\mu \left(\bigcup_{h \in x} N^+(x, h) \right) = 1. \quad (10)$$

That is, regularity means that, for any $x \in \mathcal{D}_F^+$, $U_1(\cdot, p)$ almost surely has a unique maximizer among x .

Definition 2.2. A stochastic choice $\rho : \mathcal{D}_F^+ \rightarrow \Delta(\mathcal{H})$ is said to be generated by a regular random SEU (μ, u) if, for any $x \in \mathcal{D}_F$ and $h \in \mathcal{H}$,

$$\rho^x(h) = \mu(N(x, h)).$$

The following proposition says that, if we impose an additional assumption, regularity, on (μ, u) , it provides a meaningful stochastic prediction of the ex post choice. A proof can be found in Appendix B.3.

Proposition 2.1. Any regular random SEU (μ, u) generates a unique stochastic choice ρ .

What kind of random SEU satisfies regularity? Condition (10) says that the set

$$N(x, h) \setminus N^+(x, h) = \{p \in \Delta(\Omega) | U_1(h, p) = U_1(h', p), \text{ for some } h' \in x \text{ with } h' \neq h\} \quad (11)$$

has measure zero. Since $U_1(h, p)$ is mixture linear in p , (11) is the union of finite sets which have strictly smaller “dimension” than $\Delta(\Omega)$. Thus, if μ assigns measure zero to any set with smaller dimension, it should be regular.

To state the above argument formally, we introduce some notions. Since $\#\Omega = n < \infty$, $\Delta(\Omega)$ can be identified with the $(n - 1)$ -dimensional unit simplex in $\mathbb{R}^n$. For any $F \in \mathcal{B}(\Delta(\Omega))$, let $\dim F$ denote the dimension of the affine hull of F , that is, the smallest affine space in $\mathbb{R}^n$ including F .⁴ Say that a probability measure μ over $\Delta(\Omega)$ is *full-dimensional* if $\mu(F) = 0$ whenever $\dim F < n - 1$.

Proposition 2.2. *A random SEU (μ, u) is regular if and only if μ is full-dimensional.*

For example, the uniform distribution over $\Delta(\Omega)$ is full-dimensional. Any μ with the finite support is not full-dimensional.

Proposition 2.2 is the counterpart of Lemma 3 (p. 30) of Gul and Pesendorfer [6] in the setting of acts. To show this proposition, we adapt their argument. Details are relegated to Appendix B.4.

3 Updating of a Subjective Prior

3.1 Preference Conditional on Objective Information

Imagine the situation where, after choosing a menu in period 0, the DM receives information about objective states prior to choosing an act out of the menu. The objective information typically affects the DM’s rankings over menus and hence she may prefer another menu to the predetermined menu. The preference change according to objective information presumably reveals how the DM updates the initial prior over a subjective state space in response to the objective information. Since the objective information is observable also for the analyst, this updating rule is relevant for predicting subsequent choice of the DM.

To formulate the argument as outlined above, consider a set of preferences $\{\succeq_A\}_{A \in \mathcal{A}}$, where $\mathcal{A}$ is the set of all non-empty subsets of Ω , and $\succeq_A$ is a preference relation on $\mathcal{D} \equiv \mathcal{P}(\mathcal{H})$. We call $\succeq_A$ the conditional preference on an event A , which is interpreted as the rankings over menus when the DM receives the information telling that any states outside the event A never happen. For convenience, $\succeq_\Omega$ is denoted by $\succeq$, and called the ex ante preference.

What we have in mind is the following time line:

Period 0: choose a menu x

⁴Let $\mathcal{B}(\Delta(\Omega))$ denote the Borel σ -algebra on $\Delta(\Omega)$.

Period 0¹: receive objective information $A_1 \in \mathcal{A}$ and choose another menu x^1

Period 0²: receive another information $A_2 \subset A_1$ and choose another menu x^2

$\vdots$

Period 0^t: receive another information $A_t \subset A_{t-1}$ and choose another menu x^t

Period 1⁻: receive a subjective signal s

Period 1: choose an act $h \in x^t$

Period 1⁺: An objective state in A_t is realized and the DM receives the lottery prescribed by h .

3.2 Axioms on Conditional Preferences

We impose several axioms on conditional preferences $\{\succeq_A\}_{A \in \mathcal{A}}$. The first axiom is about the ex ante preference $\succeq$.

AXIOM 3.1 (Initial Prior): $\succeq$ satisfies Axiom 2.1-2.6.

Under this axiom, Theorem 2.1 ensures that the ex ante preference $\succeq$ has an additive SEU representation with (μ_Ω, u) . The probability measure μ_Ω over $\Delta(\Omega)$ is interpreted as an initial prior over $\Delta(\Omega)$.

For any $f, g \in \mathcal{H}$ and $A \in \mathcal{A}$, let $fAg \in \mathcal{H}$ be the act which coincides with f on A and with g elsewhere. Formally,

$$fAg \equiv \begin{bmatrix} f(\omega) & \text{if } \omega \in A \\ g(\omega) & \text{if } \omega \in \Omega \setminus A \end{bmatrix}.$$

Say that an event $A \in \mathcal{A}$ is *null* with respect to $\succeq$ if, for any $h, f, g \in \mathcal{H}$,

$$\{fAh\} \sim \{gAh\}.$$

This indifference relation says that the DM does not care about the outcomes contingent on states in the event A . As mentioned in Section 2.3, commitment rankings over $\mathcal{H}$ reflect the DM's initial belief over Ω . Thus, the above indifference suggests that the DM assigns zero probability to the event A . An event $A \in \mathcal{A}$ is *non-null* if A is not a null event.

The next two axioms are imposed on each conditional preference $\succeq_A$.

AXIOM 3.2 (Conditional Order): For any non-null event $A \in \mathcal{A}$, $\succeq_A$ is complete and transitive.

Suppose the DM is told that an event A will happen. Then, she should not care about the outcomes contingent upon states in the counterfactual event $\Omega \setminus A$. To capture this behavior, for any event $A \in \mathcal{A}$ and any act $f \in \mathcal{H}$, let f_A be the act restricted on A , that is, f_A is the function from A into $\Delta(Z)$ defined by $f_A(\omega) = f(\omega)$ for all $\omega \in A$.

AXIOM 3.3 (Consequentialism): For any non-null event $A \in \mathcal{A}$ and $x, y \in \mathcal{D}$,

$$\{f_A | f \in x\} = \{f_A | f \in y\} \Rightarrow x \sim_A y.$$

The next axiom prescribes relation between the ex ante preference and a conditional preference. For any $x \in \mathcal{D}$ and $g \in \mathcal{H}$, define the menu xAg by

$$xAg \equiv \{fAg | f \in x\}.$$

AXIOM 3.4 (Dynamic Consistency): For any non-null event $A \in \mathcal{A}$, $x, y \in \mathcal{D}$, and $f \in \mathcal{H}$,

$$xAf \succ_A yAf \Leftrightarrow xAf \succ yAf.$$

The ranking $xAf \succ_A yAf$ says that the DM strictly prefers xAf to yAf if she receives the objective information A . Since the DM should not care about the consequences contingent upon $\Omega \setminus A$, the ranking reveals that x is strictly preferred to y in terms of the information A . How should two menus xAf and yAf be ranked from the ex ante perspective? If the event A happens, xAf provides a strictly better menu than does yAf , while both menus ensure the same menu $\{f\}$ when $\Omega \setminus A$ happens. Since A is non-null, A is likely to happen from the ex ante perspective. This argument leads to the ranking $xAf \succ yAf$.

The other direction can be justified by taking into account the contrapositive of the statement. That is, “ $xAf \succ yAf \Rightarrow xAf \succ_A yAf$ for any x, y, f ” is equivalent to “ $xAf \succeq_A yAf \Rightarrow xAf \succeq yAf$ for any x, y, f ”. From the above argument, if $xAf \succ_A yAf$, then the ranking, $xAf \succ yAf$, is appealing. Similarly, we can argue that, if $xAf \sim_A yAf$, then xAf and yAf should be indifferent from the ex ante perspective, that is, $xAf \sim yAf$. Therefore, Dynamic Consistency is justifiable.

3.3 Updating Rule

The following theorem shows that the axioms in Section 3.2 pin down a unique updating rule of an initial prior over the subjective states in response to objective information. A proof appears in Appendix B.5.

Theorem 3.1. *Initial Prior, Conditional Order, Consequentialism, and Dynamic Consistency are equivalent to:*

- (i) *For any non-null event $A \in \mathcal{A}$, $\succeq_A$ admits the additive SEU representation (μ_A, u) such that the risk preference u is identical across $A \in \mathcal{A}$.*

(ii) μ_A coincides with the updated probability measure of the ex ante probability μ_Ω according to the following steps:

Step 1: Adjust μ_Ω to μ^* so as to reflect the “reliance” of each $p \in \Delta(\Omega)$ in terms of the event A . Precisely, μ^* satisfies

$$\int_{\Delta(\Omega)} \varphi(p) d\mu^*(p) = \int_{\Delta(\Omega)} \varphi(p)p(A) d\mu_\Omega(p), \quad (12)$$

for all continuous function $\varphi : \Delta(\Omega) \rightarrow \mathbb{R}$.

Step 2: Update each $p \in \Delta(\Omega)$ with $p(A) > 0$ by Bayes’ Rule, that is,

$$\beta_A(p) \equiv p(\cdot|A) \equiv \frac{p(\cdot \cap A)}{p(A)}, \quad (13)$$

and derive the non-negative measure ν_A over the conditional probability measures $p(\cdot|A)$ ’s from the mapping $\beta_A : (\Delta_A(\Omega), \mu^*) \rightarrow \Delta(A)$, where

$$\Delta_A(\Omega) \equiv \{p \in \Delta(\Omega) | p(A) > 0\}.$$

Step 3: Normalize ν_A to obtain a probability measure, that is,

$$\frac{\nu_A}{\nu_A(\Delta(A))}. \quad (14)$$

Then, the probability measure (14) exactly coincides with μ_A .

Figure 1 in Section 1.4 illustrates this updating rule when $\#\Omega = 3$ and μ_Ω has the finite support.

Is the updating rule in Theorem 3.1 consistent with the Bayesian model in any sense? Once $S \equiv \Delta(\Omega)$ is taken into account as a part of the state space, the updating rule in Theorem 3.1 is consistent with Bayes’ Rule. To state this claim formally, define $\mathcal{S} \equiv S \times \Omega$ as the “full” state space. As mentioned in Section 2.3, for any $\mu_0 \in \Delta(S)$, we can define an initial prior over the full state space $P \in \Delta(\mathcal{S})$ as the unique probability measure satisfying

$$P(B \times C) = \int_B p(C) d\mu_0(p),$$

for any $B \in \mathcal{B}(S)$ and $C \in \mathcal{B}(\Omega)$. Then, for any non-null event $A \in \mathcal{A}$, the probability measure conditional on the event $S \times A$ is derived by Bayes’ Rule

$$P(E|S \times A) = \frac{P(E \cap (S \times A))}{P(S \times A)}, \quad (15)$$

for any $E \in \mathcal{B}(S \times \Omega)$.

The following proposition provides a characterization of the updating rule in Theorem 3.1. A proof can be found in Appendix B.6.

Proposition 3.1. *For any non-null event $A \in \mathcal{A}$ and any $F \in \mathcal{B}(\Delta(A))$,*

$$\mu_A(F) = P(\beta_A^{-1}(F) \times A | S \times A). \quad (16)$$

The right-hand side of (16) is interpreted as follows: For any $B \in \mathcal{B}(S)$ and $C \in \mathcal{B}(\Omega)$,

$$P(B \times C | S \times A)$$

is the conditional probability of the event $B \times C$ with respect to the Bayesian updating $P(\cdot | S \times A)$ of P . Since any state outside A never happens, any $p, p' \in S$ can be identified as long as the conditional probability of p on A coincides with that of p' . Formally, the mapping $\beta_A : S \rightarrow \Delta(A)$, defined as (13), stands for this identification. Thus, for any $F \in \mathcal{B}(\Delta(A))$ and $C \in \mathcal{B}(\Omega)$,

$$P(\beta_A^{-1}(F) \times C | S \times A) \quad (17)$$

is the distribution of $P(\cdot | S \times A)$ on $\Delta(A) \times A$ induced by the mapping β_A . This step is regarded as refinement of the full state space. Finally,

$$P(\beta_A^{-1}(F) \times A | S \times A)$$

is the marginal distribution of (17) on $\Delta(A)$. Proposition 3.1 says that the probability measure obtained by the above steps coincides with μ_A . Thus, once S is regarded as a part of the state space, the updating rule in Theorem 3.1 is consistent with the standard Bayesian updating of an initial prior over the full state space $S \times \Omega$.

3.4 Updated Stochastic Choice

When objective information A arrives, as shown in Section 3.3, the DM updates her prior over the subjective states. Now we can update the stochastic prediction of the ex post choice according to the DM's updating rule.

Once the DM is told that A is the true event, she does not care about outcomes contingent upon objective states outside A . Hence, as long as $u(h(\omega)) = u(h'(\omega))$ for all $\omega \in A$, acts h and h' should be indifferent. Thus, we focus on the domain

$$\mathcal{D}_F^{+A} \equiv \{x \in \mathcal{D}_F | \forall h, h' \in x, \exists \omega \in A, u(h(\omega)) \neq u(h'(\omega))\}.$$

That is, under given information A , $\mathcal{D}_F^{+A}$ is the set of finite menus consisting of distinct acts in terms of payoffs.

The additive SEU representation U_A with (μ_A, u) suggests that the DM chooses an act out of the given menu so as to maximize the ex post subjective expected utility, which depends on the realization of the random factor μ_A . As pointed out in Section 2.5, there may exist more than two maximizers. To derive a unique stochastic choice without an ad hoc tie-breaking rule, we focus on random SEUs admitting a unique maximizer out of any

menu with probability one. For any $A \in \mathcal{A}$, say that a random SEU (μ, u) is *regular relative to A* if, for any $x \in \mathcal{D}_F^{+A}$,

$$\mu \left(\bigcup_{h \in x} N^+(x, h) \right) = 1. \quad (18)$$

A random SEU is not necessarily regular relative to A . The next theorem, however, ensures that μ_A derived from the updating rule in Theorem 3.1 is always regular relative to A as long as the prior μ is regular. A proof appears in Appendix B.7.

Theorem 3.2. *Assume that a random SEU (μ, u) is regular, and that μ_A is the probability measure derived by the steps in Theorem 3.1. Then, (μ_A, u) is regular relative to A .*

The following is an immediate consequence of Theorem 3.1 and Theorem 3.2:

Corollary 3.3. *Assume that $\{\succeq_A\}_{A \in \mathcal{A}}$ satisfies all the axioms in Theorem 3.1 and that (μ, u) representing $\succeq$ is regular. Then, for any non-null event $A \in \mathcal{A}$, we can derive a unique stochastic choice $\rho^A : \mathcal{D}_F^{+A} \rightarrow \Delta(\mathcal{H})$.*

4 Concluding Remarks

When trying to use DLR’s model for dynamic choice, we face an obstacle. Even if the analyst know the DM’s preference over menus and derive her subjective states, the realized subjective state is not observable for the analyst. Hence, we can not predict subsequent choice based on the ex post preference.

We have proposed introducing some objective states into the Kreps’s framework. There are two added benefits of this modification. First, unlike Kreps [8, 9] and DLR, we have derived a unique probability measure over the subjective state space. This result makes stochastic prediction of subsequent choice possible.

The second benefit is that the subjective states have correlation with the objective states. Hence, the analyst can infer the ex post probability over the subjective states from the realized objective information. We have addressed how the DM updates a subjective prior over the subjective states if additional information about the objective states arrives prior to the ex post stage, and have pinned down a unique updating rule consistent with dynamic consistency.

Two remarks are in order regarding remaining questions. We have derived stochastic choice associated with the additive SEU representation with components (μ, u) only when (μ, u) is regular, equivalently, μ is full-dimensional. Otherwise, stochastic choice may not be unique. We do not know behavioral conditions ensuring full-dimensionality of μ .

As mentioned in Section 2.5, Gul and Pesendorfer [6] characterize the condition under which there exists a “random expected utility” rationalizing a given stochastic choice over lotteries. Similarly, we can examine the same question by taking a stochastic choice over acts as a primitive. The answer to this question provides a revealed preference foundation of additive SEU representations.

A Hausdorff Topology

Let $d(h, x) \equiv \inf_{h' \in x} d(h, h')$ and $e(x', x) \equiv \sup_{h' \in x'} d(h', x)$. For each $x, y \in \mathcal{P}(\mathcal{H})$, define

$$d_h(x, y) \equiv \max[e(x, y), e(y, x)].$$

Then, d_h is a pseudo-metric. That is, d_h satisfies (i) $d_h(x, y) \geq 0$, (ii) $x = y$ implies $d_h(x, y) = 0$, (iii) $d_h(x, y) = d_h(y, x)$, and (iv) $d_h(x, z) \leq d_h(x, y) + d_h(y, z)$. The Hausdorff topology is the topology generated from ε -balls with respect to d_h .

B Proofs

B.1 Proof of Theorem 2.1

Necessity of the axioms is routine. We show sufficiency. The basic procedure of the proof is the same as in DLR. For each $x \in \mathcal{P}(\mathcal{H})$, let $\text{cl}(x)$ be the closure of x . Since $\mathcal{H}$ is compact, $\text{cl}(x)$ is a compact menu. As in DLR, Continuity implies $x \sim \text{cl}(x)$. Let $\text{co}(x)$ be the convex hull of x . Under Continuity and Independence, $x \sim \text{co}(x)$ for any $x \in \mathcal{D}$. Notice that $\text{co}(x)$ is closed whenever x is closed. Hence, we can restrict our attention to the sub-domain,

$$\mathcal{D}_1 \equiv \{x \in \mathcal{D} | x = \text{co}(\text{cl}(x))\},$$

that is, the set of convex and compact menus.

Recall

$$O(h) \equiv \{h' \in \mathcal{H} | \{h(\omega)\} \succeq \{h'(\omega)\} \text{ for all } \omega\},$$

and $O(x) \equiv \cup_{h \in x} O(h)$. This defines the operation $O : \mathcal{D} \rightarrow \mathcal{D}$.

Lemma B.1.

- (i) If $x \in \mathcal{D}_1$, $O(x) \in \mathcal{D}_1$.
- (ii) $O : \mathcal{D}_1 \rightarrow \mathcal{D}_1$ is well-defined.
- (iii) $O : \mathcal{D}_1 \rightarrow \mathcal{D}_1$ is Hausdorff continuous.
- (iv) $O : \mathcal{D}_1 \rightarrow \mathcal{D}_1$ is mixture linear.

Proof. (i) We want to show $O(x)$ is compact and convex whenever x is compact and convex.

Step 1: $O(x)$ is compact.

Since $\mathcal{D}_1$ is a compact metric space, it suffices to show that $O(x)$ is closed. Let $h^n \rightarrow h$ with $h^n \in O(x)$. Then there is a sequence $\{k^n\}_{n=0}^\infty$ in x satisfying $\{k^n(\omega)\} \succeq \{h^n(\omega)\}$ for all ω . Since $\Delta(Z)$ is compact, for each ω , the sequence $\{k^n(\omega)\}_{n=0}^\infty$ has a convergent subsequence $\{k^{n_i}(\omega)\}_{i=0}^\infty$ with a limit point $l_\omega \in \Delta(Z)$. Define $k^* \in \mathcal{H}$ by $k^*(\omega) \equiv l_\omega$. From finiteness of Ω , we can find a subsequence $\{k^m\}_{m=0}^\infty$ of $\{k^n\}_{n=0}^\infty$ satisfying $k^m \rightarrow k^*$. Notice that $k^* \in x$. Since $\{k^m(\omega)\} \succeq \{h^m(\omega)\}$ for all ω , $\{k^*(\omega)\} \succeq \{h(\omega)\}$ because of Continuity. Thus, $h \in O(k^*) \subset O(x)$.

Step 2: $O(x)$ is convex.

Take $h, h' \in O(x)$. Then there are $k, k' \in x$ such that $\{k(\omega)\} \succeq \{h(\omega)\}$ and $\{k'(\omega)\} \succeq \{h'(\omega)\}$ for all ω . Since x is convex, $\lambda k + (1 - \lambda)k' \in x$ for any $\lambda \in [0, 1]$. From Order and Independence,

$$\lambda\{k(\omega)\} + (1 - \lambda)\{k'(\omega)\} \succeq \lambda\{h(\omega)\} + (1 - \lambda)\{h'(\omega)\},$$

for all ω , which is equivalent to

$$\{\lambda k(\omega) + (1 - \lambda)k'(\omega)\} \succeq \{\lambda h(\omega) + (1 - \lambda)h'(\omega)\}.$$

Hence, $\lambda h + (1 - \lambda)h' \in O(x)$.

(ii) This is a direct consequence of (i).

(iii) Let $x^n \rightarrow x$. We want to show $O(x^n) \rightarrow O(x)$. Since $\mathcal{D}_1$ is compact, the sequence $\{O(x^n)\}_{n=1}^\infty$ has a convergent subsequence $\{O(x^{n_m})\}_{m=1}^\infty$ with a limit $y \in \mathcal{D}_1$.

Step 1: $O(x) \subset y$.

Suppose otherwise. Then, there is $h \in O(x) \setminus y$. Since y is a compact subset of $\mathcal{H}$, there is an open neighborhood of h , $B(h) \subset \mathcal{H}$, satisfying $B(h) \cap y = \emptyset$. From the definition of Hausdorff metric, $B(h) \cap O(x^n) = \emptyset$ for all sufficiently large n . On the other hand, since $h \in O(x)$, there exists $\bar{h} \in x$ such that $\{\bar{h}(\omega)\} \succeq \{h(\omega)\}$ for all ω . Since $x^m \rightarrow x$, we can find a sequence $\{\bar{h}^m\}_{m=1}^\infty$ in $\mathcal{H}$ satisfying $\bar{h}^m \in x^m$ and $\bar{h}^m \rightarrow \bar{h}$ in the sense of the metric on $\mathcal{H}$, equivalently, for all ω , $\bar{h}^m(\omega) \rightarrow \bar{h}(\omega)$ in the sense of the metric on $\Delta(Z)$. Now we are going to construct a sequence $\{h^m\}_{m=1}^\infty$ in $\mathcal{H}$ with $h^m \in O(x^m)$ satisfying $h^m \rightarrow h$. Fix an arbitrary ω . There are two cases; (1) $\{\bar{h}(\omega)\} \succ \{h(\omega)\}$, and (2) $\{\bar{h}(\omega)\} \sim \{h(\omega)\}$. If case (1) holds, from Continuity, $\{\bar{h}^m(\omega)\} \succ \{h(\omega)\}$ for all sufficiently large m . Hence, define $h^m(\omega) \equiv h(\omega)$ for all sufficiently large m , and $h^m(\omega)$ can be taken to be an arbitrary point, otherwise. If case (2) holds, define $h^m(\omega) \equiv h(\omega)$ as long as $\bar{h}^m(\omega) \succeq \bar{h}(\omega) \sim h(\omega)$. Otherwise, let $k \geq 1$ be the first natural number satisfying $\bar{h}(\omega) \sim h(\omega) \succ \bar{h}^k(\omega)$. Let $l_\omega(\lambda) \equiv \lambda \bar{h}^k(\omega) + (1 - \lambda)h(\omega)$. Continuity ensures that there is λ^m such that $l_\omega(\lambda^m) \sim \bar{h}^m(\omega)$. Define $h^m(\omega) \equiv l_\omega(\lambda^m)$. From case (1) and (2), we now have a sequence $\{h^m\}_{m=1}^\infty$ such that $h^m \rightarrow h$ and $\{\bar{h}^m(\omega)\} \succeq \{h^m(\omega)\}$ for all ω , that is, $h^m \in O(x^m)$. This contradicts the fact that $B(h) \cap O(x^m) = \emptyset$ for all m sufficiently large.

Step 2: $y \subset O(x)$.

Suppose otherwise. Take $h \in y \setminus O(x)$. Since $O(x)$ is compact as long as x is compact, there exists an open neighborhood of h , $B(h)$, such that $B(h) \cap O(x) = \emptyset$. Then, we can find a sequence $h^n \in O(x^n)$ with $h^n \rightarrow h$ in the sense of the metric on $\mathcal{H}$. By definition of $O(x^n)$, there is $\bar{h}^n \in x^n$ such that $\{\bar{h}^n(\omega)\} \succeq \{h^n(\omega)\}$ for all ω . Since $\mathcal{H}$ is compact, we can assume $\{\bar{h}^n\}$ converges to a limit $h^* \in \mathcal{H}$ without loss of generality. Since $\bar{h}^n \rightarrow h^*$ and $x^n \rightarrow x$ with $\bar{h}^n \in x^n$, $h^* \in x$. From Continuity, $\{h^*(\omega)\} \succeq \{h(\omega)\}$ for all ω . Thus, $h \in O(x)$. This is a contradiction.

From Step 1 and 2, we have $O(x) = y$. That is, O is Hausdorff-continuous.

(iv) We want to show that, for any $x, x' \in \mathcal{D}_1$ and $\lambda \in [0, 1]$,

$$\lambda O(x) + (1 - \lambda)O(x') = O(\lambda x + (1 - \lambda)x').$$

Step 1: $\lambda O(x) + (1 - \lambda)O(x') \subset O(\lambda x + (1 - \lambda)x')$.

Take any $h'' \in \lambda O(x) + (1 - \lambda)O(x')$. Then there exist $h \in O(x)$ and $h' \in O(x')$ satisfying $h'' = \lambda h + (1 - \lambda)h'$. By definition of $O(x)$ and $O(x')$, there are $\bar{h} \in x$ and $\bar{h}' \in x'$ such that

$$\{\bar{h}(\omega)\} \succeq \{h(\omega)\}, \text{ and } \{\bar{h}'(\omega)\} \succeq \{h'(\omega)\},$$

for all ω . Consider $\lambda\bar{h} + (1 - \lambda)\bar{h}' \in \lambda x + (1 - \lambda)x'$. By Independence,

$$\begin{aligned} \{\lambda\bar{h}(\omega) + (1 - \lambda)\bar{h}'(\omega)\} &= \lambda\{\bar{h}(\omega)\} + (1 - \lambda)\{\bar{h}'(\omega)\} \\ &\succeq \lambda\{h(\omega)\} + (1 - \lambda)\{h'(\omega)\} \\ &= \{\lambda h(\omega) + (1 - \lambda)h'(\omega)\}. \end{aligned}$$

Thus, $h'' \in O(\lambda x + (1 - \lambda)x')$.

Step 2: $O(\lambda x + (1 - \lambda)x') \subset \lambda O(x) + (1 - \lambda)O(x')$.

Take any $h'' \in O(\lambda x + (1 - \lambda)x')$. There are $\bar{h} \in x$ and $\bar{h}' \in x'$ satisfying $\{\lambda\bar{h}(\omega) + (1 - \lambda)\bar{h}'(\omega)\} \succeq \{h''(\omega)\}$ for all ω . We are going to find $h \in O(x)$ and $h' \in O(x')$ satisfying $h'' = \lambda h + (1 - \lambda)h'$. Consider an arbitrarily fixed ω . Assume first that $\{\bar{h}(\omega)\} \succeq \{\bar{h}'(\omega)\}$. By Independence,

$$\{\bar{h}(\omega)\} \succeq \{\lambda\bar{h}(\omega) + (1 - \lambda)\bar{h}'(\omega)\} \succeq \{\bar{h}'(\omega)\}.$$

We have the following two cases: (1) $\{\bar{h}'(\omega)\} \succeq \{h''(\omega)\}$; and (2) $\{\bar{h}(\omega)\} \succeq \{h''(\omega)\} \succ \{\bar{h}'(\omega)\}$.

If case (1) holds, define $h(\omega) = h'(\omega) \equiv h''(\omega)$. Since $\{\bar{h}(\omega)\} \succeq \{\bar{h}'(\omega)\}$,

$$\{\bar{h}(\omega)\} \succeq \{h(\omega)\} \text{ and } \{\bar{h}'(\omega)\} \succeq \{h'(\omega)\}.$$

Furthermore, $h''(\omega) = \lambda h(\omega) + (1 - \lambda)h'(\omega)$.

If case (2) holds, take two lotteries l_ω and l'_ω such that $l_\omega \sim \bar{h}(\omega)$, $l'_\omega \sim \bar{h}'(\omega)$, and $h''(\omega) = \alpha l_\omega + (1 - \alpha)l'_\omega$ for some $\alpha \in (0, 1]$. From Independence, $\lambda \geq \alpha$. Define

$$h(\omega) \equiv \frac{\alpha}{\lambda} l_\omega + \left(1 - \frac{\alpha}{\lambda}\right) l'_\omega, \text{ and } h'(\omega) \equiv l'_\omega.$$

Then we have $\{\bar{h}(\omega)\} \succeq \{h(\omega)\}$, $\{\bar{h}'(\omega)\} \succeq \{h'(\omega)\}$, and $h''(\omega) = \lambda h(\omega) + (1 - \lambda)h'(\omega)$.

Consider next the case where $\{\bar{h}'(\omega)\} \succeq \{\bar{h}(\omega)\}$. Independence implies

$$\{\bar{h}'(\omega)\} \succeq \{\lambda\bar{h}(\omega) + (1 - \lambda)\bar{h}'(\omega)\} \succeq \{\bar{h}(\omega)\}.$$

We have the following two cases: (1) $\{\bar{h}(\omega)\} \succeq \{h''(\omega)\}$; and (2) $\{\bar{h}'(\omega)\} \succeq \{h''(\omega)\} \succ \{\bar{h}(\omega)\}$.

In case (1), define $h(\omega) = h'(\omega) \equiv h''(\omega)$. Since $\{\bar{h}'(\omega)\} \succeq \{\bar{h}(\omega)\}$,

$$\{\bar{h}(\omega)\} \succeq \{h(\omega)\} \text{ and } \{\bar{h}'(\omega)\} \succeq \{h'(\omega)\}.$$

Furthermore, $h''(\omega) = \lambda h(\omega) + (1 - \lambda)h'(\omega)$.

In case (2), take two lotteries l_ω and l'_ω such that $l_\omega \sim \bar{h}(\omega)$, $l'_\omega \sim \bar{h}'(\omega)$, and $h''(\omega) = \alpha l_\omega + (1 - \alpha)l'_\omega$ for some $\alpha \in [0, 1)$. From Independence, $\alpha \geq \lambda$. Define

$$h(\omega) \equiv l_\omega, \text{ and } h'(\omega) \equiv \frac{\alpha - \lambda}{1 - \lambda} l_\omega + \left(1 - \frac{\alpha - \lambda}{1 - \lambda}\right) l'_\omega.$$

Then we have $\{\bar{h}(\omega)\} \succeq \{h(\omega)\}$, $\{\bar{h}'(\omega)\} \succeq \{h'(\omega)\}$, and $h''(\omega) = \lambda h(\omega) + (1 - \lambda)h'(\omega)$.

Let h and h' be the acts defined as above. By construction, $h \in O(\bar{h}) \subset O(x)$, $h' \in O(\bar{h}') \subset O(x')$, and $h'' = \lambda h + (1 - \lambda)h'$. Therefore, $h'' \in \lambda O(x) + (1 - \lambda)O(x')$. $\square$

From Lemma B.1 (i), it is enough to consider the sub-domain

$$\mathcal{D}_2 \equiv \{x \in \mathcal{D}_1 | x = O(x)\}.$$

Since $O : \mathcal{D}_1 \rightarrow \mathcal{D}_1$ is Hausdorff continuous and mixture linear from Lemma B.1 (iii) and (iv), $\mathcal{D}_2$ is compact and convex.

First of all, Order, Continuity and Independence ensure a mixture linear representation $U : \mathcal{D}_1 \rightarrow \mathbb{R}$ of $\succeq$ because $\mathcal{D}_1$ is a mixture space. Let $u : \Delta(Z) \rightarrow \mathbb{R}$ be the restriction of U on $\Delta(Z)$, that is, $u(l) \equiv U(\{l\})$. Since $\Delta(Z)$ is compact, there exist a maximal element $\bar{l}$ and a minimal element $\underline{l}$ with respect to u . From Strong Nondegeneracy, $u(\bar{l}) > u(\underline{l})$. Without loss of generality, we can assume $u(\bar{l}) = 1$ and $u(\underline{l}) = 0$.

Let $S \equiv \Delta(\Omega)$. Since $\#\Omega = n$, S is identified with the $(n - 1)$ -dimensional unit simplex. Let $\mathcal{C}(S)$ be the set of real-valued continuous functions on S with the supnorm metric. For each $x \in \mathcal{D}_2$ and $p \in S$, let

$$\sigma_x(p) \equiv \max_{h \in x} \sum_{\omega \in \Omega} u(h(\omega))p(\omega).$$

This defines the function $\sigma : \mathcal{D}_2 \rightarrow \mathcal{C}(S)$.

Lemma B.2.

(i) σ is continuous.

(ii) σ is mixture linear.

(iii) σ is injective.

Proof. (i) Define

$$u(x) \equiv \{(u(h(\omega)))_{\omega \in \Omega} \in \mathbb{R}^n | h \in x\}.$$

Since $u : \Delta(Z) \rightarrow [0, 1]$ is continuous and mixture linear, $u(x) \subset [0, 1]^n$ is a compact and convex set. Let $\mathcal{K}([0, 1]^n)$ be the set of non-empty compact subsets of $[0, 1]^n$ with the Hausdorff metric.

Step 1: The map $\psi : \mathcal{D} \ni x \mapsto u(x) \in \mathcal{K}([0, 1]^n)$ is Hausdorff continuous.

Take a sequence $x^n \rightarrow x$ with $x^n, x \in \mathcal{D}$. Since $\mathcal{K}([0, 1]^n)$ is compact, without loss of generality, we can assume $\{\psi(x^n)\}_{n=1}^\infty$ converges to a limit $w \in \mathcal{P}([0, 1]^n)$. We want to show $\psi(x) = w$.

Suppose $\psi(x) \not\subset w$. Then, there exists $\bar{u} \in \psi(x) \setminus w$. Since w is compact, there exists an open neighborhood of $\bar{u}$, $V(\bar{u})$, separating $\bar{u}$ and w . There exists $\bar{h} \in x$ such that $\bar{u} = (u(h(\omega)))_\omega$. Since $x^n \rightarrow x$, we can find $\{h^n\}_{n=1}^\infty$ such that $h^n \rightarrow \bar{h}$ with $h^n \in x^n$. Let $u^n \equiv (u(h^n(\omega)))_\omega \in \psi(x^n)$. Since $u^n \rightarrow \bar{u}$, $u^n \in V(\bar{u})$ for all n sufficiently large. This contradicts that $\psi(x^n) \rightarrow w$. Thus, $\psi(x) \subset w$.

For the other direction, take any $\bar{u} \in w$. Since $\psi(x^n) \rightarrow w$, we can find $\{u^n\}_{n=1}^\infty$ such that $u^n \rightarrow \bar{u}$ with $u^n \in \psi(x^n)$. There exists $h^n \in x^n$ satisfying $u^n = (u(h^n(\omega)))_\omega$. Since $\mathcal{H}$ is compact, without loss of generality, assume $h^n \rightarrow \bar{h}$. Thus, $u^n \rightarrow u(\bar{h}) = \bar{u}$. Since $h^n \rightarrow \bar{h}$, $x^n \rightarrow x$ and $h^n \in x^n$, $\bar{h} \in x$. We have $\bar{u} = u(\bar{h}) \in \psi(x)$. Hence, $w \subset \psi(x)$.

Step 2: $d_{\text{supnorm}}(\sigma_x, \sigma_y) \leq d_{\text{Hausdorff}}(u(x), u(y))$.

For any $p \in S$, by definition,

$$\begin{aligned} |\sigma_x(p) - \sigma_y(p)| &= \left| \max_{h \in x} \sum_{\omega} u(h(\omega))p(\omega) - \max_{h \in y} \sum_{\omega} u(h(\omega))p(\omega) \right| \\ &= \left| \max_{u \in u(x)} u \cdot p - \max_{u \in u(y)} u \cdot p \right|. \end{aligned}$$

Let $u^{px} \in u(x)$ and $u^{py} \in u(y)$ be maximizers for the maximization problems, respectively. Without loss of generality, assume $u^{px} \cdot p \geq u^{py} \cdot p$. Let H^{py} be the hyperplane $u \cdot p = u^{py} \cdot p$ and $u^* \in H^{py}$ be a point such that $u^* \in \operatorname{argmin}_{u \in H^{py}} \|u - u^{px}\|$. Then, by the Schwarz inequality,

$$\begin{aligned} \left| \max_{u \in u(x)} u \cdot p - \max_{u \in u(y)} u \cdot p \right| &= |u^{px} \cdot p - u^{py} \cdot p| \\ &= |u^{px} \cdot p - u^* \cdot p| \\ &= |(u^{px} - u^*) \cdot p| \\ &\leq \|u^{px} - u^*\| \|p\| \\ &\leq \|u^{px} - u^*\| \\ &\leq \min_{u \in u(y)} \|u^{px} - u\| \\ &\leq d_{\text{Hausdorff}}(u(x), u(y)). \end{aligned}$$

Since this inequality holds for all p , $d_{\text{supnorm}}(\sigma_x, \sigma_y) \leq d_{\text{Hausdorff}}(u(x), u(y))$.

From Step 1 and 2, σ is continuous.

(ii) We want to show $\sigma_{\lambda x + (1-\lambda)y} = \lambda \sigma_x + (1-\lambda) \sigma_y$. Fix $p \in S$ arbitrarily. Since u is mixture linear,

$$\begin{aligned} \sigma_{\lambda x + (1-\lambda)y}(p) &= \max_{h \in \lambda x + (1-\lambda)y} \sum_{\omega} u(h(\omega))p(\omega) \\ &= \lambda \max_{h \in x} \sum_{\omega} u(h(\omega))p(\omega) + (1-\lambda) \max_{h \in y} \sum_{\omega} u(h(\omega))p(\omega) \\ &= \lambda \sigma_x(p) + (1-\lambda) \sigma_y(p). \end{aligned}$$

(iii) Take any $x, x' \in D$ with $x \neq x'$. Then, there is $\bar{h} \in x' \setminus x$. Let $\bar{u} \equiv (u(\bar{h}(\omega)))_{\omega \in \Omega} \in \mathbb{R}^n$ and $u(x) \equiv \{(u(h(\omega)))_{\omega \in \Omega} \in \mathbb{R}^n | h \in x\}$. Since u is continuous and linear, $u(x)$ is a compact and convex set in $\mathbb{R}^n$.

Suppose that $(u(x) - \bar{u}) \cap \mathbb{R}_+^n \neq \emptyset$. Then, there exists $\hat{h} \in x$ such that $u(\hat{h}(\omega)) - u(\bar{h}(\omega)) \geq 0$ for all ω . By Risk Preference Certainty, $\bar{h} \in O(x)$. This contradicts the fact that $\bar{h} \notin x = O(x)$. Hence, $(u(x) - \bar{u}) \cap \mathbb{R}_+^n = \emptyset$. By the separating hyperplane theorem, there exist $\bar{p} \in \Delta^{n-1}$ and a constant $c \in \mathbb{R}$ such that, for all $h \in x$,

$$\sum_{\omega \in \Omega} u(\bar{h}(\omega))\bar{p}(\omega) > c > \sum_{\omega \in \Omega} u(h(\omega))\bar{p}(\omega).$$

Hence, we have

$$\sigma_{x'}(\bar{p}) = \max_{h \in x'} \sum_{\omega \in \Omega} u(h(\omega))\bar{p}(\omega) > \max_{h \in x} \sum_{\omega \in \Omega} u(h(\omega))\bar{p}(\omega) = \sigma_x(\bar{p}).$$

Therefore, σ is injective. □

Let $C \subset \mathcal{C}(S)$ be the range of σ . By Lemma B.2, $\sigma : \mathcal{D} \rightarrow C$ is homeomorphic.

Lemma B.3.

- (i) C is convex.
- (ii) The zero function is in C .
- (iii) The unit function is in C .
- (iv) The supremum of any two points $\sigma, \sigma' \in C$ is also in C . That is, $\max[\sigma(p), \sigma'(p)]$ is also in C .
- (v) For all $f \in C$, $f \geq 0$.

Proof. (i) Since $\mathcal{D}_2$ is convex and σ is mixture linear, C is convex.

(ii) Let $x \equiv O(\{\underline{l}\}) \in \mathcal{D}_2$. Since $u(\underline{l}) = 0$, $\sigma_x(p) = 0$ for all p .

(iii) Let $x \equiv O(\{\bar{l}\}) \in \mathcal{D}_2$. Since $u(\bar{l}) = 1$, $\sigma_x(p) = 1$ for all p .

(iv) There exist $x', x \in \mathcal{D}_2$ such that $\sigma = \sigma_x$ and $\sigma' = \sigma_{x'}$. Let $\sigma'' \equiv \sigma_{O(\text{co}(x \cup x'))} \in C$. Then, $\sigma''(p) = \max[\sigma_x(p), \sigma_{x'}(p)]$.

(v) There exists $x \in \mathcal{D}_2$ such that $f = \sigma_x$. Since $O(\{\underline{l}\}) \subset x$, $f(p) = \sigma_x(p) \geq \sigma_{O(\{\underline{l}\})}(p) = 0$ for any p . $\square$

Define $W : C \rightarrow \mathbb{R}$ by $W(f) \equiv U(\sigma^{-1}(f))$. Notice that $W(0) = 0$ and $W(1) = 1$, where 0 and 1 are identified with the zero function and with the unit function, respectively. Since U and σ are continuous and mixture linear, so is W .

Lemma B.4. $W(\alpha f + \beta f') = \alpha W(f) + \beta W(f')$ as long as $f, f', \alpha f + \beta f' \in C$, where $\alpha, \beta \in \mathbb{R}_+$.

Proof. For any $\alpha \in [0, 1]$,

$$W(\alpha f) = W(\alpha f + (1 - \alpha)0) = \alpha W(f) + (1 - \alpha)W(0) = \alpha W(f),$$

where 0 is the zero function. For any $\alpha > 1$, let $f'' \equiv \alpha f$. Since

$$W\left(\frac{1}{\alpha}f''\right) = \frac{1}{\alpha}W(f''),$$

$\alpha W(f) = W(\alpha f)$. Finally,

$$W(f + f') = 2W\left(\frac{1}{2}f + \frac{1}{2}f'\right) = W(f) + W(f').$$

This completes the proof. $\square$

By the same argument as in DLR, we will extend $W : C \rightarrow \mathbb{R}$ on $\mathcal{C}(S)$. For any $r \geq 0$, let $rC \equiv \{rf | f \in C\}$. Let $H \equiv \cup_{r \geq 0} rC$ and

$$H^* \equiv H - H = \{f_1 - f_2 \in \mathcal{C}(S) | f_1, f_2 \in H\}.$$

For any $f \in H \setminus 0$, there is $r > 0$ satisfying $(1/r)f \in C$. Define $W(f) \equiv rW((1/r)f)$. From linearity of W on C , $W(f)$ is well-defined. That is, even if there is another $r' > 0$ satisfying $(1/r')f \in C$, $rW((1/r)f) = r'W((1/r')f)$. It is easy to see that W on H is mixture linear. By the same argument as in Lemma B.4, W is also linear.

For any $f \in H^*$, there are $f_1, f_2 \in H$ satisfying $f = f_1 - f_2$. Define $W(f) \equiv W(f_1) - W(f_2)$. We can verify $W : H^* \rightarrow \mathbb{R}$ is well-defined. Indeed, suppose that f_1, f_2, f_3 and f_4 in H satisfy $f = f_1 - f_2 = f_3 - f_4$. Since $f_1 + f_4 = f_2 + f_3$, $W(f_1) + W(f_4) = W(f_2) + W(f_3)$ by linearity of W on H .

Lemma B.5. H^* is dense in $\mathcal{C}(S)$.

Proof. From the Stone-Weierstrass theorem, it is enough to show that (i) H^* is a vector sublattice, (ii) H^* separates the points of S , that is, for any two points $p, p' \in S$, there is $f \in H^*$ such that $f(p) > f(p')$, and (iii) H^* contains the constant functions equal to one. By the exactly same argument as Lemma 11 (p.928) in DLR, (i) holds. In order to show (ii), take $p, p' \in S$ with $p \neq p'$. There is $\omega \in \Omega$ such that $p(\omega) > p'(\omega)$. Define $h \in \mathcal{H}$ by $h(\omega) = \bar{l}$ and $h(\omega') = \underline{l}$ otherwise. Then,

$$\sigma_{O(\{h\})}(p) = p(\omega) > p'(\omega) = \sigma_{O(\{h\})}(p').$$

Since $\sigma_{O(\{h\})} \in C$, (ii) holds. Finally, (iii) directly follows from Lemma B.3 (iii) and the definition of H . $\square$

Since $\mathcal{D}_2$ is compact, by the same argument as Lemma 12 (p.929) in DLR, it can be shown that there is a constant $K > 0$ such that $W(f) \leq K\|f\|$ for any $f \in H^*$. By the Hahn-Banach theorem, we can extend $W : H^* \rightarrow \mathbb{R}$ to $\bar{W} : \mathcal{C}(S) \rightarrow \mathbb{R}$ in a linear, continuous and increasing way. Since H^* is dense by Lemma B.5, this extension is unique.

Now we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{D}_2 & \xrightarrow{U} & \mathbb{R} \\ \sigma \downarrow & \nearrow \bar{W} & \\ \mathcal{C}(S) & & \end{array}$$

Since $\bar{W}$ is a positive linear functional on $\mathcal{C}(S)$, the Riesz representation theorem ensures that there exists a unique bounded countably additive non-negative measure μ_0 on S satisfying

$$\bar{W}(f) = \int_S f(p) d\mu_0(p),$$

for all $f \in \mathcal{C}(S)$. By normalization, μ_0 can be taken to be a probability measure. Thus, we have

$$U(x) = \bar{W}(\sigma(x)) = \int_S \max_{h \in x} \left(\sum_{\omega \in \Omega} u(h(\omega)) p(\omega) \right) d\mu_0(p).$$

Redefine S as the support of μ_0 . Define $\mu_1 : S \rightarrow \Delta(\Omega)$ as the identity mapping, that is, $\mu_1(p) = p$. Then, (S, μ_0, μ_1, u) is a second-order additive SEU representation.

B.2 Proof of Theorem 2.2

(i) Since u and u' are mixture linear representations of the commitment rankings over $\Delta(Z)$, that is, $\{l\} \succeq \{l'\}$, they are cardinally equivalent by the standard argument.

(ii) As shown above, u and u' are cardinally equivalent. Thus, (μ', u) also represents the same preference. Let U and U' be the canonical representations associated with (μ, u) and (μ', u) , respectively. For all $x \in \mathcal{P}(\mathcal{H})$ and $p \in \Delta(\Omega)$, let

$$\sigma_x(p) \equiv \sup_{h \in x} U_1(h, p),$$

where

$$U_1(h, p) \equiv \sum_{\omega} u(h(\omega))p(\omega).$$

Then,

$$U(x) = \int \sigma_x(p) d\mu, \quad U'(x) = \int \sigma_x(p) d\mu'.$$

Since U and U' are mixture linear functions over $\mathcal{K}(\mathcal{H})$, that is, the set of all compact menus of $\mathcal{H}$, representing the same preference, there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $U' = \alpha U + \beta$. For any lottery l ,

$$\begin{aligned} U'(\{l\}) &= \alpha U(\{l\}) + \beta \\ u(l) &= \alpha u(l) + \beta. \end{aligned}$$

Since we must have $\alpha = 1$ and $\beta = 0$,

$$\int \sigma_x(p) d\mu = \int \sigma_x(p) d\mu', \tag{19}$$

for all x . If x is convex, σ_x is the supporting function associated with x . Equation (19) holds even when σ_x is replaced with $\alpha\sigma_x - \beta\sigma_y$ for any convex menus x, y and $\alpha, \beta \geq 0$. From Lemma B.5, the set of all such functions is a dense subset of the set of real-valued continuous functions over $\Delta(\Omega)$. Hence, equation (19) still holds even if σ_x is replaced with any real-valued continuous function. Hence, the Riesz representation theorem implies $\mu = \mu'$.

B.3 Proof of Proposition 2.1

It suffices to verify that ρ^x , satisfying $\rho^x(h) = \mu(N(x, h))$, is a well-defined probability measure over $\mathcal{H}$.

Lemma B.6. $\mu(N^+(x, h)) = \mu(N(x, h))$.

Proof. Let $F(x, h) \equiv N(x, h) \setminus N^+(x, h)$. By definition, for any $p \in F(x, h)$, there exists $h' \in x$ with $h' \neq h$ such that $U_1(h, p) = U_1(h', p)$. Thus, $\cup_{h \in x} N^+(x, h)$ and $\cup_{h \in x} F(x, h)$ are mutually exclusive. Regularity implies

$$\begin{aligned} \mu(\cup_{h \in x} N^+(x, h) \cup \cup_{h \in x} F(x, h)) &= \mu(\cup_{h \in x} N^+(x, h)) + \mu(\cup_{h \in x} F(x, h)) \\ &= 1 + \mu(\cup_{h \in x} F(x, h)). \end{aligned}$$

We must have $\mu(\cup_{h \in x} F(x, h)) = 0$. Since $F(x, h) \subset \cup_{h \in x} F(x, h)$, $\mu(F(x, h)) = 0$ for any $h \in x$. Therefore, $\mu(N(x, h)) = \mu(N^+(x, h)) + \mu(F(x, h)) = \mu(N^+(x, h))$. $\square$

Since $N^+(x, h)$ and $N^+(x, h')$ are exclusive, Lemma B.6 implies

$$\sum_{h \in x} \rho^x(h) = \sum_{h \in x} \mu(N(x, h)) = \sum_{h \in x} \mu(N^+(x, h)) = \mu\left(\bigcup_{h \in x} N^+(x, h)\right) = 1.$$

Hence, ρ^x is actually a probability measure over $\mathcal{H}$ whenever μ is regular.

B.4 Proof of Proposition 2.2

Notice first that $N(x, h)$ and $N^+(x, h)$ are invariant up to affine transformation of u . Thus, without loss of generality, we can assume $u(\bar{l}) = 1$ and $u(\underline{l}) = 0$, where $\bar{l}$ and $\underline{l}$ are respectively a maximal element and a minimal element in terms of $\succeq$ on $\Delta(Z)$. We can consider $[0, 1]^n$ instead of $\mathcal{H}$ by using the continuous, mixture linear and onto mapping $\varphi : \mathcal{H} \rightarrow [0, 1]^n$ defined by $\varphi(h) = (u(h(\omega_1)), \dots, u(h(\omega_n)))$. By definition, $\varphi(h) \neq \varphi(h')$ for any $h, h' \in x$ and any $x \in \mathcal{D}_F^+$. Then,

$$\begin{aligned} N(x, h) &= \{p \in \Delta(\Omega) | \varphi(h) \cdot p \geq \varphi(h') \cdot p, \text{ for all } h' \in x\}, \\ N^+(x, h) &= \{p \in \Delta(\Omega) | \varphi(h) \cdot p > \varphi(h') \cdot p, \text{ for all } h' \in x \text{ with } h' \neq h\}, \end{aligned}$$

where ‘ $\cdot$ ’ means the inner product.

(if part) Assume μ is full-dimensional.

Lemma B.7. $\Delta(\Omega) = \cup_{h \in x} N^+(x, h) \cup F$, where F is the union of finite polyhedrons of dimension less than $n - 1$, and F and $\cup_{h \in x} N^+(x, h)$ have no intersection.

Proof. We know $\Delta(\Omega) = \cup_{h \in x} N(x, h)$. Let $F(x, h) \equiv N(x, h) \setminus N^+(x, h)$. Then, $\Delta(\Omega) = \cup_{h \in x} N^+(x, h) \cup \cup_{h \in x} F(x, h)$. By definition, $\cup_{h \in x} N^+(x, h)$ and $\cup_{h \in x} F(x, h)$ are mutually exclusive. Notice that

$$F(x, h) = \cup_{h' \in x \setminus \{h\}} \{p \in \Delta(\Omega) | \varphi(h) \cdot p = \varphi(h') \cdot p\}.$$

By definition of $\mathcal{D}_F^+$, $\varphi(h) \neq \varphi(h')$ whenever h and h' are elements of the same menu. Since $\{p \in \Delta(\Omega) | \varphi(h) \cdot p = \varphi(h') \cdot p\}$ is the intersection of the $(n - 1)$ -dimensional unit simplex and the hyperplane $(\varphi(h) - \varphi(h')) \cdot p = 0$ with non-zero normal vector, it must be a polyhedron with dimension less than $n - 1$. Hence, $F \equiv \cup_{h \in x} F(x, h)$ satisfies the requirement. $\square$

If μ is full-dimensional, $\mu(F) = 0$. Thus,

$$1 = \mu(\Delta(\Omega)) = \mu(\cup_{h \in x} N^+(x, h)) + \mu(F) = \mu(\cup_{h \in x} N^+(x, h)),$$

and hence (μ, u) is regular.

(only-if part) Assume (μ, u) is regular. Take any $F \in \mathcal{B}(\Delta(\Omega))$ with $\dim F < n - 1$. We want to show $\mu(F) = 0$. Since F has dimension less than $n - 1$, there exists a hyperplane $v' \cdot p = c$ with a normal vector $v' \in \mathbb{R}^n \setminus \{0\}$ and $c \in \mathbb{R}$ such that $F \subset \{p \in \Delta(\Omega) | v' \cdot p = c\}$. Since $\{p \in \Delta(\Omega) | v' \cdot p = c\}$ has dimension less than $n - 1$, the affine hull of $\{0\} \cup \{p \in \Delta(\Omega) | v' \cdot p = c\}$ is a hyperplane of $\mathbb{R}^n$. Hence, there exists a normal vector $v \neq 0$ such that $\{q \in \mathbb{R}^n | v \cdot q = 0\}$ coincides with the above affine hull. There exist $v^1, v^2 \in \mathbb{R}_+^n$ such that v can be rewritten as $v = v^1 - v^2$. Since $\|v\|$ can be taken to be small, we can assume $v^1, v^2 \in [0, 1]^n$. Ontoness of φ implies that there exists $h^1, h^2 \in \mathcal{H}$ satisfying $\varphi(h^1) = v^1$ and $\varphi(h^2) = v^2$. Let $x \equiv \{h^1, h^2\} \in \mathcal{D}_F^+$. Then,

$$\Delta(\Omega) = N^+(x, h^1) \cup N^+(x, h^2) \cup \{p \in \Delta(\Omega) | U_1(h^1, p) = U_1(h^2, p)\}.$$

Since (μ, u) is regular,

$$0 = \mu(\{p \in \Delta(\Omega) | U_1(h^1, p) = U_1(h^2, p)\}) = \mu(\{p \in \Delta(\Omega) | v' \cdot p = c\}) \geq \mu(F).$$

Hence, $\mu(F) = 0$.

B.5 Proof of Theorem 3.1

Lemma B.8. *For any non-null event $A \in S$ and any $\bar{f} \in \mathcal{H}$, $x \succ_A y \Leftrightarrow xA\bar{f} \succ yA\bar{f}$.*

Proof. Assume $x \succ_A y$. Take any $\bar{f} \in \mathcal{H}$. By Consequentialism, $x \sim_A xA\bar{f}$ and $y \sim_A yA\bar{f}$. From Conditional Order, $xA\bar{f} \succ_A yA\bar{f}$. Thus, Dynamic Consistency implies $xA\bar{f} \succ yA\bar{f}$.

To show the converse, assume $xA\bar{f} \succ yA\bar{f}$. Dynamic Consistency implies $xA\bar{f} \succ_A yA\bar{f}$. Since $x \sim_A xA\bar{f}$ and $y \sim_A yA\bar{f}$ by Consequentialism, $x \succ_A y$. $\square$

From Theorem 2.1, $\succeq$ admits an additive SEU representation with (μ, u) , that is,

$$U(x) = \int_S \sup_{h \in x} \left(\sum_{\omega \in \Omega} u(h(\omega))p(\omega) \right) d\mu(p).$$

represents $\succeq$. For each $p \in S$,

$$\begin{aligned} \sup_{h \in xA\bar{f}} \sum_{\Omega} u(h(\omega))p(\omega) &= \sup_{h \in xA\bar{f}} \left(\sum_A u(h(\omega))p(\omega) + \sum_{\Omega \setminus A} u(h(\omega))p(\omega) \right) \\ &= \sup_{h \in xA\bar{f}} \left(\sum_A u(h(\omega))p(\omega) + \sum_{\Omega \setminus A} u(f(\omega))p(\omega) \right) \\ &= \sup_{h \in xA\bar{f}} \sum_A u(h(\omega))p(\omega) + \sum_{\Omega \setminus A} u(f(\omega))p(\omega). \end{aligned}$$

Hence,

$$U(xA\bar{f}) = \int_S \sup_{h \in xA\bar{f}} \left(\sum_A u(h(\omega))p(\omega) \right) d\mu(p) + \int_S \sum_{\Omega \setminus A} u(f(\omega))p(\omega) d\mu(p).$$

Therefore, for all $x, y \in \mathcal{D}$, $f, g \in \mathcal{H}$,

$$\begin{aligned}
& U(xAf) > U(yAf) \\
\Leftrightarrow & \int_S \sup_{h \in xAf} \left(\sum_A u(h(\omega))p(\omega) \right) d\mu(p) > \int_S \sup_{h \in yAf} \left(\sum_A u(h(\omega))p(\omega) \right) d\mu(p) \\
\Leftrightarrow & \int_S \sup_{h \in x} \left(\sum_A u(h(\omega))p(\omega) \right) d\mu(p) > \int_S \sup_{h \in y} \left(\sum_A u(h(\omega))p(\omega) \right) d\mu(p). \tag{20}
\end{aligned}$$

Since $U(xAf)$ represents $\succeq_A$ from Lemma B.8, (20) implies

$$\tilde{U}_A(x) \equiv \int_S \sup_{h \in x} \left(\sum_A u(h(\omega))p(\omega) \right) d\mu(p)$$

also represents $\succeq_A$. By rearrangement,

$$\begin{aligned}
\tilde{U}_A(x) &= \int_{\Delta_A(\Omega)} \sup_{h \in x} \left(\sum_A u(h(\omega))p(\omega) \right) d\mu(p) \\
&= \int_{\Delta_A(\Omega)} \sup_{h \in x} \left(\sum_A u(h(\omega)) \frac{p(\omega)}{p(A)} \right) p(A) d\mu(p) \\
&= \int_{\Delta_A(\Omega)} \sup_{h \in x} \left(\sum_A u(h(\omega))\beta_A(p)(\omega) \right) p(A) d\mu(p).
\end{aligned}$$

For any $A \in \mathcal{F}$ and any continuous function $\varphi : \Delta(\Omega) \rightarrow \mathbb{R}$, let

$$\Lambda_A(\varphi) \equiv \int_{\Delta(\Omega)} \varphi(p)p(A)d\mu(p).$$

Since Λ_A is non-negative continuous linear functional, the Riesz representation theorem ensures that there exists a unique non-negative measure μ^* such that

$$\Lambda_A(\varphi) = \int_{\Delta(\Omega)} \varphi(p)d\mu^*(p).$$

Notice that $\mu^* \in \Delta_A(\Omega)$. Thus,

$$\begin{aligned}
\tilde{U}_A(x) &= \int_{\Delta_A(\Omega)} \sup_{h \in x} \left(\sum_A u(h(\omega))\beta_A(p)(\omega) \right) p(A) d\mu(p) \\
&= \int_{\Delta_A(\Omega)} \sup_{h \in x} \left(\sum_A u(h(\omega))\beta_A(p)(\omega) \right) d\mu^*(p) \\
&= \int_{\Delta(A)} \sup_{h \in x} \left(\sum_A u(h(\omega))q(\omega) \right) d\mu^* \circ (\beta_A)^{-1}(q),
\end{aligned}$$

where $\mu^* \circ (\beta_A)^{-1}$ is the non-negative measure induced by $\beta_A : \Delta_A(\Omega) \rightarrow \Delta(A)$. Finally, let

$$\mu_A \equiv \frac{\mu^* \circ (\beta_A)^{-1}}{\mu^* \circ (\beta_A)^{-1}(\Delta(A))}.$$

Then,

$$U_A(x) \equiv \int_{\Delta(A)} \sup_{h \in x} \left(\sum_A u(h(\omega)) q(\omega) \right) d\mu_A(q)$$

is the additive SEU representation with components (μ_A, u) representing $\succeq_A$.

B.6 Proof of Proposition 3.1

By definition of Bayes' Rule, for any $B \in \mathcal{B}(S)$ and $C \in \mathcal{B}(\Omega)$,

$$\begin{aligned} P(B \times C | S \times A) &= \frac{P((B \times C) \cap (S \times A))}{P(S \times A)} \\ &= \frac{\int_B p(C \cap A) d\mu_0(p)}{\int_S p(A) d\mu_0(p)}. \end{aligned} \quad (21)$$

Since the function $p(C \cap A)/p(A)$ is continuous with respect to p , we have

$$\int_B \frac{p(C \cap A)}{p(A)} d\mu^*(p) = \int_B \frac{p(C \cap A)}{p(A)} p(A) d\mu_0(p)$$

from equation (12). Hence, the numerator of expression (21) is equal to

$$\begin{aligned} \int_B \frac{p(C \cap A)}{p(A)} p(A) d\mu_0(p) &= \int_B \frac{p(C \cap A)}{p(A)} d\mu^*(p) \\ &= \int_B \beta_A(p)(C) d\mu^*(p). \end{aligned} \quad (22)$$

By taking $\varphi : S \rightarrow \mathbb{R}$ as the constant function equal to one, equation (12) implies

$$\mu^*(S) = \int_S p(A) d\mu_0(p).$$

Hence, the denominator of expression (21) is equal to

$$\begin{aligned} \int_S p(A) d\mu_0(p) &= \mu^*(S) \\ &= \mu^*(\beta_A^{-1}(\Delta(A))) \\ &= \nu_A(\Delta(A)). \end{aligned} \quad (23)$$

Thus, taking (22) and (23) together,

$$P(B \times C | S \times A) = \frac{\int_B \beta_A(p)(C) d\mu^*(p)}{\nu_A(\Delta(A))}.$$

By the change of variables, for any $F \in \mathcal{B}(\Delta(A))$,

$$\begin{aligned}
P(\beta_A^{-1}(F) \times C | S \times A) &= \frac{\int_{\beta^{-1}(F)} \beta_A(p)(C) \, d\mu^*(p)}{\nu_A(\Delta(A))} \\
&= \frac{\int_F q(C) \, d\mu^* \circ \beta_A^{-1}(q)}{\nu_A(\Delta(A))} \\
&= \frac{\int_F q(C) \, d\nu_A(q)}{\nu_A(\Delta(A))} \\
&= \int_F q(C) \, d\left(\frac{\nu_A(q)}{\nu_A(\Delta(A))}\right) \\
&= \int_F q(C) \, d\mu_A(q).
\end{aligned}$$

Thus, letting $C = A$, we have, for any $F \in \mathcal{B}(\Delta(A))$,

$$P(\beta_A^{-1}(F) \times A | S \times A) = \mu_A(F).$$

B.7 Proof of Theorem 3.2

From Proposition 2.2, μ is full-dimensional.

Step 1: μ^* satisfying (12) is full-dimensional.

We want to show $\mu^*(F) = 0$ whenever $\dim F < n - 1$. We identify $\Delta(\Omega)$ with the unit simplex in $\mathbb{R}^n$. Since F has dimension less than $n - 1$, there exists a hyperplane $v \cdot q = c$ with $v \neq 0$ and $c \in \mathbb{R}$ such that F is included the intersection of $\Delta(\Omega)$ and the hyperplane, denoted by H . Let $B^n \equiv \{p \in \Delta(\Omega) | d(H, p) < 1/n\}$, where d is the Euclidean metric, and $d(H, p) = \min_{q \in H} d(q, p)$. That is, B^n is the $1/n$ -open neighborhood of H relative to the unit simplex. For any n , there exists a continuous function $\varphi^n : \Delta(\Omega) \rightarrow [0, 1]$ such that $\varphi(p) = 1$ if $p \in H$ and $\varphi(p) = 0$ if $p \in \Delta \setminus B^n$. Then, the sequence $\{\varphi^n\}_{n=1}^\infty$ converges pointwise to the characteristic function of H , $\chi_H : \Delta(\Omega) \rightarrow \mathbb{R}$, satisfying that $\chi_H(p) = 1$ if $p \in H$ and $\chi_H(p) = 0$ otherwise. From condition (12), for all n ,

$$\int_{\Delta(\Omega)} \varphi^n(p) \, d\mu^*(p) = \int_{\Delta(\Omega)} \varphi^n(p) p(A) \, d\mu(p).$$

The dominated convergence theorem (See Royden [12].) implies

$$\begin{aligned}
\mu^*(F) &\leq \mu^*(H) \\
&= \int_{\Delta(\Omega)} \chi_H(p) \, d\mu^*(p) \\
&= \int_{\Delta(\Omega)} \chi_H(p) p(A) \, d\mu(p). \\
&= \int_H p(A) \, d\mu(p).
\end{aligned}$$

Since μ is full-dimensional, $\mu(H) = 0$. Thus, we have $\mu^*(F) = 0$.

Let $n_A \equiv \#A$. For any $A \in \mathcal{A}$, say that $\mu \in \Delta(\Omega)$ is *full-dimensional relative to A* if $\mu(F) = 0$ whenever $\dim F < n_A - 1$ with $F \in \mathcal{B}(\Delta(A))$.

Step 2: μ_A is full-dimensional relative to A .

Since μ_A satisfies

$$\mu_A(F) = \frac{\nu_A(F)}{\nu_A(\Delta(\Omega))} = \frac{\mu^*(\beta_A^{-1}(F))}{\mu^*(\Delta(\Omega))},$$

we have to show that $\mu^*(\beta_A^{-1}(F)) = 0$ for any $F \in \mathcal{B}(\Delta(A))$ with $\dim F < n_A - 1$. From Step 1, it suffices to show that $\beta_A^{-1}(F)$ has dimension less than $n - 1$.

Now $\beta_A : \Delta_A(\Omega) \rightarrow \Delta(A)$ is regarded as the composite function of the following two functions: $T_1 : \Delta_A(\Omega) \rightarrow \mathbb{R}_+^{n_A} \setminus \{0\}$ defined by $T_1(q) \equiv (q_1, \dots, q_{n_A}, 0, \dots, 0)$ and $T_2 : \mathbb{R}_+^{n_A} \setminus \{0\} \rightarrow \Delta(A)$ defined by $T_2(q) \equiv (q_1 / \sum q_i, \dots, q_{n_A} / \sum q_i)$.

For any $F \in \mathcal{B}(\Delta(A))$ with $\dim F < n_A - 1$, there exists a hyperplane $v \cdot q = c$ with $v \in \mathbb{R}^n \setminus \{0\}$ and $c \in \mathbb{R}$ such that F is included the intersection of $\Delta(A)$ and the hyperplane. Denote this intersection by E , which has dimension less than $n_A - 1$. Since the inverse image $T_2^{-1}(E)$ is equal to

$$\text{aff}(E \cup \{0\}) \setminus \{0\},$$

where $\text{aff}(X)$ means the affine hull of X , $T_2^{-1}(E)$ has dimension less than n_A . Since the projection mapping T_1 has rank n_A , $T_1^{-1}(T_2^{-1}(E))$ must have dimension less than $n - 1$. Full dimensionality of μ^* implies

$$\mu^*(\beta_A^{-1}(F)) = \mu^*(T_1^{-1}(T_2^{-1}(F))) \leq \mu^*(T_1^{-1}(T_2^{-1}(E))) = 0.$$

Thus, $\mu^*(\beta_A^{-1}(F)) = 0$.

Step 3: (μ_A, u) is regular relative to A .

First of all, as pointed out in the proof of Proposition 2.2, we can consider $[0, 1]^n$ instead of $\mathcal{H}$ by using the continuous, mixture linear and onto mapping $\varphi : \mathcal{H} \rightarrow [0, 1]^n$ defined by $\varphi(h) = (u(h(\omega_1)), \dots, u(h(\omega_n)))$. For any $x \in \mathcal{D}_F^{+A}$, $\varphi(h) \neq \varphi(h')$ for any $h, h' \in x$. For any $x \in \mathcal{D}_F^{+A}$, let

$$\begin{aligned} N^A(x, h) &= \{p \in \Delta(A) | \varphi(h) \cdot p \geq \varphi(h') \cdot p, \text{ for all } h' \in x\}, \\ N^{+A}(x, h) &= \{p \in \Delta(A) | \varphi(h) \cdot p > \varphi(h') \cdot p, \text{ for all } h' \in x \text{ with } h' \neq h\}, \end{aligned}$$

where ‘ $\cdot$ ’ means the inner product.

By the same argument in Lemma B.7, we can show that

$$\Delta(A) = \cup_{h \in x} N^{+A}(x, h) \cup F^A$$

where F^A is the union of finite polyhedrons of dimension less than $n_A - 1$, and F^A and $\cup_{h \in x} N^{+A}(x, h)$ have no intersection. Since μ_A is full-dimensional relative to A , $\mu_A(F^A) = 0$. Thus,

$$1 = \mu_A(\Delta(A)) = \mu_A(\cup_{h \in x} N^{+A}(x, h)) + \mu_A(F^A) = \mu_A(\cup_{h \in x} N^{+A}(x, h)) = \mu_A(\cup_{h \in x} N^+(x, h)),$$

and hence (μ_A, u) is regular relative to A .

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