A model of inequality aversion and private provision of public goods

Online Appendix

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1 The Kuhn-Tucker conditions of the two sub-problems

Consider the case with $\alpha > 0$, $0 < \beta < 1$ and $g_A \in (0, w)$. Here, first consider the problem with constraint $2g_A \leq G$. In this case, individual B's sub-utility is relatively low or equal to that of A:

$$\max_{G} U_B = -\alpha\gamma \log\left(w - g_A\right) + \left(1 + \alpha\right)\gamma \log\left(w - G + g_A\right) + \left(1 - \gamma\right)\log G,$$

$$\text{s.t. } G - 2g_A \geq 0.$$

Then the *Lagrangian function* can be defined as

$$\begin{aligned} L\left(G\right) = & -\alpha\gamma \log \ \left(w - g_A\right) + \left(1 + \alpha\right)\gamma \log\left(w - G + g_A\right) + \left(1 - \gamma\right)\log G \\ & + \lambda_1\left(G - 2g_A\right), \end{aligned}$$

where λ_1 is the Lagrangian multiplier. Then the Kuhn-Tucker conditions of the problem are

$$\begin{aligned} \frac{\partial L\left(G\right)}{\partial G} &= -\frac{\left(1+\alpha\right)\gamma}{w - G + g_A} + \frac{1-\gamma}{G} + \lambda_1 = 0 \\ \frac{\partial L\left(G\right)}{\partial \lambda_1} &= G - 2g_A \geq 0 \\ \lambda_1 \cdot \frac{\partial L\left(G\right)}{\partial \lambda_1} &= \lambda_1\left(G - 2g_A\right) = 0 \\ \lambda_1 &\geq 0. \end{aligned}$$

The solution for this sub-problem is

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$$g_B = \begin{cases} \frac{1-\gamma}{1+\alpha\gamma} w - \frac{(1+\alpha)\gamma}{1+\alpha\gamma} g_A & \text{for } 0 < g_A < g_A^{\alpha}, \\ g_A & \text{for } g_A^{\alpha} \leq g_A < w, \end{cases}$$

where $g_A^{\alpha} = \frac{1-\gamma}{1+\gamma+2\alpha\gamma} w$.

Next, consider the problem with the constraint of $g_A \leq G \leq 2g_A$, wherein individual B's sub-utility is relatively high. To make a problem solvable, we change $G < 2g_A$ to $G \leq 2g_A$:

$$\max_G U_B = \beta\gamma \log \ (w - g_A) + (1-\beta)\gamma \log (w - G + g_A) + (1-\gamma) \log G,$$

$$\text{s.t. } g_A \leq G \leq 2g_A.$$

Then the Lagrangian function can be defined as

$$\begin{aligned} L(G) &= \beta\gamma \log \ (w - g_A) + (1-\beta)\gamma \log (w - G + g_A) + (1-\gamma) \log G \\ &\quad + \lambda_2 (G - g_A) + \lambda_3 (2g_A - G). \end{aligned}$$

Then the Kuhn-Tucker conditions of the problem are

$$\begin{aligned} \frac{\partial L(G)}{\partial G} &= -\frac{(1-\beta)\gamma}{w - G + g_A} + \frac{1-\gamma}{G} + \lambda_2 - \lambda_3 = 0 \\ \frac{\partial L(G)}{\partial \lambda_2} &= G - g_A \geq 0 \\ \frac{\partial L(G)}{\partial \lambda_3} &= 2g_A - G \geq 0 \\ \lambda_2 \cdot \frac{\partial L(G)}{\partial \lambda_2} &= \lambda_2 (G - g_A) = 0 \\ \lambda_3 \cdot \frac{\partial L(G)}{\partial \lambda_3} &= \lambda_3 (2g_A - G) = 0 \\ \lambda_2 &\geq 0, \lambda_3 \geq 0. \end{aligned}$$

2 Comparison of the solutions to the two sub-problems

Denote the level of U_B wherein $g_B = g_A$ as

$$U_{B,g_A} = \gamma \log (w - g_A) + (1-\gamma) \log 2g_A. \tag{1}$$

This means that the level of U_B is independent of α and β if $g_B = g_A$. Denote the level of U_B wherein $g_B = 0$ as

$$U_{B,0} = \beta\gamma \log (w - g_A) + (1-\beta)\gamma \log w + (1-\gamma) \log g_A.$$

Consider a case wherein $0 < \beta < 1 - \frac{1-\gamma}{\gamma}$. Suppose that U_{B,g_A} is larger than $U_{B,0}$:

$$
\begin{aligned}
& U_{B,g_A} > U_{B,0} \\
& \qquad \leftrightarrow (1-\beta)\,\gamma \log \frac{w-g_A}{w} > -(1-\gamma)\log 2 \\
& \qquad \leftrightarrow \log \frac{w}{w-g_A} < \log 2^{\delta} \leftrightarrow \frac{w}{w-g_A} < 2^{\delta} \\
& \qquad \leftrightarrow g_A < \frac{2^{\delta}-1}{2^{\delta}} w
\end{aligned}
$$

where $\delta = \frac{1-\gamma}{(1-\beta)\gamma}$. Note that $0 < \delta < 1$ for $0 < \beta < 1 - \frac{1-\gamma}{\gamma}$ and $\frac{1}{2} < \gamma < 1$.

First, from the above, $U_{B,g_A} > U_{B,0}$ for $g_A \leq \frac{1-\gamma}{1+\gamma-2\beta\gamma} w$ since $\frac{1-\gamma}{1+\gamma-2\beta\gamma} = \frac{\delta}{2+\delta} < \frac{2^{\delta}-1}{2^{\delta}}$ for $0 < \delta < 1$. Second, in constrast to the above, $U_{B,g_A} \leq U_{B,0}$ for $\frac{1-\gamma}{(1-\beta)\gamma} w \leq g_A < w$ since $\frac{2^{\delta}-1}{2^{\delta}} < \delta$ for $0 < \delta < 1$.

To sum up, for $0 < \beta < 1 - \frac{1-\gamma}{\gamma}$, the solution of this sub-problem is

$$
g_B = \begin{cases} g_A & \text{for } 0 < g_A \leq \underline{g}_A^{\beta}, \\ \frac{1-\gamma}{1-\beta\gamma} w - \frac{(1-\beta)\gamma}{1-\beta\gamma} g_A & \text{for } \underline{g}_A^{\beta} < g_A < \overline{g}_A^{\beta}, \\ 0 & \text{for } \overline{g}_A^{\beta} \leq g_A < w, \end{cases}
$$

where $\underline{g}_A^{\beta} = \frac{1-\gamma}{1+\gamma-2\beta\gamma} w$ and $\overline{g}_A^{\beta} = \frac{1-\gamma}{(1-\beta)\gamma} w$. For $1 - \frac{1-\gamma}{\gamma} \leq \beta$, the solution of this sub-problem is

$$
g_B = \begin{cases} g_A & \text{for } 0 < g_A \leq \underline{g}_A^{\beta}, \\ \frac{1-\gamma}{1-\beta\gamma} w - \frac{(1-\beta)\gamma}{1-\beta\gamma} g_A & \text{for } \underline{g}_A^{\beta} < g_A < w. \end{cases}
$$

Note that, from the first sub-problem, $g_B = \frac{1-\gamma}{1+\alpha\gamma} w - \frac{(1+\alpha)\gamma}{1+\alpha\gamma} g_A$ gives higher utility than (1) for $0 < g_A < g_A^{\alpha}$. A same argument applies for $g_B = \frac{1-\gamma}{1-\beta\gamma} w - \frac{(1-\beta)\gamma}{1-\beta\gamma} g_A$ for $0 < g_A \leq \underline{g}_A^{\beta}$, and $g_B = 0$ for $\overline{g}_A^{\beta} \leq g_A < w$ for the second sub-problem. By using these arguments, the solutions of the two sub-problems are compared to derive the optimal response:

$$
g_B^* = \begin{cases} \frac{1-\gamma}{1+\alpha\gamma} w - \frac{(1+\alpha)\gamma}{1+\alpha\gamma} g_A & \text{for } 0 < g_A < g_A^{\alpha}, \\ g_A & \text{for } g_A^{\alpha} \leq g_A \leq \underline{g}_A^{\beta}, \\ \frac{1-\gamma}{1-\beta\gamma} w - \frac{(1-\beta)\gamma}{1-\beta\gamma} g_A & \text{for } \underline{g}_A^{\beta} < g_A < \overline{g}_A^{\beta}, \\ 0 & \text{for } \overline{g}_A^{\beta} \leq g_A < w, \end{cases}
$$

wherein $0 < \beta < 1 - \frac{1-\gamma}{\gamma}$. For $1 - \frac{1-\gamma}{\gamma} \leq \beta$, the optimal response is

$$
g_B^* = \begin{cases} \frac{1-\gamma}{1+\alpha\gamma} w - \frac{(1+\alpha)\gamma}{1+\alpha\gamma} g_A & \text{for } 0 < g_A < g_A^{\alpha}, \\ g_A & \text{for } g_A^{\alpha} \leq g_A \leq \underline{g}_A^{\beta}, \\ \frac{1-\gamma}{1-\beta\gamma} w - \frac{(1-\beta)\gamma}{1-\beta\gamma} g_A & \text{for } \underline{g}_A^{\beta} < g_A < w. \end{cases}
$$