From Cantor to Semi-hyperbolic Parameters along External Rays

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March 2, 2019

Abstract

For the quadratic family $f_c(z) = z^2 + c$ with c in the exterior of the Mandelbrot set, it is known that every point in the Julia set moves holomorphically. Let $\hat{c}$ be a semi-hyperbolic parameter in the boundary of the Mandelbrot set. In this paper we prove that for each $z = z(c)$ in the Julia set, the derivative $dz(c)/dc$ is uniformly $O(1/\sqrt{|c - \hat{c}|})$ when c belongs to a parameter ray that lands on $\hat{c}$. We also characterize the degeneration of the dynamics along the parameter ray.

1 Introduction and main results

Let $\mathbb{M}$ be the *Mandelbrot set*, the connectedness locus of the quadratic family

$$\{f_c : z \mapsto z^2 + c\}_{c \in \mathbb{C}}.$$

That is, the Julia set $J(f_c)$ is connected if and only if $c \in \mathbb{M}$. For $c \notin \mathbb{M}$, it is well-known that the Julia set $J(f_c)$ is a Cantor set, and the critical point $z = 0$ does not belong to the Julia set. Moreover, f_c with $c \notin \mathbb{M}$ is *hyperbolic*: i.e., there exist positive numbers α_c and β_c such that $|Df_c^n(z)| \geq \alpha_c(1 + \beta_c)^n$ for any $n \geq 0$ and $z \in J(f_c)$.

Holomorphic motion of the Cantor Julia sets. For $c \notin \mathbb{M}$, because of hyperbolicity, every point in $z \in J(f_c)$ moves holomorphically with c. In other words, we have a *holomorphic motion* ([BR, L, Mc, MSS]) of the Cantor Julia sets over any simply connected domain in $\mathbb{C} - \mathbb{M}$. In this paper, we obtain some results regarding limiting behavior of this holomorphic motion when c approaches the boundary of $\mathbb{M}$.

Let us describe it more precisely: For a technical reason, we consider the holomorphic motion of a Cantor Julia set over the topological disk $\mathbb{X} = \mathbb{C} - \mathbb{M} \cup \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the set of positive real numbers. For any base point $c_0 \in \mathbb{X}$, there exists a unique map $H : \mathbb{X} \times J(f_{c_0}) \to \mathbb{C}$ such that

(1) $H(c_0, z) = z$ for any $z \in J(f_{c_0})$;

(2) For any $c \in \mathbb{X}$, the map $z \mapsto H(c, z)$ is injective on $J(f_{c_0})$ and it extends to a quasiconformal map on $\overline{\mathbb{C}}$.

(3) For any $z_0 \in J(f_{c_0})$, the map $c \mapsto H(c, z_0)$ is holomorphic on $\mathbb{X}$.

(4) For any $c \in \mathbb{X}$, the map $h_c(z) := H(c, z)$ satisfies $h_c(J(f_{c_0})) = J(f_c)$ and $f_c \circ h_c = h_c \circ f_{c_0}$ on $J(f_{c_0})$.

See [Mc, §4] for more details.

2010 Mathematics Subject Classification. Primary 37F45; Secondary 37F99.

Parameter rays. Let $\mathbb{D}$ denote the open disk of radius one centered at the origin. There is a unique biholomorphic function $\Phi_{\mathbb{M}}$ from $\overline{\mathbb{C}} - \mathbb{M}$ to $\overline{\mathbb{C}} - \overline{\mathbb{D}}$ satisfying $\Phi_{\mathbb{M}}(c)/c \to 1$ $(c \to \infty)$ with which the set

$$\mathcal{R}_{\mathbb{M}}(\theta) := \{\Phi_{\mathbb{M}}^{-1}(re^{i2\pi\theta}) \,:\, 1 < r < \infty\}$$

is defined and called the *parameter ray* of angle $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ of the Mandelbrot set $\mathbb{M}$. (This is a hyperbolic geodesic of the simply connected domain $\overline{\mathbb{C}} - \mathbb{M}$ starting at infinity.) See Figure 1. Given θ, if the limit $\hat{c} = \lim_{r\to 1^+} \Phi_{\mathbb{M}}^{-1}(re^{i2\pi\theta})$ exists, then $\hat{c} \in \partial\mathbb{M}$ is called the *landing point* of the parameter ray $\mathcal{R}_{\mathbb{M}}(\theta)$. We also say that θ is an *external angle* of the parameter $\hat{c}$.

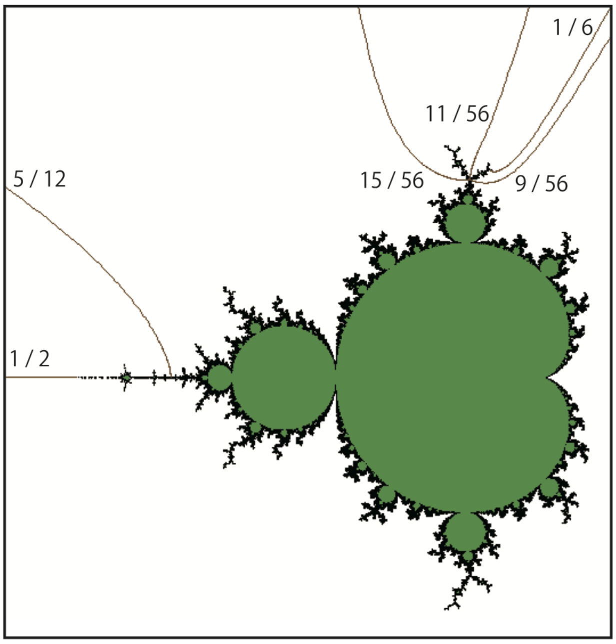

Figure 1: The Mandelbrot set and the parameter rays of angles 9/56, 1/6, 11/56, 15/56, 5/12, and 1/2.

Example (Real Cantor Julia sets)**.** When $c \notin \mathbb{M}$ approaches $\hat{c} = -2$ along the real axis (equivalently, along the parameter ray of angle 1/2), $J(f_c)$ is contained in the real axis and its motion is depicted in Figure 2.

Semi-hyperbolic parameters and Misiurewicz points. We are concerned with boundary behavior of the holomorphic motion given by the map H above, along the parameter rays that land on a fairly large subset of $\partial\mathbb{M}$.

We say a parameter $\hat{c}$ in $\partial\mathbb{M}$ is *semi-hyperbolic* if the critical point is non-recurrent and belongs to the Julia set. For each semi-hyperbolic parameter $\hat{c} \in \partial\mathbb{M}$, there exists at least one parameter ray $\mathcal{R}_{\mathbb{M}}(\theta)$ landing at $\hat{c}$. (See [D2, Theorem 2]. Indeed, there are at most finite number of parameter rays landing at $\hat{c}$. See Remark 7.2.) Note that for the quadratic polynomial $z^2 + c$ (more generally, unicritical polynomials of the form $z^d + c$), $\hat{c} \in \partial\mathbb{M}$ being semi-hyperbolic implies it is a Collet-Eckmann parameter. (See [PRLS, Main Theorem &

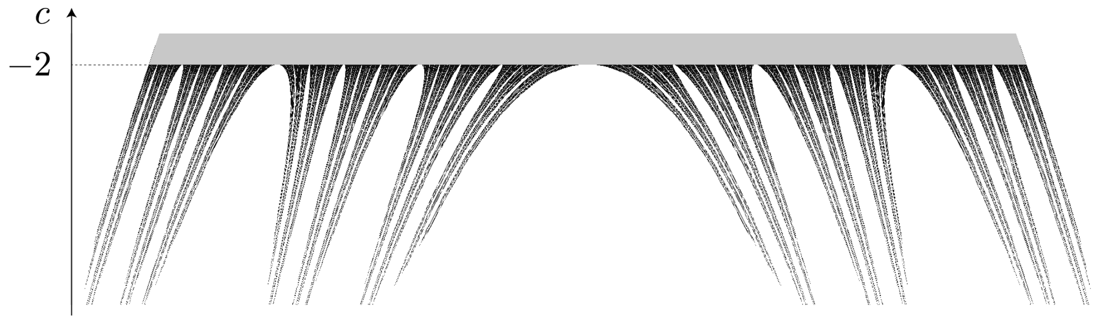

Figure 2: Each horizontal slice of the black part is the Julia set of parameter $c \in [-2.733, -2)$. The gray part is the real slice of $J(f_c)$ for $c \in [-2, -1.875]$. Note that $J(f_{-2}) = [-2, 2]$.

p.51] also [RL, p.291 & 299].) Shishikura [Shi] showed that for any open set U intersecting with $\partial\mathbb{M}$, the semi-hyperbolic parameters in U form a dense subset of Hausdorff dimension 2 of $U \cap \partial\mathbb{M}$. By a result of Douady [D2], the parameter ray $\mathcal{R}_{\mathbb{M}}(\theta)$ lands on a semi-hyperbolic parameter if and only if $\theta \in \mathbb{T}$ is non-recurrent under the angle-doubling $t \mapsto 2t$ (mod 1). Hence every interval of $\mathbb{T}$ contains uncountably many angles for which the parameter rays land on semi-hyperbolic parameters. The geometric and dynamical properties of the Julia sets of semi-hyperbolic parameters are deeply investigated in a work of Carleson-Jones-Yoccoz [CJY]. For example, if $\hat{c} \in \partial\mathbb{M}$ is semi-hyperbolic, then $J(f_{\hat{c}})$ is a locally connected dendrite such that $\overline{\mathbb{C}} - J(f_{\hat{c}})$ is a John domain.

A typical example of semi-hyperbolic parameter is a Misiurewicz point: We say a parameter $\hat{c}$ is *Misiurewicz* if the critical point of $f_{\hat{c}}$ is a pre-periodic point. (By a *pre-periodic* point z we mean $f_{\hat{c}}^l(z) = f_{\hat{c}}^{l+p}(z)$ for some integers l and p but $f_{\hat{c}}^n(z) \neq z$ for all $n \geq 1$.) It is known that such a Misiurewicz point $\hat{c}$ eventually lands on a repelling periodic cycle in the dynamics of $f_{\hat{c}}$, and that the Misiurewicz points are contained in the boundary of the Mandelbrot set. It is also known that the parameter $\hat{c}$ is Misiurewicz if and only if $\hat{c}$ is the landing point of $\mathcal{R}_{\mathbb{M}}(\theta)$ for some rational θ of even denominator. (See [DH1, Éxposé VIII] and [CG, VIII, 6] for example.) Holomorphic motions of the Julia sets along such rays are depicted in Figure 3.

Main results. Let z_0 be any point in $J(f_{c_0})$. Then the map $c \mapsto z(c) := H(c, z_0)$ is holomorphic over $\mathbb{X} = \mathbb{C} - \mathbb{M} \cup \mathbb{R}_+$. If we choose a semi-hyperbolic parameter $\hat{c} \in \partial\mathbb{M}$, there exists a parameter ray $\mathcal{R}_{\mathbb{M}}(\theta) \subset \mathbb{X}$ of angle $\theta \in \mathbb{T} - \{0\}$ that lands on $\hat{c}$. As c moves along the parameter ray $\mathcal{R}_{\mathbb{M}}(\theta)$ toward $\hat{c}$, $z(c) = H(c, z_0)$ moves along an analytic curve in the plane.

Our main theorem states that the speed of such a motion is uniformly bounded by a function of $|c - \hat{c}|$:

Theorem 1.1 (Main Theorem). *Let $\hat{c} \in \partial\mathbb{M}$ be a semi-hyperbolic parameter that is the landing point of $\mathcal{R}_{\mathbb{M}}(\theta)$. Then there exists a constant $K > 0$ that depends only on $\hat{c}$ such that for any $c \in \mathcal{R}_{\mathbb{M}}(\theta)$ sufficiently close to $\hat{c}$ and any $z = z(c) \in J(f_c)$, the point $z(c)$ moves holomorphically with*

$$\left| \frac{dz(c)}{dc} \right| \leq \frac{K}{\sqrt{|c - \hat{c}|}}.$$

The proof is given in Section 5. By this theorem we obtain one-sided Hölder continuity of the holomorphic motion along the parameter ray:

Theorem 1.2 (Holomorphic Motion Lands). *Let $\hat{c} \in \partial\mathbb{M}$ be a semi-hyperbolic parameter that is the landing point of $\mathcal{R}_{\mathbb{M}}(\theta)$, and let $c = c(r) := \Phi_{\mathbb{M}}^{-1}(re^{i2\pi\theta})$ with parameter $r \in (1, 2]$. Then for any $z(c(2))$ in $J(f_{c(2)})$, the improper integral*

$$z(\hat{c}) := z(c(2)) + \lim_{\delta \to +0} \int_2^{1+\delta} \frac{dz(c)}{dc} \frac{dc(r)}{dr}\, dr$$

exists in the Julia set $J(f_{\hat{c}})$. In particular, $z(c)$ is uniformly one-sided Hölder continuous of exponent $1/2$ at $c = \hat{c}$ along $\mathcal{R}_{\mathbb{M}}(\theta)$. More precisely, there exists a constant K' depending only on $\hat{c}$ such that

$$|z(c) - z(\hat{c})| \le K' \sqrt{|c - \hat{c}|}$$

for any $c = c(r) \in \mathcal{R}_{\mathbb{M}}(\theta)$ with $1 < r \le 2$.

This theorem implies:

Theorem 1.3 (From Cantor to Semi-hyperbolic). *For any semi-hyperbolic parameter $\hat{c} \in \partial\mathbb{M}$ and any parameter ray $\mathcal{R}_{\mathbb{M}}(\theta)$ landing at $\hat{c}$, the conjugacy $H(c, \cdot) = h_c : J(f_{c_0}) \to J(f_c)$ converges uniformly to a semiconjugacy $h_{\hat{c}} : J(f_{c_0}) \to J(f_{\hat{c}})$ from f_{c_0} to $f_{\hat{c}}$ as $c \to \hat{c}$ along $\mathcal{R}_{\mathbb{M}}(\theta)$.*

The proofs of these theorems are given Section 8. In Theorem 1.6 below, we will specify where the semiconjugacy $h_{\hat{c}} : J(f_{c_0}) \to J(f_{\hat{c}})$ fails to be injective. Indeed, the semiconjugacy is injective except on a countable subset.

By Theorems 1.2 and 1.3, we have a semiconjugacy $h_{\hat{c}} \circ h_c^{-1} : J(f_c) \to J(f_{\hat{c}})$ with $|h_{\hat{c}} \circ h_c^{-1}(z) - z| = O(\sqrt{|c - \hat{c}|})$ as $c \to \hat{c}$ along $\mathcal{R}_{\mathbb{M}}(\theta)$. Thus we obtain:

Corollary 1.4 (Hausdorff Convergence). *The Hausdorff distance between $J(f_c)$ and $J(f_{\hat{c}})$ is $O(\sqrt{|c - \hat{c}|})$ as $c \to \hat{c}$ along $\mathcal{R}_{\mathbb{M}}(\theta)$.*

This result is compatible with a result by Rivera-Letelier [RL]. See Remark 1.7.

Symbolic dynamics. Let

$$\Sigma_3 := \Big\{ \mathbf{s} = \{s_0, s_1, s_2, \ldots\} \; : \; s_n = *, \; 0 \text{ or } 1 \text{ for all } n \ge 0 \Big\}$$

be the space consisting of sequences of $*$'s, 0's and 1's with the product topology, and σ be the left shift in Σ_3, $\sigma(\mathbf{s}) = \mathbf{s}' = (s_0', s_1', s_2', \cdots)$ with $s_i' = s_{i+1}$. Let

$$\Sigma_2 := \Big\{ \mathbf{s} = \{s_0, s_1, s_2, \ldots\} \; : \; s_n = 0 \text{ or } 1 \text{ for all } n \ge 0 \Big\} \subset \Sigma_3$$

be a closed subspace of Σ_3. A point $\mathbf{e} \in \Sigma_2$ is said to be *aperiodic* if $\sigma^n(\mathbf{e}) \ne \mathbf{e}$ for any $n \ge 0$. Two points $\mathbf{a}$ and $\mathbf{s}$ in Σ_2 are said to be *equivalent* with respect to aperiodic $\mathbf{e} \in \Sigma_2$, denoted by $\mathbf{a} \sim_{\mathbf{e}} \mathbf{s}$, if there is $k \ge 0$ such that $a_n = s_n$ for all $n \ne k$ and $\sigma^{k+1}(\mathbf{a}) = \sigma^{k+1}(\mathbf{s}) = \mathbf{e}$. It is plain to verify that the relation $\sim_{\mathbf{e}}$ is indeed an equivalence relation, and is the smallest equivalence relation that identifies $0\mathbf{e}$ with $1\mathbf{e}$.

Note that for $c \notin \mathbb{M}$ the dynamics of f_c on the Julia set is conjugate to that of σ on Σ_2. We will use an aperiodic $\mathbf{e}$ to represent the (itinerary of the) non-recurrent critical orbit of the semi-hyperbolic $f_{\hat{c}}$. Then $\mathbf{a}$ and $\mathbf{s}$ in Σ_2 are equivalent with respect to this $\mathbf{e}$ if and only if the points in $J(f_c)$ corresponding to $\mathbf{a}$ and $\mathbf{s}$ will degenerate to a point that eventually lands on the critical value $\hat{c}$ in $J(f_{\hat{c}})$ as c moves along the parameter ray landing on $\hat{c}$.

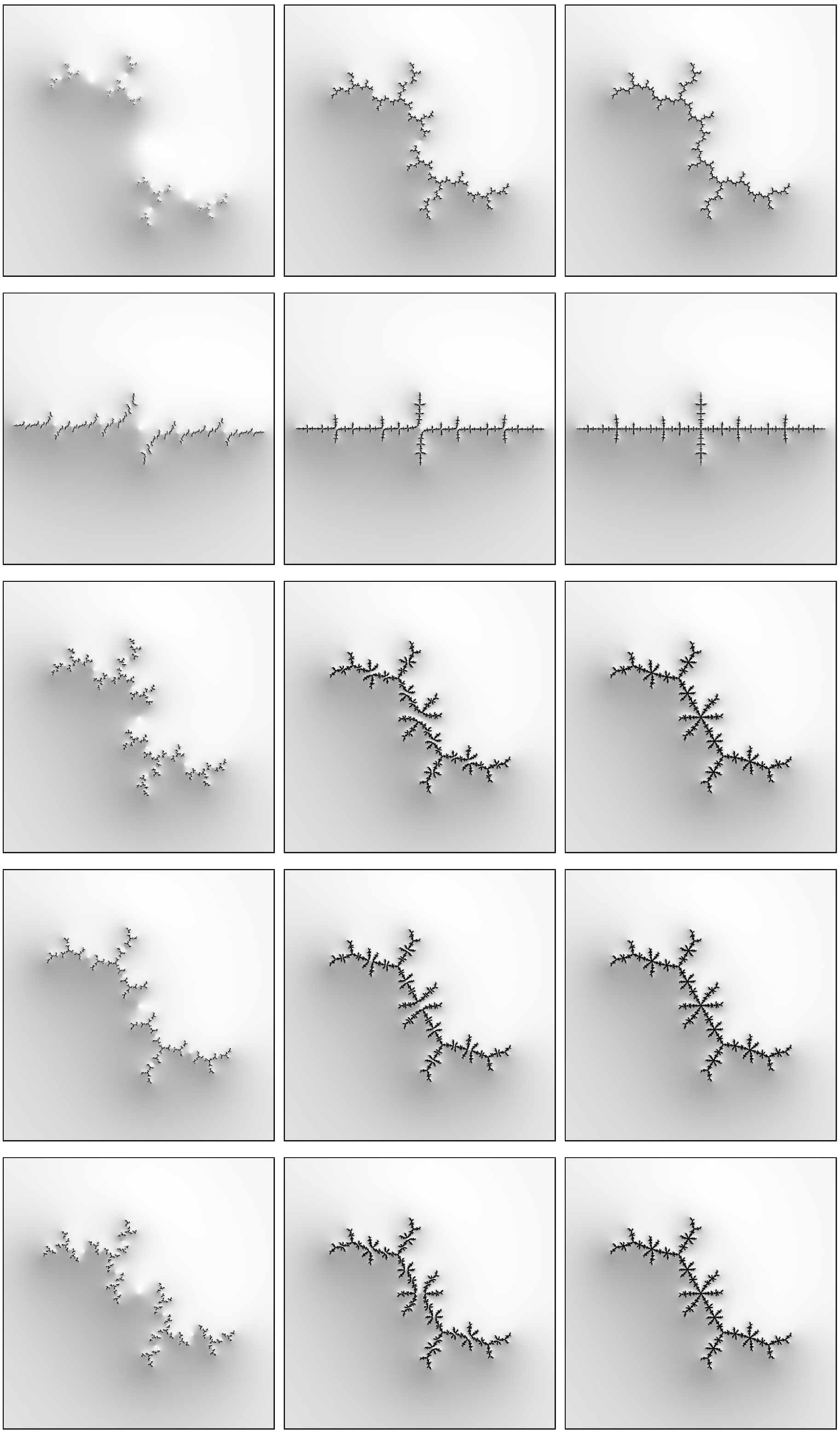

Figure 3: Holomorphic motion along the parameter rays of angles 1/6, 5/12, 9/56, 11/56, and 15/56.

Let $\mathcal{T} : \mathbb{T} \to \mathbb{T}$, $t \mapsto 2t \pmod 1$ be the angle-doubling map. Fix $\theta \in \mathbb{T} - \{0\}$, the two points $\theta/2$ and $(\theta+1)/2$ divide $\mathbb{T}$ into two open semi-circles $\mathbb{T}_0^\theta$ and $\mathbb{T}_1^\theta$ with $\theta \in \mathbb{T}_0^\theta$. Define the *itinerary* of a point t under $\mathcal{T}$ with respect to θ as $\mathcal{E}^\theta(t) = \{\mathcal{E}^\theta(t)_n\}_{n \geq 0}$ with

$$\mathcal{E}^\theta(t)_n = \begin{cases} 0 & \text{for } \mathcal{T}^n(t) \in \mathbb{T}_0^\theta \\ 1 & \text{for } \mathcal{T}^n(t) \in \mathbb{T}_1^\theta \\ * & \text{for } \mathcal{T}^n(t) \in \left\{\frac{\theta}{2}, \frac{\theta+1}{2}\right\}. \end{cases}$$

The itinerary of θ itself, $\mathcal{E}^\theta(\theta)$, is called the *kneading sequence* of θ.

Another consequence of Theorem 1.2 is as follows.

Theorem 1.5 (Symbolic Dynamics at Semi-hyperbolic Parameter)**.** *Let $\hat{c}$ be a semi-hyperbolic parameter with an external angle θ and $\mathbf{e} = \mathcal{E}^\theta(\theta)$ be the kneading sequence of θ. Then $(J(f_{\hat{c}}), f_{\hat{c}})$ is topologically conjugate to $(\Sigma_2/\sim_{\mathbf{e}}, \tilde{\sigma})$, where $\tilde{\sigma}$ is induced by the shift transformation σ of Σ_3.*

Theorem 1.5 also implies that the semiconjugacy in Theorem 1.3 is one-to-one except at countable points where it is two-to-one.

Theorem 1.6 (Almost Injectivity)**.** *Let $h_{\hat{c}} : J(f_{c_0}) \to J(f_{\hat{c}})$ be the semiconjugacy given in Theorem 1.3. For any $w \in J(f_{\hat{c}})$, the preimage $h_{\hat{c}}^{-1}(\{w\})$ has at most two points, and it consists of two distinct points if and only if $f_{\hat{c}}^n(w) = 0$ for some $n \geq 0$.*

We prove these two theorems above in Section 17. More precise properties of the semiconjugacy can be found in Corollary 16.2.

Structure of the paper. The structure of this paper is a little complicated, but we belive this presentation requires less memory of the readers. In Section 2 we briefly summarize the notation and properties of the dynamics of $f_c(z) = z^2 + c$ with semi-hyperbolic parameters. Section 3 is devoted for "the derivative formula", which is a key tool for our estimate. In Section 4 we introduce the notion of "Z-cycle" to describe the behavior of the orbits. We also present Lemmas A, B, and C about Z-cycles, whose proofs are given later. In Section 5 we prove the Main Theorem by assuming these lemmas. In Section 6 we introduce the notion of "S-cycle" and "the S-cycle decompositions" of Z-cycles. We also present Lemmas A', B', and C', whose proofs are given later as well. Section 7 is devoted for Proposition S about stability of landing dynamic rays, and some lemmas that come from the assumption that the parameter c moves along the parameter ray. In Section 8 we prove Theorems 1.2 and 1.3. Then by assuming Lemmas A', B', and C', we prove Lemmas A and B in Sections 9 and 10 respectively. Section 11 is devoted for some lemmas on hyperbolic metrics, and by using them, we prove Lemmas B', A', C', and C in Sections 12, 13, 14, and 15 respectively. In Section 16 we work with symbolic dynamics, and finally in Section 17 we give proofs of Theorems 1.5 and 1.6.

Remark 1.7.

- The estimate in the Main Theorem is optimal. For example, if $\hat{c} = -2$ (that is the Misiurewicz parameter with $f_{\hat{c}}^2(0) = f_{\hat{c}}^3(0) = 2$), then for $c = -2 - \epsilon$ with $\epsilon > 0$ the repelling fixed point on the positive real axis is given by $(1 + \sqrt{9 + 4\epsilon})/2 = 2 + \epsilon/3 + o(\epsilon)$. Hence its preimages near the critical point are $z = \pm\sqrt{2\epsilon/3}(1 + o(\epsilon))$, whose derivatives are $dz/d\epsilon = \pm(1/\sqrt{6\epsilon})(1 + o(\epsilon))$. This implies that $|dz/dc|$ is compatible with $1/\sqrt{|c - \hat{c}|}$. See Figure 2.

- The results and the proofs in this paper are easily generalized to the unicritical family $\{z \mapsto z^d + c : c \in \mathbb{C}\}$, simply by replacing the square root (" $\sqrt{|c-\hat{c}|}$ ") by the dth root (" $|c-\hat{c}|^{1/d}$ ") in the Main Theorem.

- In [CK] the authors give a simple proof of the Main Theorem for $\hat{c} = -2$.

- In [D1], Douady showed that the Julia set $J(f_c)$ continuously depends on c at any semi-hyperbolic parameter $\hat{c}$ in the sense of Hausdorff topology. Moreover, in [RL], Rivera-Letelier showed that the Hausdorff distance between $J(f_c)$ and $J(f_{\hat{c}})$ is $O(|c-\hat{c}|^{1/2})$ for c close enough to $\hat{c}$, and that the Hausdorff dimension of the Julia set $J(f_c)$ converges to that of $\hat{c}$ if c tends to $\hat{c}$ along the parameter ray. Our results, in addition, give the convergence of the dynamics.

- It is known that any parameter ray of odd denominator has a landing point $\hat{c}$ on $\partial\mathbb{M}$ such that $f_{\hat{c}}$ has a parabolic periodic point. However, when c moves along such a parameter ray, $J(f_c)$ does not converge in the Hausdorff topology. The discontinuity comes from the "parabolic implosion", which is also described in Douady's article [D1].

- Suppose $\hat{c} \in \partial\mathbb{M}$ and $\hat{c} \in J(f_{\hat{c}})$, and suppose $\hat{c}$ has an external angle θ. There have been several results concerning the quotient dynamics for $f_{\hat{c}}$ by kneading sequences. If the kneading sequence $\mathcal{E}^\theta(\theta)$ is aperiodic, then the same statement as Theorem 1.5 that $(J(f_{\hat{c}}), f_{\hat{c}})$ is topologically conjugate to $(\Sigma_2/\sim_{\mathcal{E}^\theta(\theta)}, \tilde{\sigma})$ has been known by Bandt and Keller [BK]. Let $\approx_\theta$ be the smallest equivalence relation that if t, t' are points in $\mathbb{T}$ such that for every n either $\mathcal{E}^\theta(t)_n = \mathcal{E}^\theta(t')_n$ or $\mathcal{E}^\theta(t)_n = *$ or $\mathcal{E}^\theta(t')_n = *$, then t is equivalent to t'. They also showed that $(J(f_{\hat{c}}), f_{\hat{c}})$ is topologically conjugate to $(\mathbb{T}/\approx_\theta, \widetilde{\mathcal{T}})$ as well, where $\widetilde{\mathcal{T}}$ is induced by the angle-doubling map $\mathcal{T}$ on $\mathbb{T}/\approx_\theta$. Besides, for $f_{\hat{c}}$ with locally connected Julia set and no irrational indifferent cycles, Kiwi [K2] defined $\equiv_{\hat{c}}$ to be the smallest equivalence relation in $\mathbb{T}$ which identifies t and t' whenever the landing points of the dynamic rays $\mathcal{R}_{\hat{c}}(t)$ and $\mathcal{R}_{\hat{c}}(t')$ coincide. (See Section 7 for the definition of the dynamic rays.) Then he showed that $(J(f_{\hat{c}}), f_{\hat{c}})$ is topologically conjugate to $(\mathbb{T}/\equiv_{\hat{c}}, \widehat{\mathcal{T}})$, where $\widehat{\mathcal{T}}$ is induced by $\mathcal{T}$ on $\mathbb{T}/\equiv_{\hat{c}}$. (For $\hat{c}$ a Misiurewicz parameter, Kiwi's result has been obtained earlier in [AK]. However, in [K2] more general cases were considered including non-locally connected Julia sets.)

2 Misiurewicz and semi-hyperbolic parameters

In this section we briefly summarize the notation and properties of the dynamics of $f_c(z) = z^2 + c$ with semi-hyperbolic parameters.

Notation.

- Let $\mathbb{N}$ denote the set of positive integers. We denote the set of non-negative integers by $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

- Let $\mathbb{D}(a, r)$ denote the disk in $\mathbb{C}$ centered at a and of radius $r > 0$. When $a = 0$ we denote it by $\mathbb{D}(r)$.

- Let $\mathrm{N}(A, r)$ denote the open r-neighborhood of the set $A \subset \mathbb{C}$ for $r > 0$. That is, $\mathrm{N}(A, r) := \bigcup_{a \in A} \mathbb{D}(a, r)$.

- For non-negative variables X and Y, by $X \asymp Y$ we mean there exists an implicit constant $C > 1$ independent of X and Y such that $X/C \leq Y \leq CX$.

- When we say "for any $X \ll 1$" it means that "for any sufficiently small $X > 0$".
- Let c be a parameter for the quadratic family $\{f_c(z) = z^2 + c \,:\, c \in \mathbb{C}\}$. By $c \approx \hat{c}$ we mean there exists an implicit constant $\delta > 0$ independent of $c \neq \hat{c}$ such that $|c - \hat{c}| < \delta$. When we say "the constant C independent of $c \approx \hat{c}$" it means that C does not depend on $c \neq \hat{c}$ but it *may* depend on $\hat{c}$.

Misiurewicz and semi-hyperbolic parameters. Let $\hat{c} \in \partial\mathbb{M}$ be a Misiurewicz point with $f_{\hat{c}}^l(0) = f_{\hat{c}}^{l+p}(0)$ where we choose the minimal l and p in $\mathbb{N}$. Then it is known that $f_{\hat{c}}^l(0)$ is actually a repelling periodic point.

More generally, suppose that $\hat{c} \in \partial\mathbb{M}$ is semi-hyperbolic, and set $\hat{b}_n := f_{\hat{c}}^n(0)$ for each $n \geq 0$. Let $\Omega(\hat{c})$ denote the set of accumulation points of the set $\{\hat{b}_n\}_{n \geq 0}$, i.e., the ω-limit set of 0. Moreover, by a result of Carleson, Jones, and Yoccoz [CJY], $\Omega(\hat{c})$ is a *hyperbolic set* in the sense of [Shi]: i.e., $\Omega(\hat{c})$ is compact; $f_{\hat{c}}(\Omega(\hat{c})) \subset \Omega(\hat{c})$ (indeed, we have $f_{\hat{c}}(\Omega(\hat{c})) = \Omega(\hat{c})$); and there exist constants $\alpha, \beta > 0$ such that $|Df_{\hat{c}}^n(z)| \geq \alpha(1+\beta)^n$ for any $z \in \Omega(\hat{c})$ and $n \geq 0$. For example, if $\hat{c}$ is Misiurewicz, the set $\Omega(\hat{c})$ is the repelling cycle on which the orbit of 0 lands.

For $\hat{c} \in \partial\mathbb{M}$ a semi-hyperbolic parameter, it is proved in [CJY] that there are constants $\epsilon > 0$, $C > 0$, and $0 < \eta < 1$ such that for all $z \in J(f_{\hat{c}})$, $n \geq 0$, and any connected component $B_n(z,\epsilon)$ of $f_{\hat{c}}^{-n}(\mathbb{D}(z,\epsilon))$, we have

$$\operatorname{diam} B_n(z,\epsilon) < C\,\eta^n. \tag{1}$$

In what follows we fix a $p \in \mathbb{N}$ such that $|Df_{\hat{c}}^p(z)| \geq 3$ for any $z \in \Omega(\hat{c})$. [1] We first check:

Proposition 2.1 (Critical Orbit Lands)**.** *The critical orbit $\hat{b}_n = f_{\hat{c}}^n(0)$ $(n \in \mathbb{N}_0)$ eventually lands on $\Omega(\hat{c})$. That is, there exists a minimal integer l such that $\hat{b}_l = f_{\hat{c}}^l(0) \in \Omega(\hat{c})$.*

Proof. Suppose that $\hat{b}_n \notin \Omega(\hat{c})$ for every $n \in \mathbb{N}$. Since $|Df_{\hat{c}}^p(x)| \geq 3$ for any $x \in \Omega(\hat{c})$ we apply the Koebe distortion theorem (see [Du]) to find a $\delta > 0$ such that if $\hat{b}_n \in \mathrm{N}(\Omega(\hat{c}),\delta) - \Omega(\hat{c})$, we have

$$\operatorname{dist}(\hat{b}_{n+p}, \Omega(\hat{c})) \geq 2\operatorname{dist}(\hat{b}_n, \Omega(\hat{c})).$$

(We also used compactness and invariance of $\Omega(\hat{c})$. See also Remark 2.3.) Hence there exists an accumulation point of the critical orbit in $\overline{\mathbb{C}} - \mathrm{N}(\Omega(\hat{c}),\delta)$. However, it contradicts to the definition of $\Omega(\hat{c})$. ■

Another remarkable fact is that the hyperbolic set $\Omega(\hat{c})$ moves holomorphically and preserves the dynamics (See [Shi, §1]):

Proposition 2.2 (Holomorphic Motion of $\Omega(\hat{c})$)**.** *There exist a neighborhood Δ of $\hat{c}$ in the parameter plane $\mathbb{C}$ and a map $\chi : \Delta \times \Omega(\hat{c}) \to \mathbb{C}$ with the following properties:*

(1) *$\chi(\hat{c}, z) = z$ for any $z \in \Omega(\hat{c})$;*

(2) *For any $c \in \Delta$, the map $z \mapsto \chi(c,z)$ is injective on $\Omega(\hat{c})$ and it extends to a quasiconformal map on $\overline{\mathbb{C}}$.*

(3) *For any $z_0 \in \Omega(\hat{c})$, the map $c \mapsto \chi(c, z_0)$ is holomorphic on Δ.*

(4) *For any $c \in \Delta$, the map $\chi_c(z) := \chi(c,z)$ satisfies $f_c \circ \chi_c = \chi_c \circ f_{\hat{c}}$ on $\Omega(\hat{c})$.*

[1] Of course "3" does not have particular meaning. Any constant bigger than one will do.

Definition of V_j's. Now we give a fundamental setting for the proofs of our results that will be assumed in what follows.

- Set $\Omega(c) := \chi_c(\Omega(\hat{c}))$ for each $c \in \Delta$ given in Proposition 2.2. Then $\Omega(c)$ is a hyperbolic subset of the Julia set $J(f_c)$. Since $J(f_c)$ is a Cantor set when $c \notin \mathbb{M}$, $\Omega(c)$ is a totally disconnected set for any $c \in \Delta$.

- Set $U_l := \mathrm{N}(\Omega(\hat{c}), R_l)$ for a sufficiently small $R_l > 0$, such that

  – there is a constant $\mu \geq 2.5$ such that for any $c \approx \hat{c}$ and $z \in U_l$ we have
  $$|Df_c^p(z)| \geq \mu; \quad \text{and}$$

  – for any $c \approx \hat{c}$, $U_l \Subset f_c^p(U_l)$.

  Such an R_l exists because $|Df_{\hat{c}}^p(z)| \geq 3$ on $\Omega(\hat{c})$ and the function $(c, z) \mapsto |Df_c^p(z)|$ is continuous.

- We set $b_j(c) := \chi_c(\hat{b}_j) \in \Omega(c)$ for each $j \geq l$ and $c \in \Delta$. By taking a smaller Δ if necessary, we can also find an analytic family of pre-landing points $b_0(c)$, $b_1(c)$, $\cdots$, $b_{l-1}(c)$ over Δ such that $b_{j+1}(c) = f_c(b_j(c))$ and $\hat{b}_j = b_j(\hat{c})$ for each $j = 0, 1, \cdots , l-1$. (For $j = 0$, $b_0(c)$ is defined as a branch of $f_c^{-1}(b_1(c))$.)

- Choose disjoint topological disks V_j for $j = 0, 1, \cdots , l-1$ such that

  – $V_0 := \mathbb{D}(0, \nu)$ for some $\nu \ll 1$. We will add more conditions for ν later.

  – For each $j = 1, \cdots , l-1$, the topological disk V_j contains $\hat{b}_j$ and satisfies $\operatorname{diam} V_j \asymp \nu^2$. More precisely, there exists a constant $C_0 > 1$ independent of j such that $\nu^2/C_0 \leq \operatorname{diam} V_j \leq C_0\nu^2$.

  – For any $c \approx \hat{c}$ and each $j = 0, 1, \cdots , l-2$, we have $f_c(V_j) \Subset V_{j+1}$.

  We also take a constant $C_0' > 1$ such that for any $c \approx \hat{c}$,

  – the set $V_l := \mathrm{N}(\Omega(\hat{c}), C_0'\nu^2)$ contains the topological disk $f_c(V_{l-1})$; and

  – at least for each $j = 0, 1, \cdots , p-1$, $f_c^j(V_l) \Subset U_l$.

  We assume that ν is sufficiently small such that $V_j \cap V_l = \emptyset$ for each $j = 0, 1, \cdots , l-1$. Let $\mathcal{V}$ denote the union $V_1 \cup V_2 \cup \cdots \cup V_{l-1} \cup V_l$. See Figure 4.

- Let ξ be the distance from 0 to the closure of the set
  $$\{\hat{b}_1, \hat{b}_2, \cdots , \hat{b}_{l-1}\} \cup U_l.$$
  Since 0 is not recurrent (i.e., $0 \notin \Omega(\hat{c})$), we have $\xi > 0$ if we take R_l small enough. We may assume in addition that $0 < \xi \leq 1$ if we reset $\xi := 1$ when $\xi > 1$. If necessary, we replace ν so that R_l and $C_0\nu^2$ are smaller than $\xi/2$. Then we have $|Df_c(z)| = 2|z| \geq \xi$ for any $z \in \mathcal{V} \cup U_l$ and $c \approx \hat{c}$.

Remark 2.3. The backward dynamics of f^p near $\Omega(\hat{c})$ is uniformly shrinking with respect to the Euclidean metric. For example, *one can find an $R > 0$ depending only on $\hat{c}$ such that for any $x \in \Omega(\hat{c})$ there exists a univalent branch g of $f_{\hat{c}}^{-p}$ on $\mathbb{D}(f_{\hat{c}}^p(x), R)$ satisfying $g(f_{\hat{c}}^p(x)) = x$ and $g(\mathbb{D}(f_{\hat{c}}^p(x), R)) \subset \mathbb{D}(x, R/2)$.* Indeed, we first take an $R_0 > 0$ such that $f_{\hat{c}}^p$ is univalent for any $\mathbb{D}(x, R_0)$ with $x \in \Omega(\hat{c})$. By the Koebe 1/4 theorem, $f_{\hat{c}}^p(\mathbb{D}(x, R_0))$

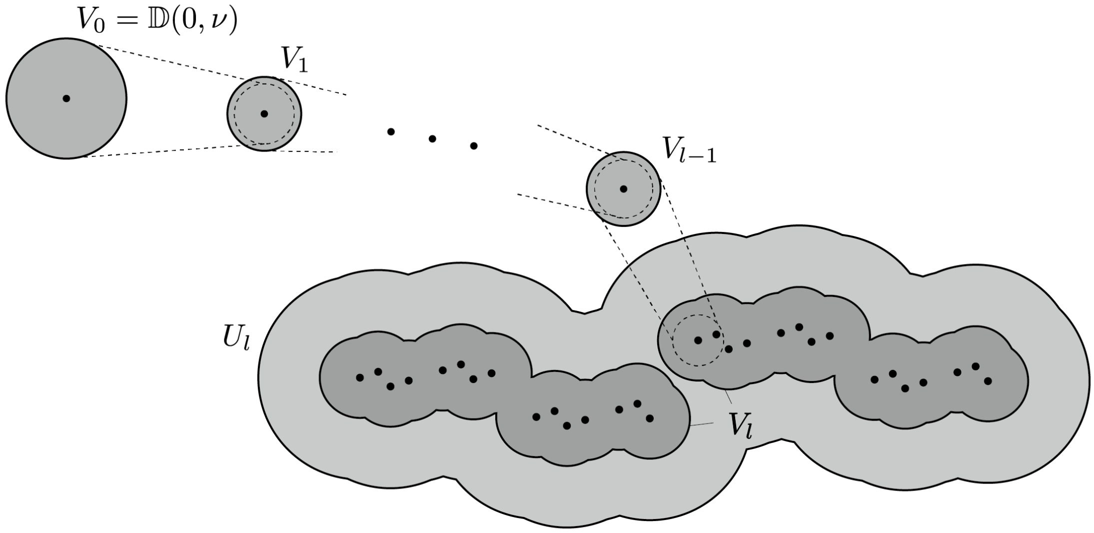

Figure 4: V_0, V_1, $\cdots$, V_l and U_l.

contains $\mathbb{D}(f_{\hat{c}}^p(x), R_0|Df_{\hat{c}}^p(x)|/4)$. Since $|Df_{\hat{c}}^p(x)| \ge 3$ on $\Omega(\hat{c})$, there is a univalent branch g of $f_{\hat{c}}^{-p}$ on $\mathbb{D}(f_{\hat{c}}^p(x), 3R_0/4)$ with $g(f_{\hat{c}}^p(x)) = x$ and $|Dg(f_{\hat{c}}^p(x))| \le 1/3$. The Koebe distortion theorem implies that g maps the disk $\mathbb{D}(f_{\hat{c}}^p(x), R)$ into $\mathbb{D}(x, R/2)$ by taking a sufficiently small $R < 3R_0/4$.

We assume that the R_l in the definition of U_l is relatively smaller than this R, and we will implicitly apply this type of argument to the backward dynamics of f_c near U_l for $c \approx \hat{c}$.

3 The derivative formula

Recall that the map $H : \mathbb{X} \times J(f_{c_0}) \to \mathbb{C}$ in Section 1 gives a holomorphic motion of the Julia set $J(f_{c_0})$ over the simply connected domain $\mathbb{X} = \mathbb{C} - \mathbb{M} \cup \mathbb{R}_+$ with the base point $c_0 \in \mathbb{X}$. For a given point $z_0 \in J(f_{c_0})$, we want to have some estimates for the derivative of the holomorphic function $z(c) = H(c, z_0)$ at $c \in \mathbb{X}$.

In fact, such a holomorphic motion always exists for any simply connected domain $\mathbb{Y}$ in $\mathbb{C} - \mathbb{M}$ with any base point $c_0 \in \mathbb{Y}$. For a given $c \in \mathbb{C} - \mathbb{M}$, the derivative of such a motion at c is independent of the choice of the domain $\mathbb{Y}$ containing c and the basepoint c_0. For example, it is convenient to consider the motion over the simply connected domain $\mathbb{Y} := \mathbb{C} - \mathbb{M} \cup \mathbb{R}_-$ (where $\mathbb{R}_-$ is the set of negative real numbers) and assume that $\mathbb{X}$ and $\mathbb{Y}$ share the base point $c_0 \in \mathbb{Y} \cap \mathbb{X} = \mathbb{C} - \mathbb{M} \cup \mathbb{R}$.

Now we prove:

Proposition 3.1. *For any $c \notin \mathbb{M}$ and $z = z(c) \in J(f_c)$, we have*

$$\left| \frac{d}{dc} z(c) \right| \le \frac{1 + \sqrt{1 + 6\,|c|}}{\mathrm{dist}\,(c, \partial\mathbb{M})}.$$

In particular, $|dz/dc| = O(1/\sqrt{|c|})$ as $c \to \infty$.

Proof. Let $\delta_c := \operatorname{dist}(c, \partial\mathbb{M})$ and $d_c := (1 + \sqrt{1 + 4|c|})/2$ for $c \in \mathbb{C}$. Let $s(z) := \sup_{n \geq 0} |f_c^n(z)|$ for $z \in J(f_c)$. Since $f_c^{n+1}(z) = (f_c^n(z))^2 + c$, we have $s(z) \geq s(z)^2 - |c|$ and this implies $s(z) \leq d_c$. Hence the Julia set $J(f_c)$ is contained in $\overline{\mathbb{D}(d_c)}$ for any $c \in \mathbb{C}$.

Now assume that $c \notin \mathbb{M}$. Then the disk $\mathbb{D}(c, \delta_c)$ is contained in either $\mathbb{X} = \mathbb{C} - \mathbb{M} \cup \mathbb{R}_+$ or $\mathbb{Y} = \mathbb{C} - \mathbb{M} \cup \mathbb{R}_-$, and the motion of $J(f_{c_0})$ restricted to this disk is well-defined. Let us consider a parameter $\zeta \in \mathbb{D}(c, \delta_c)$ such that $|\zeta - c| = \delta_c/2$. Since $\delta_c \leq |c|$, we have $|\zeta| \leq 3|c|/2$ and thus the Julia set $J(f_\zeta)$ is contained in $\overline{\mathbb{D}(d_{3|c|/2})}$. By applying the Cauchy integral formula, we obtain

$$\left|\frac{d}{dc}z(c)\right| = \left|\frac{1}{2\pi i}\int_{|\zeta - c| = \delta_c/2} \frac{z(\zeta)}{(\zeta - c)^2}\, d\zeta\right| \leq \frac{2\, d_{3|c|/2}}{\delta_c} = \frac{1 + \sqrt{1 + 6|c|}}{\operatorname{dist}(c, \partial\mathbb{M})}.$$

Since $\mathbb{M}$ is contained in $\overline{\mathbb{D}(2)}$, we have $|c| - 2 \leq \delta_c \leq |c|$. This implies $|dz/dc| = O(1/\sqrt{|c|})$ as $c \to \infty$

■

The derivative formula. Our main theorem is based on the following formula (see also [CKLY]):

Proposition 3.2 (The Derivative Formula)**.** *For any $c \notin \mathbb{M}$ and $z = z(c) \in J(f_c)$, we have*

$$\frac{d}{dc}z(c) = -\sum_{n=1}^{\infty} \frac{1}{Df_c^n(z(c))}.$$

Proof. Set $f := f_c$ and $z_n = z_n(c) := f^n(z(c))$. Then the relation $z_{n+1} = {z_n}^2 + c$ implies

$$\frac{dz_{n+1}}{dc} = 2z_n \cdot \frac{dz_n}{dc} + 1 \iff \frac{dz_n}{dc} = -\frac{1}{Df(z_n)} + \frac{1}{Df(z_n)}\frac{dz_{n+1}}{dc}.$$

Hence we have

$$\begin{aligned}
\frac{d}{dc}z(c) = \frac{dz_0}{dc} &= -\frac{1}{Df(z_0)} + \frac{1}{Df(z_0)}\frac{dz_1}{dc} \\
&= -\frac{1}{Df(z_0)} + \frac{1}{Df(z_0)}\left(-\frac{1}{Df(z_1)} + \frac{1}{Df(z_1)}\frac{dz_2}{dc}\right) \\
&= -\frac{1}{Df(z_0)} - \frac{1}{Df^2(z_0)} + \frac{1}{Df^2(z_0)}\frac{dz_2}{dc} \\
&= -\sum_{n=1}^{N} \frac{1}{Df^n(z(c))} + \frac{1}{Df^N(z_0)}\frac{dz_N}{dc}.
\end{aligned}$$

By letting $N \to \infty$ we formally have the desired formula. The series actually converges since $|dz_N/dc|$ is uniformly bounded by a constant depending only on c (by Proposition 3.1) and $|Df^N(z_0)|$ grows exponentially by hyperbolicity of $f = f_c$. ■

Remark 3.3.

- The estimate in Proposition 3.1 is valid for any $c \in \mathbb{C} - \partial\mathbb{M}$. Moreover, the derivative formula is also valid for any hyperbolic parameter in $\mathbb{M}$.

- Proposition 3.1 implies an estimate

$$\left|\frac{dz}{dc}(c)\right| = O\Big(|c-\hat{c}|^{-1-\beta}\Big)$$

if c approaches $\hat{c} \in \partial\mathbb{M}$ in such a way that

$$\mathrm{dist}(c, \partial\mathbb{M}) \geq C|c-\hat{c}|^{1+\beta}$$

for some constant $C > 0$. The smallest possible value that β can take is zero, for example, when $c \to \hat{c} = -2$ along the negative real axis. Typically β is positive, for example, $\beta = 1/2$ in the main theorem of [RL].

In general, when c approaches semi-hyperbolic $\hat{c} \in \partial\mathbb{M}$ along a parameter ray landing at $\hat{c}$, it satisfies $\mathrm{dist}(c, \partial\mathbb{M}) \geq C|c-\hat{c}|$ for some $C > 0$, and thus $\beta = 0$. (This is a combination of two facts: the John property of the complement of the Julia set $J(f_{\hat{c}})$ by [CJY] and the asymptotic similarity between $J(f_{\hat{c}})$ and $\mathbb{M}$ at $\hat{c}$ by [RL].) This observation implies that our main theorem is stronger and it does not come from the geometry of the Mandelbrot set. We need the dynamics (the derivative formula) to prove it.

4 Z-cycles

For $c \approx \hat{c}$, choose any $z = z_0 \in J(f_c)$. The orbit $z_n := f_c^n(z_0)$ $(n \in \mathbb{N}_0)$ may land on V_0 (or more precisely, on $V_0 \cap J(f_c)$), and go out, then it may come back again. To describe the behavior of such an orbit, we introduce the notion of "Z-cycle" for the orbit of z, where "Z" indicates that the orbit comes close to "zero".

We set $f := f_c$ for brevity.

Definition (Z-cycle). A *finite Z-cycle* of the orbit $z_n = f^n(z_0)$ $(n \in \mathbb{N}_0)$ is a finite subset of $\mathbb{N}_0$ of the form

$$\mathsf{Z} = \big\{n \in \mathbb{N}_0 \,:\, N \leq n < N'\big\} = [N, N') \cap \mathbb{N}_0,$$

such that z_N, $z_{N'} \in V_0$ but $z_n \notin V_0$ if $N < n < N'$. An *infinite Z-cycle* is an infinite subset of $\mathbb{N}_0$ of the form

$$\mathsf{Z} = \{n \in \mathbb{N}_0 \,:\, N \leq n < \infty\} = [N, \infty) \cap \mathbb{N}_0,$$

such that $z_N \in V_0$ but $z_n \notin V_0$ for all $n > N$. By a *Z-cycle* we mean a finite or infinite Z-cycle. In both cases, we denote them $\mathsf{Z} = [N, N')$ or $\mathsf{Z} = [N, \infty)$ for brevity.

Decomposition of the orbit by Z-cycles. For a given orbit $z_n = f^n(z_0)$ $(n \in \mathbb{N}_0)$ of $z_0 \in J(f_c)$, the set $\mathbb{N}_0$ of indices is uniquely decomposed by using finite or infinite Z-cycles in one of the following three types:

- The first type is of the form

$$\mathbb{N}_0 = [0, N_1) \sqcup \mathsf{Z}_1 \sqcup \mathsf{Z}_2 \sqcup \cdots, \tag{2}$$

where $z_n \notin V_0$ for $n \in [0, N_1)$ and $\mathsf{Z}_k := [N_k, N_{k+1})$ is a finite Z-cycle for each $k \geq 1$.

- The second type is of the form

$$\mathbb{N}_0 = [0, N_1) \sqcup \mathsf{Z}_1 \sqcup \mathsf{Z}_2 \sqcup \cdots \sqcup \mathsf{Z}_{k_0}, \tag{3}$$

where $k_0 \geq 1$ such that $z_n \notin V_0$ for $n \in [0, N_1)$; $\mathsf{Z}_k := [N_k, N_{k+1})$ is a finite Z-cycle for each $1 \leq k \leq k_0 - 1$; and $\mathsf{Z}_{k_0} = [N_{k_0}, \infty)$ is an infinite Z-cycle.

- The third type is just $\mathbb{N}_0 = [0, N_1)$ with $N_1 = \infty$, where $z_n \notin V_0$ for all $n \in \mathbb{N}$.

In the first and second types it is possible that $N_1 = 0$ and $[0, N_1)$ is empty. For the second and third types, we set $\mathsf{Z}_k := \emptyset$ for any $k \geq 1$ for which Z_k is not defined yet. Hence we always assume that $\mathbb{N}_0$ formally has an infinite decomposition of the form (2) associated with the orbit of $z_0 \in J(f_c)$.

The three lemmas. In what follows we assume the following "parameter ray condition" without (or with) mentioning:

> **"Parameter ray condition"**. *The parameter c is always in the parameter ray $\mathcal{R}_{\mathbb{M}}(\theta)$ that lands on $\hat{c}$.*

Now we present three principal lemmas about Z-cycle. (The proofs will be given later.)

Lemma A. *There exists a constant $K_{\mathrm{A}} > 0$ such that for any $c \approx \hat{c}$, any $z = z_0 \in J(f_c)$, and for any Z-cycle $\mathsf{Z} = [N, N')$ of the orbit $z_n = f_c^n(z)$ $(n \in \mathbb{N}_0)$, we have*

$$\sum_{i=1}^{N'-N} \frac{1}{|Df_c^i(z_N)|} \leq \frac{K_{\mathrm{A}}}{\sqrt{|c - \hat{c}|}}, \tag{4}$$

where we set $N' - N := \infty$ if $N' = \infty$.

Lemma B. *There exists a constant $K_{\mathrm{B}} > 0$ such that for any $c \approx \hat{c}$ and any $N \leq \infty$, if $z = z_0 \in J(f_c)$ satisfies $z_n \notin V_0$ for any $n \in [0, N)$, then we have*

$$\sum_{i=1}^{N} \frac{1}{|Df_c^i(z_0)|} \leq K_{\mathrm{B}}. \tag{5}$$

In fact, K_{B} depends only on the choices of $\hat{c}$ and ν. Hence we have:

Corollary 4.1. *For any $c \approx \hat{c}$ and any $z = z_0 \in J(f_c)$, if the orbit of z never lands on $V_0 = \mathbb{D}(\nu)$, then the derivative satisfies*

$$\left|\frac{dz}{dc}\right| \leq \sum_{n=1}^{\infty} \frac{1}{|Df_c^n(z_0)|} \leq K_{\mathrm{B}}. \tag{6}$$

Lemma C (Z-cycles Expand Uniformly)**.** *There exists a constant $\Lambda > 1$ such that for any $c \approx \hat{c}$, any $z = z_0 \in J(f_c)$, and for any finite Z-cycle $\mathsf{Z} = [N, N')$ of the orbit $z_n = f_c^n(z)$ $(n \in \mathbb{N}_0)$, we have*

$$|Df_c^{N'-N}(z_N)| \geq \Lambda. \tag{7}$$

This Λ also depends only on the choice of ν. Indeed, Λ is bounded by a constant compatible with ν^{-1}.

5 Proof of the main theorem assuming Lemmas A, B, and C

We will use the derivative formula (Proposition 3.2) and Lemmas A, B, and C to show the inequality.

For a given $c \approx \hat{c}$ and $z = z_0 \in J(f_c)$, we consider the decomposition $\mathbb{N}_0 = [0, N_1) \sqcup \mathsf{Z}_1 \sqcup \mathsf{Z}_2 \sqcup \cdots$ as in (2). Set $f := f_c$. Then we have

$$\left|\frac{dz}{dc}\right| \le \sum_{n=1}^{\infty} \frac{1}{|Df^n(z_0)|} = \sum_{n=1}^{N_1} \frac{1}{|Df^n(z_0)|} + \sum_{k\ge 1}\sum_{n\in \mathsf{Z}_k} \frac{1}{|Df^{n+1}(z_0)|}$$
$$= \sum_{n=1}^{N_1} \frac{1}{|Df^n(z_0)|} + \sum_{k\ge 1, \mathsf{Z}_k\neq\emptyset} \sum_{i=1}^{N_{k+1}-N_k} \frac{1}{|Df^{N_k}(z_0)|\,|Df^i(z_{N_k})|}.$$

By Lemma B, we obviously have $1/|Df^{N_1}(z_0)| \le K_{\mathrm{B}}$. By Lemma C, we have

$$|Df^{N_k}(z_0)| = |Df^{N_k - N_{k-1}}(z_{N_{k-1}})| \cdots |Df^{N_2-N_1}(z_{N_1})|\,|Df^{N_1}(z_0)| \ge \Lambda^{k-1}/K_{\mathrm{B}}$$

as long as $\mathsf{Z}_k \neq \emptyset$. Hence by Lemma A, we have

$$\sum_{n=1}^{\infty} \frac{1}{|Df^n(z_0)|} \le K_{\mathrm{B}} + \sum_{k\ge 1} \frac{K_{\mathrm{B}}}{\Lambda^{k-1}} \cdot \frac{K_{\mathrm{A}}}{\sqrt{|c-\hat{c}|}} = K_{\mathrm{B}} + \frac{K_{\mathrm{B}}\Lambda}{\Lambda - 1} \cdot \frac{K_{\mathrm{A}}}{\sqrt{|c-\hat{c}|}}.$$

We may assume that $|c - \hat{c}| \le 1$ such that $K_{\mathrm{B}} \le K_{\mathrm{B}}/\sqrt{|c-\hat{c}|}$. Hence by setting $K := K_{\mathrm{B}} + \dfrac{K_{\mathrm{B}}K_{\mathrm{A}}\Lambda}{\Lambda - 1}$, we have $\left|\dfrac{dz}{dc}\right| \le \dfrac{K}{\sqrt{|c-\hat{c}|}}$ for any $c \approx \hat{c}$.

■

6 S-cycles

To show Lemmas A, B, and C, we introduce the notion of "S-cycle".

For $c \approx \hat{c}$, set $f := f_c$ and choose any $z = z_0 \in J(f_c)$. The orbit $z_n := f^n(z_0)$ $(n \in \mathbb{N}_0)$ may land on $\mathcal{V}$. Unless it lands exactly on the hyperbolic set $\Omega(c)$, it will follow some orbit of $\Omega(c)$ for a while, and be repelled out of U_l eventually. Then it may come back to $\mathcal{V}$, or land on V_0. We define such a process as an "S-cycle", where "S" indicates that orbit stays near the "singularity" of the hyperbolic metric γ to be defined in Section 11, or the cycle is relatively "short" compared to Z-cycle.

Definition (S-cycle)**.** A *finite S-cycle* $\mathsf{S} = [M, M')$ of the orbit $z_n = f^n(z_0)$ $(n \in \mathbb{N}_0)$ is a finite subset of $\mathbb{N}_0$ with the following properties:

(S1) $z_M \in V_j \subset \mathcal{V}$ for some $j = 1, 2, \ldots, l$. If $M > 0$ then $z_{M-1} \notin \mathcal{V}$.

(S2) There exists a minimal $m \ge 1$ such that for $n = M + (l-j) + mp$, $z_{n-p} \in U_l$ but $z_n \notin U_l$.

(S3) $M' = M + (l-j) + mp + L$ for some $L \in [1, \infty)$ such that $z_n \notin V_0 \cup \mathcal{V}$ for $n = M + (l-j) + mp + i$ $(0 \le i < L)$ and $z_{M'} \in V_0 \cup \mathcal{V}$.

Note that in (S1), z_{M-1} may be contained in V_0. Note also that in (S2), some of $z_{n-p+1}, \cdots, z_{n-1}$ may *not* be contained in U_l.

An *infinite S-cycle* $\mathsf{S} = [M, \infty)$ of the orbit $z_n = f^n(z_0)$ $(n \in \mathbb{N}_0)$ is an infinite subset of $\mathbb{N}_0$ satisfying either

- Type (I): (S1), (S2), and

  (S3)' $z_n \notin V_0 \cup \mathcal{V}$ for all $n \geq M + (l-j) + mp$;

or

- Type (II): (S1) and

  (S2)' either $z_M = b_j(c)$ for $j < l$ or $z_M \in \Omega(c)$ for $j = l$. Equivalently, $z_n \in U_l$ for every $n = M + (l-j) + kp$ with $k \in \mathbb{N}$.

Decomposition of Z-cycles by S-cycles. Every Z-cycle $\mathsf{Z} = [N, N')$ ($N \leq \infty$) of the orbit $z_n = f^n(z_0)$ ($n \in \mathbb{N}_0$) has a unique decomposition by finite or infinite S-cycles.

For a finite Z-cycle $\mathsf{Z} = [N, N')$, there exists a finite decomposition

$$\mathsf{Z} = \{N\} \sqcup \mathsf{S}_1 \sqcup \mathsf{S}_2 \sqcup \cdots \sqcup \mathsf{S}_{k_0},$$

where $\mathsf{S}_k := [M_k, M_{k+1})$ is a finite S-cycle for each $k = 1, \cdots, k_0$ satisfying $N + 1 = M_1$ and $N' = M_{k_0+1}$.

For an infinite Z-cycle $\mathsf{Z} = [N, \infty)$, there exists either a finite decomposition

$$\mathsf{Z} = \{N\} \sqcup \mathsf{S}_1 \sqcup \mathsf{S}_2 \sqcup \cdots \sqcup \mathsf{S}_{k_0},$$

where $\mathsf{S}_k := [M_k, M_{k+1})$ is finite for $k = 1, \cdots, k_0 - 1$ but infinite for $k = k_0$; or an infinite decomposition

$$\mathsf{Z} = \{N\} \sqcup \mathsf{S}_1 \sqcup \mathsf{S}_2 \sqcup \cdots$$

where $\mathsf{S}_k := [M_k, M_{k+1})$ is finite for any $k \geq 1$.

When we have a finite decomposition $\mathsf{Z} = \{N\} \sqcup \mathsf{S}_1 \sqcup \mathsf{S}_2 \sqcup \cdots \sqcup \mathsf{S}_{k_0}$, we set $\mathsf{S}_k := \emptyset$ for $k > k_0$ and we assume that any Z-cycle formally has an infinite decomposition of the form $\mathsf{Z} = \{N\} \sqcup \mathsf{S}_1 \sqcup \mathsf{S}_2 \sqcup \cdots$. We call this *the S-cycle decomposition* of Z.

The three lemmas for S-cycles. Now we present three lemmas for S-cycles, that are parallel to Lemmas A, B, and C for Z-cycles:

Lemma A'. *There exists a constant $\kappa_{\mathrm{A}} > 0$ such that for any $c \approx \hat{c}$, any $z = z_0 \in J(f_c)$, and for any S-cycle $\mathsf{S} = [M, M')$ of the orbit $z_n = f_c^n(z)$ ($n \in \mathbb{N}_0$), we have*

$$\sum_{i=1}^{M'-M} \frac{1}{|Df_c^i(z_M)|} \leq \kappa_{\mathrm{A}}, \tag{8}$$

where we set $M' - M := \infty$ if $M' = \infty$.

Lemma B'. *There exists a constant $\kappa_{\mathrm{B}} > 0$ such that for any $c \approx \hat{c}$ and any $M \leq \infty$, if $z = z_0 \in J(f_c)$ satisfies $z_n \notin V_0 \cup \mathcal{V}$ for $n \in [0, M)$, then*

$$\sum_{i=1}^{M} \frac{1}{|Df_c^i(z_0)|} \leq \kappa_{\mathrm{B}}. \tag{9}$$

Lemma C' (S-cycles Expand Uniformly). *By choosing a sufficiently small ν, there exists a constant $\lambda > 1$ such that for any $c \approx \hat{c}$, any $z = z_0 \in J(f_c)$, and for any finite S-cycle $\mathsf{S} = [M, M')$ of the orbit $z_n = f_c^n(z)$ $(n \in \mathbb{N}_0)$, we have*

$$|Df_c^{M'-M}(z_M)| \geq \lambda. \tag{10}$$

The proofs of these lemmas will be given later.

7 Some lemmas concerning the parameter ray condition

This section is devoted for some lemmas related to the condition that c is always on the parameter ray $\mathcal{R}_{\mathbb{M}}(\theta)$ landing at $\hat{c}$ (the "parameter ray condition").

Dynamic rays for Cantor Julia sets. (See [CG, VIII, 3], [M, Appendix A].) For any parameter $c \in \mathbb{C}$, the *Böttcher coordinate* at infinity is a unique conformal map Φ_c defined near ∞ such that $\Phi_c(f_c(z)) = \Phi_c(z)^2$ and $\Phi_c(z)/z \to 1$ as $z \to \infty$. Let $K(f_c)$ be the set of z whose orbit is never captured in the domain of Φ_c. Then the boundary of $K(f_c)$ coincides with the Julia set $J(f_c)$.

When $c \in \mathbb{M}$, the set $K(f_c)$ is connected and the Böttcher coordinate extends to a conformal isomorphism $\Phi_c : \mathbb{C} - K(f_c) \to \mathbb{C} - \overline{\mathbb{D}}$. The *dynamic ray of angle* $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the analytic curve

$$\mathcal{R}_c(t) := \left\{\Phi_c^{-1}(re^{2\pi i t}) \,:\, r > 1\right\}.$$

We say that $\mathcal{R}_c(t)$ *lands at* $z \in K(f_c)$ if $\Phi_c^{-1}(re^{2\pi i t})$ tends to z as $r \searrow 1$.

When $c \notin \mathbb{M}$, the set $K(f_c)$ coincides with $J(f_c)$ which is a Cantor set. There exists a minimal $r_c > 1$ such that the inverse Φ_c^{-1} extends to a conformal embedding of $\mathbb{C} - \overline{\mathbb{D}(r_c)}$ into $\mathbb{C}$ whose image contains the critical value $c = f_c(0)$. (The Douady-Hubbard uniformization $\Phi_{\mathbb{M}} : \mathbb{C} - \mathbb{M} \to \mathbb{C} - \overline{\mathbb{D}}$ is given by setting $\Phi_{\mathbb{M}}(c) := \Phi_c(c)$.) The dynamic ray of angle $t \in \mathbb{T}$ is partially defined in $\Phi_c^{-1}(\mathbb{C} - \overline{\mathbb{D}(r_c)})$, and it extends to an analytic curve $\mathcal{R}_c(t)$ landing at a point in $K(f_c)$ unless $2^n t = t_c$ for some $n \geq 1$, where $t_c := (2\pi)^{-1} \arg \Phi_c(c)$.

Our setting and notation. Let us go back to our setting with semi-hyperbolic $\hat{c} \in \partial\mathbb{M}$ where $\mathcal{R}_{\mathbb{M}}(\theta)$ lands. We will use the following facts and notations:

- There is no interior point in $K(f_{\hat{c}})$ and thus $K(f_{\hat{c}}) = J(f_{\hat{c}})$. Moreover, $J(f_{\hat{c}})$ is connected and locally connected ([CJY]). By Carathéodory's theorem, $\Phi_{\hat{c}}^{-1}$ extends continuously to $\mathbb{C} - \mathbb{D}$ and the dynamic ray $\mathcal{R}_{\hat{c}}(t)$ of any angle t lands.
- The angle θ is not recurrent under the angle doubling $t \mapsto 2t$ ([D2, Thm.2]). Set

$$\Theta := \left\{2^{n+l-1}\theta \in \mathbb{T} \,:\, n \geq 0\right\}$$

  and let $\widehat{\Theta}$ denote its closure in $\mathbb{T}$, where l is the minimal l with $f_{\hat{c}}^{l-1}(\hat{c}) \in \Omega(\hat{c})$. For $t \in \widehat{\Theta}$ the dynamic ray $\mathcal{R}_{\hat{c}}(t)$ lands on a point in the hyperbolic set $\Omega(\hat{c})$. (See Step 1 of Proposition S below.) In particular, $\mathcal{R}_{\hat{c}}(2^{n+l-1}\theta)$ lands on $\hat{b}_{n+l} \in \Omega(\hat{c})$ for each $n \geq 0$.
- Let us fix an $r_0 > 1$ and consider the compact set

$$E_0 := \left\{re^{2\pi i t} \,:\, t \in \widehat{\Theta},\, r \in [r_0^{1/2^p}, r_0]\right\} \subset \mathbb{C} - \overline{\mathbb{D}}.$$

  By choosing r_0 close enough to 1, the set $E(\hat{c}) := \Phi_{\hat{c}}^{-1}(E_0)$ is contained in U_l.

- The parameter ray condition $c \in \mathcal{R}_{\mathbb{M}}(\theta)$ is equivalent to $c \in \mathcal{R}_c(\theta)$, or to $2\pi t_c = \arg \Phi_c(c) = 2\pi\theta$. Non-recurrence of θ assures that the dynamic rays $\mathcal{R}_c(t)$ with $t \in \widehat{\Theta}$ are always defined and land on the Julia set.

- Since the Böttcher coordinate $\Phi_c(z)$ is holomorphic in both c and z as long as it is defined, $E(c) := \Phi_c^{-1}(E_0)$ is well-defined for each $c \approx \hat{c}$ and also contained in U_l. More precisely, we choose the disk Δ in Proposition 2.2 small enough and assume that both $E(c)$ and $\Omega(c)$ moves holomorphically in U_l for any $c \in \Delta$.

Let us check the following proposition, that is interesting in its own right:

Proposition S (Stability of Landing Rays)**.** *For any $c \in \Delta$ (without assuming the parameter ray condition) and any $t \in \widehat{\Theta}$, the dynamic ray $\mathcal{R}_c(t)$ lands on a point in the hyperbolic set $\Omega(c)$ and $\mathcal{R}_c(t) \cap U_l$ has uniformly bounded length. In particular, $\mathcal{R}_c(2^{n+l-1}\theta)$ lands on $b_{n+l}(c) \in \Omega(c)$ for each $n \geq 0$. Moreover, the set*

$$\widehat{\mathcal{R}}(c) := \overline{\bigcup_{t \in \widehat{\Theta}} \mathcal{R}_c(t)} \subset \overline{\mathbb{C}}$$

moves continuously in the Hausdorff topology on the Riemann sphere as $c \to \hat{c}$.

Proof. The proof breaks into three steps.

Step 1. We first consider the case of $c = \hat{c}$. We claim: *For any angle $t \in \widehat{\Theta}$, the dynamic ray $\mathcal{R}_{\hat{c}}(t)$ lands on $\Omega(\hat{c})$ and $\mathcal{R}_{\hat{c}}(t) \cap U_l$ has uniformly bounded length.*

Let $x = x(t)$ denote the landing point of $\mathcal{R}_{\hat{c}}(t)$. By the Carathéodory theorem, $x(t)$ depends continuously on the angle t. Since $x(2^{l-1}\theta) = \hat{b}_l \in \Omega(\hat{c})$ and any angle $t \in \widehat{\Theta}$ is an accumulation point of the orbit of $2^{l-1}\theta$ by the angle doubling, we obtain $x(t) \in \Omega(\hat{c})$. (Note that $\Omega(\hat{c})$ is forward invariant and compact.)

Let us set $\mathcal{R} := \mathcal{R}_{\hat{c}}(t)$ and

$$\mathcal{R}(n) := \left\{ z \in \mathcal{R} \,:\, |\Phi_{\hat{c}}(z)|^{2^{np}} \in [r_0^{1/2^p}, r_0] \right\}$$

for $n \geq 0$ such that $f_{\hat{c}}^{np}(\mathcal{R}(n)) = f_{\hat{c}}^{np}(\mathcal{R}) \cap E(\hat{c})$ and the union

$$\mathcal{R}(0) \cup \mathcal{R}(1) \cup \mathcal{R}(2) \cup \cdots$$

coincides with the bounded arc $\mathcal{R} - \Phi_{\hat{c}}^{-1}(\{w \in \mathbb{C} \,:\, |w| > r_0\})$. Note that the arc $f_{\hat{c}}^{np}(\mathcal{R}(n)) \subset E(\hat{c}) \subset U_l$ has uniformly bounded length. By the Koebe distortion theorem and the condition $|Df_{\hat{c}}^p(z)| \geq \mu$ in U_l, we have

$$\mathrm{length}(\mathcal{R}(n)) = O(\mu^{-n}),$$

where the implicit constant is independent of the angle t. Hence the dynamic ray $\mathcal{R}$ has uniformly bounded length in U_l.

Step 2. Next we claim: *For any $c \approx \hat{c}$ and angle $t \in \widehat{\Theta}$, the dynamic ray $\mathcal{R}_c(t)$ lands on $\chi_c(x(t)) \in \Omega(c)$ and $\mathcal{R}_c(t) \cap U_l$ has uniformly bounded length.*

Set $\mathcal{R}' := \mathcal{R}_c(t)$ and

$$\mathcal{R}'(n) := \left\{ z \in \mathcal{R}' \,:\, |\Phi_c(z)|^{2^{np}} \in [r_0^{1/2^p}, r_0] \right\}$$

such that $f_c^{np}(\mathcal{R}'(n)) = f_c^{np}(\mathcal{R}') \cap E(c)$. We also set $x' := \chi_c(x)$ where $x = x(t)$ is the landing point of $\mathcal{R} = \mathcal{R}_{\hat{c}}(t)$ in $\Omega(\hat{c})$. Since $\Omega(c)$ and $E(c)$ move holomorphically in U_l with respect to $c \approx \hat{c}$, we may assume that the disk $D := \mathbb{D}(f_{\hat{c}}^{np}(x), R_l)$ contains the point $f_c^{np}(x') = \chi_c(f_{\hat{c}}^{np}(x))$ and the arcs $f_{\hat{c}}^{np}(\mathcal{R}(n))$ and $f_c^{np}(\mathcal{R}'(n))$. Since there exists a univalent branch g_c of f_c^{-np} defined on D such that it sends $f_c^{np}(x')$ to x' and $f_c^{np}(\mathcal{R}'(n))$ to $\mathcal{R}'(n)$, and since $|Df_c^p(z)| \geq \mu$ in U_l, we have

$$\mathrm{dist}\,(x', \mathcal{R}'(n)) = O(\mu^{-n}).$$

It follows that $\mathcal{R}' = \mathcal{R}_c(t)$ lands at $x' = \chi_c(x)$ and $\mathcal{R}' \cap U_l$ has uniformly bounded length independent of $c \approx \hat{c}$ and $t \in \widehat{\Theta}$.

Step 3. Finally we show the continuity of the set $\widehat{\mathcal{R}}(c)$. It is enough to show: *For any $c \approx \hat{c}$ there exists a homeomorphism $\phi_c : \widehat{\mathcal{R}}(\hat{c}) \to \widehat{\mathcal{R}}(c)$ such that $\phi_c \to \mathrm{id}$ uniformly as $c \to \hat{c}$ in the spherical metric.*

By Step 2, the homeomorphism ϕ_c is naturally defined by $\phi_c(\infty) = \infty$, $\phi_c := \chi_c$ on $\Omega(\hat{c})$, and $\phi_c := \Phi_c^{-1} \circ \Phi_{\hat{c}}$ on each ray $\mathcal{R}_{\hat{c}}(t)$ with $t \in \widehat{\Theta}$.

Now suppose that there exists an $\epsilon > 0$ such that for any $k \in \mathbb{N}$, we can find a pair of c_k and z_k such that $|c_k - \hat{c}| \leq 1/k$, $z_k \in \widehat{\mathcal{R}}(\hat{c})$, and the spherical distance between $\phi_{c_k}(z_k)$ and z_k exceeds ϵ. By taking a subsequence, we may assume that z_k has a limit $\zeta = \lim_{k\to\infty} z_k$ in $\widehat{\mathcal{R}}(\hat{c})$.

Since the map $\Phi_c^{-1}(w)$ is continuous in both c and w, the map ϕ_c converges to identity as $c \to \hat{c}$ locally uniformly near each point of $\widehat{\mathcal{R}}(\hat{c}) - \Omega(\hat{c}) \cup \{\infty\}$. The convergence of ϕ_c near ∞ is uniform as well in the spherical metric because Φ_c is tangent to identity near ∞. Hence the limit ζ above belongs to $\Omega(\hat{c})$.

Let $W(n)$ denote the bounded subset of $\widehat{\mathcal{R}}(\hat{c})$ given by

$$W(n) := \Omega(\hat{c}) \cup \bigcup_{t \in \widehat{\Theta}} \left\{ \Phi_{\hat{c}}^{-1}(re^{2\pi i t}) \,:\, r \leq r_0^{1/2^{np}} \right\}.$$

For any n, there exists a $k_n \in \mathbb{N}$ such that $z_k \in W(n)$ for any $k \geq k_n$. Now we define a point x_k in $\Omega(\hat{c})$ as follow: let $x_k := z_k$ if $z_k \in \Omega(\hat{c})$. Otherwise z_k belongs to a dynamic ray $\mathcal{R}_{\hat{c}}(t_k)$ for some $t_k \in \widehat{\Theta}$, and we let $x_k = x(t_k)$ be its landing point. Then we obtain

$$|\phi_{c_k}(z_k) - z_k| \leq |\phi_{c_k}(z_k) - \phi_{c_k}(x_k)| + |\phi_{c_k}(x_k) - x_k| + |x_k - z_k|,$$

where both $|\phi_{c_k}(z_k) - \phi_{c_k}(x_k)|$ and $|x_k - z_k|$ are $O(\mu^{-n})$ by Steps 1 and 2, and $|\phi_{c_k}(x_k) - x_k| = |\chi_{c_k}(x_k) - x_k| = O(|c_k - \hat{c}|) = O(1/k)$. (See [BR, Corollary 2].) Hence $|\phi_{c_k}(z_k) - z_k|$ is bounded by $\epsilon/2$ by taking sufficiently large n and k. This is a contradiction. ■

The next lemma will be used in the proof of Lemma A:

Lemma T. *Let $\hat{c} \in \partial\mathbb{M}$ be a semi-hyperbolic parameter. There exists a positive constant $C_{\mathrm{T}} = C_{\mathrm{T}}(\hat{c})$ such that* $\mathrm{dist}\,(0, J(f_c)) \geq C_{\mathrm{T}}\sqrt{|c - \hat{c}|}$ *for any $c \approx \hat{c}$ on the parameter ray $\mathcal{R}_{\mathbb{M}}(\theta)$ that lands at $\hat{c}$.*

Proof. Since $f_c(z) - f_c(0) = (z - 0)^2$, it is equivalent to show

$$\mathrm{dist}\,(c, J(f_c)) \geq C'_{\mathrm{T}}|c - \hat{c}|$$

for some constant $C'_{\mathrm{T}} = C_{\mathrm{T}}^2 > 0$ independent of $c \approx \hat{c}$ with $c \in \mathcal{R}_{\mathbb{M}}(\theta)$.

Set $a(c) := f_c^l(0)$ and $b(c) := b_l(c)$ for $c \approx \hat{c}$. Since f_c^{l-1} is univalent near c, we have

$$\mathrm{dist}\,(c, J(f_c)) \asymp \mathrm{dist}\,(a(c), J(f_c))$$

by the Koebe distortion theorem. By a result of Rivera-Letelier [RL, Appendix 2] and van Strien [vS, Theorem.1.1] (see also Douady and Hubbard [DH2, p.333, Lemma 1] for Misiurewicz case), there exists a constant $B_0 \neq 0$ such that

$$a(c) - b(c) = B_0(c - \hat{c}) + O((c - \hat{c})^2)$$

for $c \approx \hat{c}$. Hence it is enough to show that there exists a constant $C''_{\mathrm{T}} > 0$ such that

$$\mathrm{dist}\,(a(c), J(f_c)) \geq C''_{\mathrm{T}}|a(c) - b(c)| \tag{11}$$

for $c \approx \hat{c}$ with $c \in \mathcal{R}_{\mathbb{M}}(\theta)$.

For each $z \in E(c) = \Phi_c^{-1}(E_0)$ defined in the proof of Proposition S, there exists an angle $t \in \widehat{\Theta}$ such that $\arg \Phi_c(z) = 2\pi t$. By Proposition S, the external ray $R_c(t)$ lands on a point $L_c(z)$ in $\Omega(c)$. Now we define a constant $\Gamma(c)$ for each $c \approx \hat{c}$ by

$$\Gamma(c) := \inf \left\{ \frac{\mathrm{dist}\,(z, J(f_c))}{|z - L_c(z)|} \in (0, 1] \,:\, z \in E(c) \right\}$$

and claim that its infimum

$$\Gamma := \inf \{\Gamma(c) \,:\, c \in \Delta\}$$

is a positive constant if we choose sufficiently small disk Δ centered at $\hat{c}$. Indeed, if there exists a sequence $c_k \to \hat{c}$ in Δ such that $\Gamma(c_k) \to 0$, then we have $\mathrm{dist}\,(z_k, J(f_{c_k})) \to 0$ for some $z_k \in E(c_k)$. (Note that $|z - L_c(z)|$ is always bounded because $E(c)$ and $J(c)$ are uniformly bounded for $c \in \Delta$.) However, it is impossible because $E(c)$ and $J(f_c)$ move continuously at $c = \hat{c}$ and $E(\hat{c})$ has a definite distance from $J(f_{\hat{c}})$. Hence we obtain

$$\mathrm{dist}\,(z, J(f_c)) \geq \Gamma\, |z - L_c(z)|$$

for each $z \in E(c)$ and $c \in \Delta$.

Suppose that $c \in R_{\mathbb{M}}(\theta) \cap \Delta$ and $f_c^{np}(a(c)) \in E(c)$ for some $n \in \mathbb{N}_0$. Since $L_c(f_c^{np}(a(c))) = f_c^{np}(b(c))$, we have

$$\mathrm{dist}\,(f_c^{np}(a(c)), J(f_c)) \geq \Gamma\, |f_c^{np}(a(c)) - f_c^{np}(b(c))|.$$

By Proposition S, if we choose sufficiently small r_0, then the length of the arc in the dynamic ray joining any $z \in E(c)$ and $L_c(z) \in \Omega(c)$ is uniformly and arbitrarily small. Thus there exists a univalent branch of f_c^{-np} on the disk $\mathbb{D}(f_c^{np}(b(c)), 2R_l)$ that sends both $f_c^{np}(a(c))$ and $f_c^{np}(b(c))$ to $a(c)$ and $b(c)$ respectively. By the Koebe distortion theorem, we have (11). ∎

Remark 7.1. This proof is based on the argument to show that the basin at infinity of $f_{\hat{c}}$ is a John domain. See [CJY, §3] and [CG, p.118].

The next lemma will be used in the proof of Lemma C:

Lemma U. *There exists a constant $C_{\mathrm{U}} > 0$ with the following property: for any $c \approx \hat{c}$ with $c \in \mathcal{R}_{\mathbb{M}}(\theta)$ and any $z_0 \in V_0 \cap J(f_c)$ such that $z_{n-p} \in U_l$ and $z_n \notin U_l$, we have $|Df_c^n(z_0)| \geq C_{\mathrm{U}}/|z_0|$.*

Proof. By Lemma T (and its proof), we have $|z_0| \geq \operatorname{dist}(0, J(f_c)) \geq C_{\mathrm{T}}\sqrt{|c-\hat{c}|}$ and $|b_0(c)| \asymp \sqrt{|b_l(c) - f_c^{l-1}(c)|} \asymp \sqrt{|c-\hat{c}|}$. Hence we have $|z_0| \geq C_1|b_0(c)|$ for some constant $C_1 > 0$ and it follows that

$$|z_1 - b_1(c)| = |z_0^2 - b_0(c)^2| \leq C_2|z_0|^2$$

where $C_2 := 1 + C_1^2$.

Now $z_n \notin U_l$ means that $|z_n - \hat{b}_n| \geq \operatorname{dist}(z_n, \Omega(\hat{c})) \geq R_l$. Since $z_{n-p} \in U_l$, z_n is still close to $\Omega(\hat{c})$ and by taking a smaller R_l if necessary, we may assume that there exists an $R > R_l$ independent of $c \approx \hat{c}$ and $z_0 \in V_0 \cap J(f_c)$ such that $z_n \in \mathbb{D}(\hat{b}_n, R)$. Since we may assume that $|\hat{b}_n - b_n(c)| = |\hat{b}_n - \chi_c(\hat{b}_n)| \leq R_l/2$ for any $c \approx \hat{c}$, we have

$$|z_n - b_n(c)| \geq |z_n - \hat{b}_n| - |\hat{b}_n - b_n(c)| \geq R_l/2.$$

Let G be a univalent branch of $f_c^{-(n-1)}$ defined on $\mathbb{D}(\hat{b}_n, 2R)$ (by taking smaller R and R_l if necessary) that maps $b_n(c)$ to $b_1(c)$ and z_n to z_1. By the Koebe distortion theorem, we have

$$|DG(z_n)| \asymp |DG(b_n(c))|$$

and

$$|z_1 - b_1(c)| = |G(z_n) - G(b_n(c))| \asymp |DG(b_n(c))|\ |z_n - b_n(c)|.$$

Since $|z_1 - b_1(c)| \leq C_2|z_0|^2$ and $|z_n - b_n(c)| \geq R_l/2$, we have $|Df_c^{n-1}(z_1)| = |DG(z_n)|^{-1} \geq C_3/|z_0|^2$, where C_3 is a constant independent of $c \approx \hat{c}$. Hence we have

$$|Df_c^n(z_0)| = |Df_c^{n-1}(z_1)|\ |Df_c(z_0)| \geq \frac{C_3}{|z_0|^2} \cdot (2|z_0|) = \frac{2C_3}{|z_0|}.$$

Set $C_{\mathrm{U}} := 2C_3$. ■

Geometry of the parameter ray. The following lemma will be used in the proof of Theorem 1.2:

Lemma V. *Let $\hat{c} \in \partial\mathbb{M}$ be a semi-hyperbolic parameter and $\mathcal{R}_{\mathbb{M}}(\theta)$ a parameter ray landing on $\hat{c}$. Then the sequence $\{c_n\}_{n\geq 0}$ in $\mathcal{R}_{\mathbb{M}}(\theta)$ defined by*

$$c_n := \Phi_{\mathbb{M}}^{-1}\left(r_0^{1/2^{np}} e^{2\pi i\theta}\right)$$

satisfies the following properties:

(1) $|c_{n+k} - \hat{c}| = O(\mu^{-k})|c_n - \hat{c}|$ *for any n and $k \geq 0$.*

(2) *Let $\mathcal{R}_{\mathbb{M}}(n)$ be the subarc of $\mathcal{R}_{\mathbb{M}}(\theta)$ bounded by c_n and c_{n+1}. Then*

$$|c_{n+1} - c_n| \asymp \operatorname{length}(\mathcal{R}_{\mathbb{M}}(n)) = O(\mu^{-n}).$$

In particular, $\mathcal{R}_{\mathbb{M}}(\theta)$ has finite length in a neighborhood of $\hat{c}$.

Proof. By a result by Rivera-Letelier [RL], there exists a constant $\hat{\lambda} \neq 0$ such that $\Psi := \Phi_{\mathbb{M}}^{-1} \circ \Phi_{\hat{c}} : \mathbb{C} - J(f_{\hat{c}}) \to \mathbb{C} - \mathbb{M}$ is of the form

$$\Psi(z) = \hat{c} + \hat{\lambda}(z - \hat{c}) + O(|z - \hat{c}|^{3/2})$$

when $z \in \mathbb{C} - J(f_{\hat{c}})$ and $z \approx \hat{c}$. In particular, Ψ maps the dynamic ray $\mathcal{R}_{\hat{c}}(\theta)$ to the parameter ray $\mathcal{R}_{\mathbb{M}}(\theta)$ conformally near the landing point $\hat{c}$. Hence it is enough to check that *the points*

$$z_n := \Psi^{-1}(c_n) = \Phi_{\hat{c}}^{-1}\left(r_0^{1/2^{np}} e^{2\pi i\theta}\right)$$

satisfies

(1') $|z_{n+k} - \hat{c}| = O(\mu^{-k})|z_n - \hat{c}|$ *for* $k \geq 0$*; and*

(2') *the length of the subarc of* $\mathcal{R}_{\hat{c}}(\theta)$ *bounded by* z_n *and* z_{n+1} *is compatible with* $|z_{n+1} - z_n|$ *and is* $O(\mu^{-n})$

for sufficiently large n.

For each $t \in \widehat{\Theta}$ and $n \geq 0$, set $z_n(t) := \Phi_{\hat{c}}^{-1}\left(r_0^{1/2^{np}} e^{2\pi i t}\right)$ such that the sequence $\{z_n(t)\}_{n\geq 0}$ converges along the external ray $\mathcal{R}_{\hat{c}}(t)$ to the landing point $x(t)$. Note that $z_0(t)$ and $z_1(t)$ bound the arc $\mathcal{R}_{\hat{c}}(t) \cap E(\hat{c})$. Since $E(\hat{c})$ and $\widehat{\Theta}$ are compact, we have

(a) $|z_0(t) - x(t)| \asymp 1$; and

(b) $|z_0(t) - z_1(t)| \asymp \text{length}(\mathcal{R}_{\hat{c}}(t) \cap E(\hat{c}))$,

where the implicit constants are independent of $t \in \widehat{\Theta}$.

Now suppose that n is large enough such that $np \geq l - 1$ and thus $t_n := 2^{np}\theta \in \widehat{\Theta}$. Then we can find a univalent branch of $f_{\hat{c}}^{-np}$ defined on a disk centered at $x(t_n)(= \hat{b}_{np+1})$ with a definite radius independent of n that maps $z_0(t_n)$, $z_k(t_n)$ and $x(t_n)$ univalently to z_n, z_{n+k} and $\hat{c}$ respectively. By the Koebe distortion theorem and (a) we have

$$\frac{|z_{n+k} - \hat{c}|}{|z_n - \hat{c}|} \asymp \frac{|z_k(t_n) - x(t_n)|}{|z_0(t_n) - x(t_n)|} \asymp |z_k(t_n) - x(t_n)|.$$

We can find a univalent inverse branch G_k of $f_{\hat{c}}^{kp}$ defined on a disk centered at $x(t_{n+k})(= \hat{b}_{(n+k)p+1})$ with a definite radius independent of n and k that maps $z_0(t_{n+k})$ and $x(t_{n+k})$ univalently to $z_k(t_n)$ and $x(t_n)$. Hence by Koebe again we have

$$|z_k(t_n) - x(t_n)| \asymp |DG_k(x(t_{n+k}))|\, |z_0(t_{n+k}) - x(t_{n+k})| = O(\mu^{-k}).$$

It follows that $|z_{n+k} - \hat{c}| = O(\mu^{-k})|z_n - \hat{c}|$ and we obtain (1').

By (b) and the same argument as above, the length of the subarc of $\mathcal{R}_{\hat{c}}(\theta)$ bounded by z_n and z_{n+1} is uniformly compatible with $|z_{n+1} - z_n|$ for any $n \geq 0$. As a corollary of Step 1 of Proposition S, we conclude that the length is $O(\mu^{-n})$. Thus we obtain (2'). ■

Remark 7.2. Since there exist at most finitely many dynamic rays of the Julia set $J(f_{\hat{c}})$ landing at $\hat{c}$ (see Thurston [Th, Theorem II.5.2] or Kiwi [K1, Theorem 1.1]), the asymptotic similarity between $J(f_{\hat{c}})$ and $\mathbb{M}$ at $\hat{c}$ by Rivera-Letelier [RL] implies that $\mathbb{M}$ has the same finite number of parameter rays landing at $\hat{c}$. (cf. [CG, VIII, 6]. See also [Mc, Chapter 6].)

8 Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. We combine the Main Theorem and Lemma V. It is enough to show the existence of the improper integral

$$z(c(r_0)) + \lim_{\delta\to+0}\int_{r_0}^{1+\delta}\frac{dz(c)}{dc}\frac{dc(r)}{dr}\,dr = z(c(r_0)) + \sum_{n\geq 0}\int_{\mathcal{R}_{\mathbb{M}}(n)}\frac{dz(c)}{dc}\,dc,$$

where $r_0 > 1$ is a constant given in the definition of the set E_0 in the previous section, and $\mathcal{R}_{\mathbb{M}}(n)$ is the subarc of $\mathcal{R}_{\mathbb{M}}(\theta)$ bounded by c_n and c_{n+1} defined in Lemma V. Note that by Lemma V, we obtain

$$\text{length}\mathcal{R}_{\mathbb{M}}(n) \asymp |c_{n+1}-c_n| \leq |c_{n+1}-\hat{c}| + |c_n-\hat{c}| = O(|c_n-\hat{c}|)$$

and

$$|c_n-\hat{c}| \leq \sum_{m\geq n}\text{length}\mathcal{R}_{\mathbb{M}}(m) = O(\mu^{-n}).$$

Note also that

$$|c_n-\hat{c}| \asymp |c-\hat{c}| \tag{12}$$

for any $c \in \mathcal{R}_{\mathbb{M}}(n)$, where the implicit constant is independent of n by the Koebe distortion theorem, applied in the same way as the proof of Lemma V.

By the Main Theorem we obtain

$$\begin{aligned}\sum_{n\geq 0}\int_{\mathcal{R}_{\mathbb{M}}(n)}\left|\frac{dz(c)}{dc}\right||dc| &\leq \sum_{n\geq 0}\int_{\mathcal{R}_{\mathbb{M}}(n)}\frac{K}{\sqrt{|c-\hat{c}|}}\,|dc| \\ &\asymp \sum_{n\geq 0}\frac{K}{\sqrt{|c_n-\hat{c}|}}\,\text{length}\mathcal{R}_{\mathbb{M}}(n) \\ &= \sum_{n\geq 0} O\left(\frac{1}{\sqrt{|c_n-\hat{c}|}}\,|c_n-\hat{c}|\right) \\ &= \sum_{n\geq 0} O(\mu^{-n/2}) < \infty.\end{aligned}$$

Hence the improper integral above converges absolutely to some $z(\hat{c})$.

To show the one-sided Hölder continuity, it is enough to check $|z(c_n)-z(\hat{c})| = O(\sqrt{|c_n-\hat{c}|})$ for each c_n by (12). The same argument as above yields

$$|z(c_n)-z(\hat{c})| \leq \sum_{k\geq 0}\int_{\mathcal{R}_{\mathbb{M}}(n+k)}\left|\frac{dz(c)}{dc}\right||dc| \leq \sum_{k\geq 0} O(\sqrt{|c_{n+k}-\hat{c}|}).$$

By (1) of Lemma V, we have $|c_{n+k}-\hat{c}| = O(\mu^{-k})|c_n-\hat{c}|$ for each $k \geq 0$ and thus $|z(c_n)-z(\hat{c})| = \sum_{k\geq 0} O(\mu^{-k/2})\sqrt{|c_n-\hat{c}|} = O(\sqrt{|c_n-\hat{c}|})$.

Since it is clear that $z(\hat{c})$ is confined in a bounded region, to show $z(\hat{c}) \in J(f_{\hat{c}})$, we only need to show $\lim_{c\to\hat{c}}(z(c)^2+c) = (\lim_{c\to\hat{c}} z(c))^2 + \lim_{c\to\hat{c}} c$, but this follows from the continuity of the quadratic map. ∎

Proof of Theorem 1.3 For each $z_0 \in J(f_{c_0})$ and its motion $z(c) = h_c(z_0) = H(c, z_0)$ along the parameter ray $\mathcal{R}_{\mathbb{M}}(\theta)$, we define $h_{\hat{c}}(z_0)$ by the limit $z(\hat{c})$ given in Theorem 1.2. Since h_c is continuous and the convergence of h_c to $h_{\hat{c}}$ as $c \to \hat{c}$ along the parameter ray $\mathcal{R}_{\mathbb{M}}(\theta)$ is uniform, $h_{\hat{c}}$ is continuous as well. Hence $f_{\hat{c}} \circ h_{\hat{c}} = h_{\hat{c}} \circ f_{c_0}$ is obvious and it is enough to show the surjectivity of $h_{\hat{c}} : J(f_{c_0}) \to J(f_{\hat{c}})$. First we take any repelling periodic point $x \in J(f_{\hat{c}})$. Since there is a holomorphic family $x(c)$ of repelling periodic points for c sufficiently close to $\hat{c}$ such that $x = x(\hat{c})$, we have some $z_0 \in J(f_{c_0})$ with $h_c(z_0) = x(c)$ for any $c \approx \hat{c}$ with $c \in \mathcal{R}_{\mathbb{M}}(\theta)$. In particular, we have $h_{\hat{c}}(z_0) = x$. Next we take any $w \in J(f_{\hat{c}})$ and a sequence of repelling periodic points x_n of $f_{\hat{c}}$ that converges to w as $n \to \infty$. (Such a sequence exists since repelling periodic points are dense in the Julia set.) Let $z_n \in J(f_{c_0})$ be the repelling periodic point with $h_{\hat{c}}(z_n) = x_n$. Then any accumulation point y of the sequence z_n satisfies $h_{\hat{c}}(y) = w$ by continuity. ■

9 Proof of Lemma A assuming Lemmas A' and C'

Without loss of generality we may assume that $N = 0$, i.e., $z = z_0 \in V_0 \cap J(f_c)$. We set $f := f_c$. Now consider the S-cycle decomposition $\mathsf{Z} = \{0\} \sqcup \mathsf{S}_1 \sqcup \mathsf{S}_2 \sqcup \cdots$ of $\mathsf{Z} = [0, N')$ where $\mathsf{S}_k = [M_k, M_{k+1})$ if $\mathsf{S}_k \neq \emptyset$, and $M_1 = 1$. Then we have

$$\begin{aligned}
\sum_{i=1}^{N'} \frac{1}{|Df^i(z)|} &= \frac{1}{|Df(z)|} + \sum_{k \geq 1} \sum_{n \in \mathsf{S}_k} \frac{1}{|Df^{n+1}(z)|} \\
&= \frac{1}{2|z|} + \sum_{k \geq 1,\, \mathsf{S}_k \neq \emptyset} \sum_{i=1}^{M_{k+1}-M_k} \frac{1}{|Df^i(z_{M_k})|\,|Df^{M_k}(z)|} \\
&\leq \frac{1}{2|z|} + \sum_{k \geq 1,\, \mathsf{S}_k \neq \emptyset} \frac{\kappa_{\mathrm{A}}}{|Df^{M_k}(z)|}
\end{aligned}$$

by Lemma A'. If $\mathsf{S}_k \neq \emptyset$, then by Lemma C',

$$|Df^{M_k}(z)| = |Df^{M_k - M_{k-1}}(z_{M_{k-1}})| \cdots |Df^{M_2 - M_1}(z_{M_1})|\ |Df(z)| \geq \lambda^{k-1} \cdot 2|z|,$$

where $M_1 = 1$. Hence we have $|Df^{M_k}(z)|^{-1} \leq 1/(\lambda^{k-1} \cdot 2|z|)$ for any k. Moreover, by Lemma T, we have $\mathrm{dist}\,(0, J(f_c)) \geq C_{\mathrm{T}}\sqrt{|c - \hat{c}|}$ for $c \approx \hat{c}$ on the parameter ray, and thus

$$\begin{aligned}
\sum_{i=1}^{N'} \frac{1}{|Df^i(z)|} &\leq \frac{1}{2|z|} + \sum_{k=1}^{\infty} \frac{\kappa_{\mathrm{A}}}{\lambda^{k-1} \cdot (2|z|)} \\
&\leq \frac{1}{2 \cdot \mathrm{dist}\,(0, J(f_c))} \left\{ 1 + \sum_{k=1}^{\infty} \frac{\kappa_{\mathrm{A}}}{\lambda^{k-1}} \right\} \\
&\leq \frac{1}{2C_{\mathrm{T}}\sqrt{|c - \hat{c}|}} \left\{ 1 + \kappa_{\mathrm{A}} \frac{\lambda}{\lambda - 1} \right\}.
\end{aligned}$$

Hence by setting $K_{\mathrm{A}} := (2C_{\mathrm{T}})^{-1}\{1 + \kappa_{\mathrm{A}}\lambda/(\lambda - 1)\}$, we have the claim. ■

10 Proof of Lemma B assuming Lemmas A', B' and C'

Just like the S-cycle decompositions of Z-cycles, we have a finite or infinite decomposition of the form

$$[0, N) = [0, M_1) \sqcup \mathsf{S}_1 \sqcup \mathsf{S}_2 \sqcup \cdots$$

where we have the following three cases:

1. $N = M_1 \le \infty$ and $z_n \notin V_0 \cup \mathcal{V}$ for any $0 \le n < M_1$. Hence $\mathsf{S}_k = \emptyset$ for all $k \in \mathbb{N}$.

2. $z_n \notin V_0 \cup \mathcal{V}$ for $0 \le n < M_1$, and there exists a $k_0 \in \mathbb{N}$ such that $\mathsf{S}_k := [M_k, M_{k+1})$ is an S-cycle for each $k \le k_0$ and $\mathsf{S}_k = \emptyset$ for all $k > k_0$.

3. $z_n \notin V_0 \cup \mathcal{V}$ for $0 \le n < M_1$, and $\mathsf{S}_k := [M_k, M_{k+1})$ is a finite S-cycle for any $k \in \mathbb{N}$.

Set $f = f_c$. For all cases, we have

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{|Df^n(z)|} &= \sum_{n=1}^{M_1} \frac{1}{|Df^n(z)|} + \sum_{k \ge 1, \mathsf{S}_k \neq \emptyset} \sum_{i=1}^{M_{k+1}-M_k} \frac{1}{|Df^i(z_{M_k})|\,|Df^{M_k}(z)|} \\ &\le \kappa_{\mathrm{B}} + \sum_{k \ge 1, \mathsf{S}_k \neq \emptyset} \frac{\kappa_{\mathrm{A}}}{|Df^{M_k}(z)|}\end{aligned}$$

by Lemmas A' and B'. By Lemma B' again, we obviously have $|Df^{M_1}(z)|^{-1} < \kappa_{\mathrm{B}}$. Hence by Lemma C', we have

$$|Df^{M_k}(z)| = |Df^{M_k - M_{k-1}}(z_{M_{k-1}})| \cdots |Df^{M_2 - M_1}(z_{M_1})|\,|Df^{M_1}(z)| \ge \lambda^{k-1}/\kappa_{\mathrm{B}}.$$

Hence we have

$$\sum_{n=1}^{N} \frac{1}{|Df^n(z)|} \le \kappa_{\mathrm{B}} + \sum_{k \ge 1} \frac{\kappa_{\mathrm{A}}\kappa_{\mathrm{B}}}{\lambda^{k-1}} < \kappa_{\mathrm{B}} + \kappa_{\mathrm{A}}\kappa_{\mathrm{B}} \frac{\lambda}{\lambda - 1} =: K_{\mathrm{B}}.$$

■

11 Hyperbolic metrics

For the proofs of Lemmas A', B', C' and C, we will use the hyperbolic metrics and the expansion of f_c with respect to these metrics.

For a domain Ω in $\mathbb{C}$ with $\#(\mathbb{C} - \Omega) \ge 2$, there exists a *hyperbolic metric* $\rho(z)|dz|$ on Ω of constant curvature -4 induced by the metric $|dz|/(1 - |z|^2)$ on the universal covering $\mathbb{D} = \widetilde{\Omega}$. We first recall the following standard fact:

Lemma W. *Let Ω_0 be a domain in $\mathbb{C}$ with $\#(\mathbb{C} - \Omega_0) \ge 2$ and $\rho_0(z)|dz|$ be its hyperbolic metric. Then for any domain $\Omega \subset \Omega_0$, the hyperbolic metric $\rho(z)|dz|$ of Ω satisfies*

$$\rho_0(z) \le \rho(z) \le \frac{1}{\operatorname{dist}(z, \partial\Omega)},$$

where $\operatorname{dist}(z, \partial\Omega)$ *is the Euclidean distance between z and $\partial\Omega$.*

See [Ah, Theorems 1.10 & 1.11] for more details.

Postcritical sets. The *postcritical set* $P(f_c)$ of the polynomial $f_c(z) = z^2 + c$ is defined by

$$P(f_c) := \overline{\{f_c(0),\, f_c^2(0),\, f_c^3(0),\, \cdots\}}.$$

For example, we have

$$P(f_{\hat{c}}) = \{\hat{b}_1,\, \hat{b}_2,\, \cdots,\, \hat{b}_{l-1}\} \cup \Omega(\hat{c})$$

when $c = \hat{c}$ and this set is finite if $\hat{c}$ is a Misiurewicz point. Moreover, for any $c \approx \hat{c}$, we have $\sharp P(f_c) \geq 2$ and the universal covering of (each component of) $\mathbb{C} - P(f_c)$ is the unit disk [2].

Let $\gamma = \gamma(z)|dz|$ denote the hyperbolic metric of $\mathbb{C} - P(f_{\hat{c}})$, which is induced by the metric $|dz|/(1 - |z|^2)$ on the unit disk $\mathbb{D}$. The metric $\gamma = \gamma(z)|dz|$ has the following properties:

(i) $\gamma : \mathbb{C} - P(f_{\hat{c}}) \to \mathbb{R}_+$ is real analytic and diverges on $P(f_{\hat{c}}) \cup \{\infty\}$.

(ii) if both z and $f_{\hat{c}}(z)$ are in $\mathbb{C} - P(f_{\hat{c}})$, we have

$$\frac{\gamma(f_{\hat{c}}(z))}{\gamma(z)} |Df_{\hat{c}}(z)| > 1.$$

Lemma X. *If the constant ν is sufficiently small, there exists a constant $C_{\mathrm{X}} \asymp \nu^2$ with the following property: For any $c \approx \hat{c}$, we have*

$$\frac{\gamma(z)}{\gamma(\zeta)} \geq C_{\mathrm{X}}$$

if either

(1) $z, \zeta \in J(f_c) - \mathcal{V}$*; or*

(2) $z \in J(f_c) - V_0 \cup \mathcal{V}$ *and* $\zeta \in V_1 - f_c(V_0)$.

Proof. We may assume that there exists an $R_0 > 0$ such that $J(f_c) \subset \overline{\mathbb{D}(R_0)}$ for any $c \approx \hat{c}$. Since γ diverges only at the postcritical set $P(f_{\hat{c}})$ in $\overline{\mathbb{D}(R_0)}$, there exists a constant $C_4 > 0$ such that $\gamma(w) \geq C_4$ for any $w \in \overline{\mathbb{D}(2R_0)} - P(f_{\hat{c}})$. In particular, we have $\gamma(z) \geq C_4$ in both cases (1) and (2). Moreover, for these cases, we can find a constant C_5 independent of $\nu \ll 1$ and $c \approx \hat{c}$ such that

$$\mathrm{dist}\,(\zeta, P(f_{\hat{c}})) \geq C_5 \nu^2.$$

Hence if ν is sufficiently small, then Lemma W implies that that $\gamma(\zeta) \leq 1/(C_5\nu^2)$. Now we have $\gamma(z)/\gamma(\zeta) \geq C_4 C_5 \nu^2 =: C_{\mathrm{X}}$. ■

Lemma Y. *There exists a constant $A > 1$ such that for $c \approx \hat{c}$, if $z, f_c(z), \ldots, f_c^n(z)$ are all contained in $J(f_c) - \mathcal{V}$, we have*

$$|Df_c^n(z)| \geq C_{\mathrm{X}} A^n.$$

This estimate also holds if $z, f_c(z), \ldots, f_c^{n-1}(z)$ are all contained in $J(f_c) - V_0 \cup \mathcal{V}$ and $f_c^n(z) \in V_1 - f_c(V_0)$.

[2] Without the parameter ray condition, f_c may have Siegel disks and the set $\mathbb{C} - P(f_c)$ may contain the disks.

Proof. Since the Julia set is uniformly bounded when $c \approx \hat{c}$, we may assume that there exists a constant $A > 1$ such that for any $c \approx \hat{c}$,

$$\frac{\gamma(f_c(w))}{\gamma(w)} |Df_c(w)| \geq A$$

if either $w,\ f_c(w) \in J(f_c) - \mathcal{V}$; or $w \in J(f_c) - \mathcal{V} \cup V_0$ and $f_c(w) \in V_1 - f_c(V_0)$.

By the chain rule, we have

$$|Df_c^n(z)| = \prod_{i=0}^{n-1} |Df_c(f_c^i(z))| \geq \prod_{i=0}^{n-1} \frac{\gamma(f_c^i(z))}{\gamma(f_c^{i+1}(z))} A \geq \frac{\gamma(z)}{\gamma(f_c^n(z))} A^n.$$

By applying Lemma X with $\zeta := f_c^n(z)$, we obtain the desired inequality. ■

12 Proof of Lemma B'

Set $f = f_c$. Suppose that $M < \infty$. Since we have $z_i \notin V_0 \cup \mathcal{V}$ for all $i \leq M - 1$, we can apply Lemma Y and we have

$$|Df^i(z_0)| \geq \frac{\gamma(z_0)}{\gamma(z_i)} \cdot A^i \geq C_{\mathrm{X}} A^i.$$

If $z_M \notin V_0 \cup \mathcal{V}$ or $z_M \notin V_1 - f_c(V_0)$, then we can apply Lemma Y again and we have $|Df^M(z_0)| \geq C_{\mathrm{X}} A^M \geq C_{\mathrm{X}}$. Otherwise $z_M \in V_j$ for some $j \neq 1$. Since $z_{M-1} \notin V_0 \cup \mathcal{V}$, we may assume that $|z_{M-1}| \geq \xi_0$ for some constant $0 < \xi_0 \leq 1/2$ depending only on $\hat{c}$ and independent of $\nu \ll 1$, $c \approx \hat{c}$, and $z_0 \in J(f_c)$. Hence we have

$$|Df^M(z_0)| = |Df^{M-1}(z_{M-1})|\ |Df(z_{M-1})| \geq C_{\mathrm{X}} A^{M-1} \cdot 2\xi_0 \geq 2\xi_0 C_{\mathrm{X}}.$$

Thus

$$\sum_{i=1}^{M} \frac{1}{|Df^i(z_0)|} \leq \sum_{i=1}^{M-1} \frac{1}{C_{\mathrm{X}} A^i} + \frac{1}{2\xi_0 C_{\mathrm{X}}} < \frac{1}{C_{\mathrm{X}}} \left(\frac{1}{A-1} + \frac{1}{2\xi_0} \right) =: \kappa_{\mathrm{B}}.$$

If $M = \infty$, then the same estimate as above yields

$$\sum_{i=1}^{\infty} \frac{1}{|Df^i(z_0)|} \leq \sum_{i=1}^{\infty} \frac{1}{C_{\mathrm{X}} A^i} < \frac{1}{C_{\mathrm{X}}(A-1)} < \kappa_{\mathrm{B}}.$$

■

13 Proof of Lemma A'

Set $f = f_c$. For a given S-cycle $\mathsf{S} = [M, M')$, we may assume that $M = 0$ without loss of generality. We divide the proof in two cases.

Case 1. Suppose that S is either a finite S-cycle or an infinite S-cycle of type (I). Then there exist $j \in \{1, 2, \cdots, l\}$, $m \in \mathbb{N}$, and $L \in \mathbb{N} \cup \{\infty\}$ such that

- $z = z_0 \in V_j$;
- $z_{n-p} \in U_l$ when $n = (l - j) + mp$, but $z_n \notin U_l$;
- $z_{n+i} \notin V_0 \cup \mathcal{V}$ if $0 \leq i < L$.

- $M' < \infty$ iff $L < \infty$ and $M' = (l-j) + mp + L$.

Hence we have the following estimates of $|Df^n(z)|$:

- When $n = 1, \cdots, l-j-1$, we have $z_n \in V_{j+n}$ and
$$|Df^n(z)| \geq \xi^n \geq \xi^{l-1}.$$
- When $n = (l-j) + kp + i$ with $0 \leq k < m$ and $0 \leq i < p$,
$$\begin{aligned}|Df^n(z)| &= |Df^{l-j}(z)|\ |Df^{kp}(z_{l-j})|\ |Df^i(z_{(l-j)+kp})| \\ &\geq \xi^{l-j} \cdot \mu^k \cdot \xi^i \\ &\geq \xi^{(l-1)+(p-1)} \mu^k.\end{aligned}$$
- When $n = (l-j) + mp + i$ with $0 \leq i < L \leq \infty$,
$$\begin{aligned}|Df^n(z)| &= |Df^{(l-j)+mp}(z)|\ |Df^i(z_{(l-j)+mp})| \\ &\geq \xi^{l-j} \cdot \mu^m \cdot \frac{\gamma(z_{(l-j)+mp})}{\gamma(z_n)} \cdot A^i \\ &\geq \xi^{l-1}\, C_{\mathrm{X}}\, A^i.\end{aligned}$$
Here the constant A above is the same as that of Lemma Y.
- When $L < \infty$ and $n = M' = (l-j) + mp + L$, the point $z_{M'}$ satisfies either $z_{M'} \in V_1 - f_c(V_0)$; or $z_{M'} \in V_j$ for some $j \neq 1$. By the same argument as in the proof of Lemma B', there exists a constant $0 < \xi_0 \leq 1/2$ depending only on $\hat{c}$ such that
$$\begin{aligned}|Df^n(z)| = |Df^{M'}(z)| &= |Df^{(l-j)+mp}(z)|\ |Df^L(z_{(l-j)+mp})| &(13)\\ &\geq \xi^{l-j} \cdot \mu^m \cdot \min\{C_{\mathrm{X}}\, A^L,\ C_{\mathrm{X}}\, A^{L-1} \cdot 2\xi_0\} \\ &\geq 2\xi^{l-1}\, \xi_0\, C_{\mathrm{X}} &(14)\end{aligned}$$

By these estimates, when $M' < \infty$, we have:

$$\begin{aligned}&\sum_{i=1}^{M'} \frac{1}{|Df^i(z)|} \\ =& \sum_{i=1}^{l-j-1} \frac{1}{|Df^i(z)|} + \sum_{k=0}^{m-1} \sum_{i=0}^{p-1} \frac{1}{|Df^{l-j}(z)|\ |Df^{kp+i}(z_{l-j})|} \\ &\qquad + \sum_{i=0}^{L-1} \frac{1}{|Df^{(l-j)+mp}(z)|\ |Df^i(z_{(l-j)+mp})|} + \frac{1}{|Df^{M'}(z)|} \\ \leq& \frac{l-2}{\xi^{l-1}} + \sum_{k=0}^{m-1} \frac{p}{\xi^{(l-1)+(p-1)} \cdot \mu^k} + \sum_{i=0}^{L-1} \frac{1}{\xi^{l-1}\, C_{\mathrm{X}}\, A^i} + \frac{1}{2\xi^{l-1}\, \xi_0\, C_{\mathrm{X}}} \\ \leq& \frac{l-2}{\xi^{l-1}} + \frac{p}{\xi^{(l-1)+(p-1)}} \cdot \frac{\mu}{\mu-1} + \frac{1}{\xi^{l-1}\, C_{\mathrm{X}}} \cdot \frac{A}{A-1} + \frac{1}{2\xi^{l-1}\, \xi_0\, C_{\mathrm{X}}} \\ =:& \kappa_{\mathrm{A}}.\end{aligned}$$

Note that κ_{A} does not depend on j, m, and L.

If $M' = \infty$, then $L = \infty$ and one can easily check

$$\sum_{i=1}^{\infty} \frac{1}{|Df^i(z)|} \leq \frac{l-2}{\xi^{l-1}} + \frac{p}{\xi^{(l-1)+(p-1)}} \cdot \frac{\mu}{\mu-1} + \frac{1}{\xi^{l-1}\, C_{\mathrm{X}}} \cdot \frac{A}{A-1} < \kappa_{\mathrm{A}}.$$

Case 2. Suppose that $\mathsf{S} = [0, \infty)$ is an infinite S-cycle of type (II). Then there exists a $j \in \{1, 2, \cdots, l\}$ such that $z = z_0 \in V_j$ and $z = b_j(c)$ if $j < l$ and $z \in \Omega(c)$ if $j = l$. Hence for any $k \in \mathbb{N}$ we have $z_{(l-j)+kp} \in U_l$. By the same estimates as in Case 1, we have

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{1}{|Df^i(z)|} &= \sum_{i=1}^{l-j-1} \frac{1}{|Df^i(z)|} + \sum_{k=0}^{\infty}\sum_{i=0}^{p-1} \frac{1}{|Df^{l-j}(z)|\,|Df^{kp+i}(z_{l-j})|} \\ &\le \frac{l-2}{\xi^{l-1}} + \sum_{k=0}^{\infty} \frac{p}{\xi^{(l-1)+(p-1)} \cdot \mu^k} \\ &= \frac{l-2}{\xi^{l-1}} + \frac{p}{\xi^{(l-1)+(p-1)}} \cdot \frac{\mu}{\mu-1} < \kappa_A.\end{aligned}$$

■

14 Proof of Lemma C'

Set $f = f_c$. We will show that $|Df^{M'-M}(z_M)| \ge \kappa_{\mathrm{C}}/\nu$ for some constant κ_{C} that depends only on $\hat{c}$. By choosing ν sufficiently small, we have $\lambda := \kappa_{\mathrm{C}}/\nu > 1$.

As in the proof of Lemma A', we assume that $M = 0$ and set $M' := (l-j) + mp + L$ where $z_0 \in V_j$ for some $1 \le j \le l$. We also set $n := (l-j) + mp$, then by the chain rule we have

$$|Df^{M'}(z_0)| = |Df^n(z_0)| \cdot |Df^L(z_n)|. \tag{15}$$

First let us give an estimate of $|Df^n(z_0)|$. We can find an $\tilde{R}_l > 0$ such that

$$f^p(\mathbb{D}(\hat{w}, R_l)) \Subset \mathbb{D}(\hat{w}_p, \tilde{R}_l/2) \Subset \mathbb{D}(\hat{w}_p, \tilde{R}_l)$$

for any $\hat{w} \in \Omega(\hat{c})$ if we choose R_l small enough, where $\hat{w}_p = f_{\hat{c}}^p(\hat{w})$. Let $\hat{x} := \hat{b}_j$ if $z_0 \in V_j$ and $j \ne l$, or $\hat{x} := \hat{w}$ if $z_0 \in \mathbb{D}(\hat{w}, C_0'\nu^2) \subset V_l$ for some $\hat{w} \in \Omega(\hat{c})$. (The choice of $\hat{w}$ is not unique.) Let $x_0(c) = b_j(c)$ if $j < l$, or $x_0(c) = \chi_c(\hat{x})$ if $j = l$. Note that for any $c \approx \hat{c}$, we have $b_j(c) \in V_j$, $\chi_c(\hat{w}) \in \mathbb{D}(\hat{w}, C_0'\nu^2)$, and $\chi_c(\hat{w}_p) \in \mathbb{D}(\hat{w}_p, C_0'\nu^2)$. In particular, we may assume that $|z_0 - x_0(c)| \le \max(C_0, 2C_0') \cdot \nu^2$ and $|\hat{x}_n - x_n(c)| \le R_l/2$, where $\hat{x}_n = f_{\hat{c}}^n(\hat{x})$ and $x_n(c) = \chi_c(\hat{x}_n) = f_c^n(x_0(c))$. Thus, $|z_n - x_n(c)| \ge |z_n - \hat{x}_n| - |\hat{x}_n - x_n(c)| \ge R_l/2$.

Now we take the inverse branch G of f^n defined on $\mathbb{D}(\hat{x}_n, \tilde{R}_l)$ that maps $x_n(c)$ to $x_0(c)$, and z_n to z_0. By the Koebe distortion theorem, we have

$$|DG(z_n)| \asymp |DG(x_n(c))|$$

and

$$|z_0 - x_0(c)| = |G(z_n) - G(x_n(c))| \asymp |DG(x_n(c))||z_n - x_n(c)|.$$

Since $|z_0 - x_0(c)| \le \max(C_0\,\nu^2, 2C_0'\,\nu^2)$ and $|z_n - x_n(c)| \ge R_l/2$, we have $|DG(z_n)| \le C_6\,\nu^2/R_l$, where C_6 is a constant independent of $c \approx \hat{c}$, $\nu \ll 1$, and $z_0 \in J(f_c)$. Hence $|Df^n(z_0)| \ge R_l/(C_6\,\nu^2)$.

Next we give an estimate of the form $|Df^L(z_n)| \ge C_7\nu$, where C_7 is a constant independent of $c \approx \hat{c}$, $\nu \ll 1$, and $z_0 \in J(f_c)$. (Then by (15) the proof is done.) The estimate relies on the geometry of (and dynamics on) the postcritical set $P(f_{\hat{c}})$: Take any $i \in [0, L)$, then by Lemmas W and Y we obtain

$$\begin{aligned}|Df^L(z_n)| &= |Df^{L-i}(z_n)||Df^i(z_{M'-i})| \\ &\ge \frac{\gamma(z_n)}{\gamma(z_{M'-i})} A^{L-i} \cdot |Df^i(z_{M'-i})| \\ &\ge \gamma(z_n) \cdot \mathrm{dist}\,(z_{M'-i}, P(f_{\hat{c}})) \cdot |Df^i(z_{M'-i})|.\end{aligned}$$

By taking a small enough R_l, we may assume that $f_c^p(U_l)$ is disjoint from $P(f_{\hat{c}})-\Omega(\hat{c})$. Hence z_n has a definite distance from $P(f_{\hat{c}})$ (more precisely, dist $(z_n, P(f_{\hat{c}}))$ is bigger than a positive constant independent of $c \approx \hat{c}$, $\nu \ll 1$, and $z_0 \in J(f_c)$) and we always have $\gamma(z_n) \asymp 1$.

Thus it is enough to show: *There exists an $i \in [0,\, l+p)$ such that*

(1) *$z_{M'-i}$ has a definite distance from $P(f_{\hat{c}})$; and*

(2) *$|Df^i(z_{M'-i})| \geq C_8\nu$ for some constant C_8 depending only on $\hat{c}$.*

Note that if $z_{M'} \in V_0$, then $z_{M'}$ already has a definite distance from $P(f_{\hat{c}})$ by semi-hyperbolicity. This situation corresponds to $i=0$ and condition (2) is ignored.

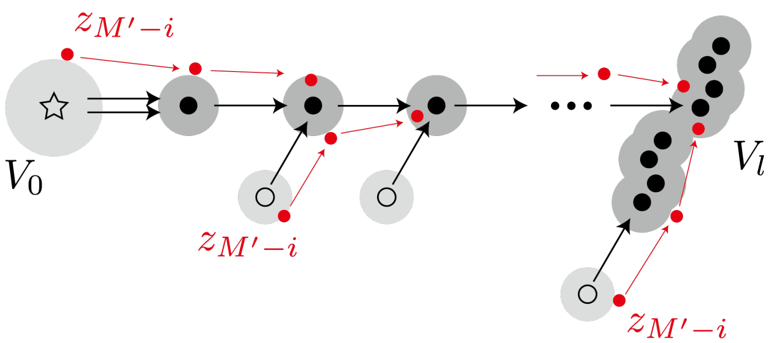

Figure 5: Black heavy dots indicate the critical orbit. Some possible behaviors of $z_{M'-i} \mapsto z_{M'-i+1} \mapsto \cdots \mapsto z_{M'}$ are indicated by smaller dots (in red).

If $z_{M'} \in V_{j'}$ with $1 \leq j' \leq l$, then such an i can be found in $[1, l+p)$ by the following procedure (Figure 5). Suppose that $z_{M'} \in V_1$. Then $z_{M'-1}$ is contained in $f^{-1}(V_1) - V_0$, and thus $|z_{M'-1}| \geq \nu$. By setting $i=1$, it follows that $z_{M'-1}$ has a definite distance from $P(f_{\hat{c}})$, and we have $|Df(z_{M'-1})| \geq 2\nu$.

Suppose that $z_{M'} \in V_2$. Then $f^{-1}(V_2)$ has two components containing $\pm\hat{b}_1$ for any $c \approx \hat{c}$. If $z_{M'-1}$ is in the component containing $-\hat{b}_1$, then $|z_{M'-1} - (-\hat{b}_1)| \asymp \nu^2$ and it has a definite distance from $P(f_{\hat{c}})$. Now set $i=1$. Since $|Df(-\hat{b}_1)| = 2|\hat{b}_1| \geq \xi$ by definition of ξ in Section 2, we have $|Df(z_{M'-1})| \asymp |Df(-\hat{b}_1)| \geq \xi > \nu$ for $\nu \ll 1$. If $z_{M'-1}$ is in the component containing $\hat{b}_1$, then $z_{M'-1}$ is necessarily contained in $f^{-1}(V_2) - V_1$, and then $|z_{M'-1} - \hat{b}_1| \asymp \nu^2$. In this situation $|z_{M'-2}| \asymp \nu$ and $z_{M'-2}$ has a definite distance from $P(f_{\hat{c}})$. Set $i=2$. Then

$$|Df^2(z_{M'-2})| = |Df(z_{M'-2})||Df(z_{M'-1})| \asymp \nu \cdot |Df(\hat{b}_1)| \geq \xi\nu.$$

Suppose that $z_{M'} \in V_{j'}$ with $j' = 3, \cdots, l-1$. As in the situation of $z_{M'} \in V_2$, either

- $|z_{M'-i} - (-\hat{b}_{j'-i})| \asymp \nu^2$ for some $i < j'$ and $z_{M'-i}$ has a definite distance from $P(f_{\hat{c}})$; or
- $|z_{M'-j'}| \asymp \nu$ and $z_{M'-j'}$ has a definite distance from $P(f_{\hat{c}})$. We set $i := j'$ in this case.

In both cases, we have $|z_{M'-k} - \hat{b}_{j'-k}| \asymp \nu^2$ for each $k = 1, \cdots, i-1$. In particular, since $2|\hat{b}_n| \geq \xi$ for $n \in \mathbb{N}$, we have:

- If $i < j'$, then $|Df^i(z_{M'-i})| \asymp 2|\hat{b}_{j'-i}| \cdot 2|\hat{b}_{j'-i+1}| \cdots 2|\hat{b}_{j'-1}| \geq \xi^i \geq \xi^{l-2}$.

- If $i = j'$, then $|Df^i(z_{M'-i})| \asymp 2\nu \cdot 2|\hat{b}_1| \cdots 2|\hat{b}_{j'-1}| \geq 2\xi^{j'-1}\nu \geq 2\xi^{l-2}\nu$.

In both cases, we have $|Df^i(z_{M'-i})| \geq C_8\nu$ for some constant $C_8 > 0$ independent of $c \approx \hat{c}$, $\nu \ll 1$, and $z_0 \in J(f_c)$.

Finally suppose that $z_{M'} \in V_l$, i.e., $\mathrm{dist}\,(z_{M'}, \Omega(\hat{c})) < C_0'\nu^2$ by definition of V_l. Now we claim: *there exists a* $k' \leq p$ *such that* $\mathrm{dist}\,(z_{M'-k'}, \Omega(\hat{c})) \asymp R_l$.

Indeed, if there exists some $1 \leq k' < p$ such that $z_{M'}, z_{M'-1}, \cdots, z_{M'-k'+1} \in U_l$ but $z_{M'-k'} \notin U_l$, then $\mathrm{dist}\,(z_{M'-k'}, \Omega(\hat{c})) \asymp R_l$. Now suppose that all $z_{M'}, z_{M'-1}, z_{M'-2}, \ldots, z_{M'-p+1}$ remain in U_l (but not in V_l except $z_{M'}$). Let us show that $z_{M'-p} \notin U_l$ by contradiction.

Assume that $z_{M'-p} \in U_l$. Since $|Df^p(z)| \geq \mu > 2.5$ for $z \in U_l$, by the Koebe distortion theorem and invariance of $\Omega(c)$ by $f^p = f_c^p$, we obtain $2 \cdot \mathrm{dist}\,(z_{M'-p}, \Omega(c)) < \mathrm{dist}\,(z_{M'}, \Omega(c))$ if $\nu \ll 1$. (Note that we have $z_{M'} \notin \Omega(c)$ since S is a finite S-cycle.) Since $\Omega(\hat{c})$ moves holomorphically, we may assume that $\mathrm{dist}\,(\Omega(c), \Omega(\hat{c})) \leq C_0'\nu^2/4$ for $c \approx \hat{c}$. Hence we obtain

$$\begin{aligned}
\mathrm{dist}\,(z_{M'-p}, \Omega(\hat{c})) &\leq \mathrm{dist}\,(z_{M'-p}, \Omega(c)) + C_0'\nu^2/4 \\
&< \mathrm{dist}\,(z_{M'}, \Omega(c))/2 + C_0'\nu^2/4 \\
&\leq (\mathrm{dist}\,(z_{M'}, \Omega(\hat{c})) + C_0'\nu^2/4)/2 + C_0'\nu^2/4 \\
&< C_0'\nu^2.
\end{aligned}$$

It would imply $z_{M'-p} \in V_l$, contradicting the construction of the S-cycle $[M, M')$. It follows that $z_{M'-p} \notin U_l$ and thus $\mathrm{dist}\,(z_{M'-k'}, \Omega(\hat{c})) \asymp R_l$ for $k' = p$.

The point $z_{M'-k'}$ above has a definite distance from $\Omega(\hat{c})$. It also has a definite distance from $P(f_{\hat{c}})$, unless $|z_{M'-k'} - \hat{b}_{l-1}| \asymp \nu^2$. However, in this case we may apply the same argument as in the case of $1 \leq j' \leq l-1$ and there exists an $i \in [k', k'+l)$ such that $z_{M'-i}$ has a definite distance from $P(f_{\hat{c}})$. Moreover, since i is bounded by $p + l$, we have $|Df^i(z_{M'-i})| \geq C_8\nu$ by replacing the above C_8 if necessary. ■

15 Proof of Lemma C

This proof is similar to that of Lemma C'. We will show that $|Df_c^{N'-N}(z_0)| \geq K_{\mathrm{C}}/\nu$ for some constant K_{C} that depends only on $\hat{c}$, and we set $\Lambda := K_{\mathrm{C}}/\nu > 1$ by choosing $\nu \ll 1$.

Without loss of generality we may assume that $N = 0$. Set $n := l + mp$ and $L := N' - n$ such that $z_{n-p} \in U_l$, $z_n \notin U_l$, $z_{n+i} \notin V_0$ for $0 \leq i < L$, and $z_{n+L} \in V_0$.

By the chain rule, we have

$$|Df_c^{N'}(z_0)| = |Df_c^n(z_0)| \cdot |Df_c^L(z_n)|. \tag{16}$$

By Lemma U, we have $|Df_c^n(z_0)| \geq C_{\mathrm{U}}/|z_0| \geq C_{\mathrm{U}}/\nu$ where the constant $C_{\mathrm{U}} > 0$ is independent of $c \approx \hat{c}$ and $z_0 \in J(f_c) \cap V_0$. Hence it is enough to show that $|Df_c^L(z_n)| \geq \eta$ for some constant $\eta > 0$ that is independent of $\nu \ll 1$, $c \approx \hat{c}$ and $z_0 \in V_0 \cap J(f_c)$. (Then we have $|Df_c^{N'}(z)| \geq C_{\mathrm{U}}\eta/\nu$ by (16) and the proof is done.)

To show this, we use the hyperbolic metric. Let $\rho(z)|dz| = \rho_c(z)|dz|$ be the hyperbolic metric on $\mathbb{C} - P(f_c)$, where

$$P(f_c) = \{c,\, f_c(c),\, f_c^2(c),\, \ldots\}$$

is the postcritical set of f_c for $c \approx \hat{c}$ with $c \notin \mathbb{M}$.

Since $J(f_c) \cap P(f_c) = \emptyset$ when $c \notin \mathbb{M}$, we have

$$\frac{\rho(f(z))}{\rho(z)}|Df_c(z)| \geq 1$$

for any $z \in J(f_c)$. (See [Mc, Theorem 3.5] for example.) We also have $\rho(z) \le \mathrm{dist}\,(z, P(f_c))^{-1}$ by Lemma W. Hence $|Df^L(z_n)| \ge \rho(z_n)/\rho(z_{N'}) \ge \rho(z_n) \cdot \mathrm{dist}\,(z_{N'}, P(f_c))$. To complete the proof, we show that both $\rho(z_n)$ and $\mathrm{dist}\,(z_{N'}, P(f_c))$ are uniformly bounded from below for any $c \approx \hat{c}$ and for any $z_0 \in V_0 = \mathbb{D}(\nu)$.

Let us work with $\mathrm{dist}\,(z_{N'}, P(f_c))$ first: Let $\widetilde{\mathcal{R}}(c)$ denote the closure of the union of the forward images of the dynamic ray $\mathcal{R}_c(\theta)$. By using the set $\widehat{\mathcal{R}}(c)$ defined in Section 7, we have

$$\widetilde{\mathcal{R}}(c) = \overline{\mathcal{R}_c(\theta) \cup \mathcal{R}_c(2\theta) \cup \cdots \cup \mathcal{R}_c(2^{l-1}\theta) \cup \widehat{\mathcal{R}}(c)}.$$

By Proposition S, this set moves continuously as $c \to \hat{c}$ along $c \in \mathcal{R}_{\mathbb{M}}(\theta)$ with respect to the Hausdorff distance on the sphere. Since the postcritical set $P(f_c)$ is contained in $\widetilde{\mathcal{R}}_c$, we obtain

$$\mathrm{dist}\,(z_{N'}, P(f_c)) \ge \mathrm{dist}\,(z_{N'}, \widetilde{\mathcal{R}}_c) \ge \mathrm{dist}\,(0, \widetilde{\mathcal{R}}_c) - |z_{N'}| \ge \mathrm{dist}\,(0, \widetilde{\mathcal{R}}_c) - \nu,$$

where $\mathrm{dist}\,(0, \widetilde{\mathcal{R}}_c)$ tends to $\mathrm{dist}\,(0, \widetilde{\mathcal{R}}_{\hat{c}}) > 0$ as $c \to \hat{c}$ with $c \in \mathcal{R}_{\mathbb{M}}(\theta)$. Now we choose sufficiently small ν and we conclude that $\mathrm{dist}\,(z_{N'}, P(f_c))$ is bounded by a positive constant that is independent of $c \to \hat{c}$ with the parameter ray condition and $z_{N'} \in V_0$.

Next we work with $\rho(z_n)$: Let $T_c : \mathbb{C} \to \mathbb{C}$ $(c \neq 0)$ be a complex affine map with $T_c(c) = \hat{c}$ and $T_c(f_c(c)) = f_{\hat{c}}(\hat{c})$ such that $T_c(z) \to z$ uniformly on compact sets as $c \to \hat{c}$. Set $g_c := T_c \circ f_c \circ T_c^{-1}$. Then g_c is a quadratic map whose postcritical set is

$$P(g_c) = T_c(P(f_c)) = \{\hat{c},\, f_{\hat{c}}(\hat{c}) = g_c(\hat{c}),\, g_c^2(\hat{c}),\, \ldots\}.$$

Hence the hyperbolic metrics ρ'_c on $\mathbb{C} - P(g_c)$ and $\hat{\rho}$ on $\mathbb{C} - \{\hat{c},\, f_{\hat{c}}(\hat{c})\}$ satisfy $T_c^* \rho'_c = \rho_c$ and $\hat{\rho} \le \rho'_c$ for all c, where T_c^* is the pull-back.

As in the proof of Lemma C', if we choose R_l small enough, then we can find an $\tilde{R}_l > 0$ such that $f_c^p(U_l) \Subset \mathrm{N}(\Omega(\hat{c}), \tilde{R}_l)$ for any $c \approx \hat{c}$ and that the closure E of the set $\mathrm{N}(\Omega(\hat{c}), \tilde{R}_l) - V_l$ contains neither $\hat{c}$ nor $f_{\hat{c}}(\hat{c})$. (Note that $f_{\hat{c}}(\hat{c})$ may belong to $\Omega(\hat{c})$ and be contained in V_l.) It follows that z_n is contained in E for $c \approx \hat{c}$, and hence so is $z'_n := T_c(z_n)$. Thus we obtain

$$\rho'_c(z'_n) \ge \hat{\rho}(z'_n) \ge \min_{w \in E} \hat{\rho}(w) > 0.$$

Since $\rho_c(z_n) = \rho'(T_c(z_n))|DT_c(z_n)| = \rho'(z'_n)|DT_c(z_n)|$ and $DT_c(w) \to 1$ uniformly on E as $c \to \hat{c}$, we conclude that $\rho(z_n)$ is bounded by a positive constant from below that is independent of $\nu \ll 1$, $c \approx \hat{c}$ and the original choice of $z_0 \in V_0 \cap J(f_c)$. ■

16 Itinerary sequences

When $c \notin \mathbb{M}$, the critical value c has a well defined external angle $t_c = (2\pi)^{-1} \arg \Phi_c(c)$. The angle t_c is not equal to zero when $c \in \mathbb{X} = \mathbb{C} - \mathbb{M} \cup \mathbb{R}_+$. For $c \in \mathbb{X}$, the dynamic rays $\mathcal{R}_c(t_c/2)$ and $\mathcal{R}_c((t_c+1)/2)$ together with the critical point 0 separate the complex plane $\mathbb{C}$ into two disjoint open sets, say $W_0 = W_0(c)$ and $W_1 = W_1(c)$. Let the one that contains c be W_0. If $t_c = \theta$ and $\mathcal{R}_{\mathbb{M}}(\theta)$ lands at a semi-hyperbolic parameter $\hat{c}$, then $\mathcal{R}_{\hat{c}}(\theta)$ lands at $\hat{c}$, and both $\mathcal{R}_{\hat{c}}(\theta/2)$ and $\mathcal{R}_{\hat{c}}((\theta+1)/2)$ land at 0. Moreover, as c approaches $\hat{c}$ along $\mathcal{R}_{\mathbb{M}}(\theta)$, in a large disk centered at 0, rays $\mathcal{R}_c(\theta/2)$ and $\mathcal{R}_c((\theta+1)/2)$ move continuously to $\mathcal{R}_{\hat{c}}(\theta/2)$ and $\mathcal{R}_{\hat{c}}((\theta+1)/2)$, respectively.

Assume $z \in J(f_c)$. Define its *itinerary* or *itinerary sequence* $I_c(z) = \{I_c(z)_n\}_{n \ge 0}$ by $I_c(z)_n = 0$ if $f_c^n(z) \in W_0$, $I_c(z)_n = 1$ if $f_c^n(z) \in W_1$, and $I_c(z)_n = *$ if $f_c^n(z) = 0$. If the critical point 0 belongs to the Julia set, $I_c(f_c(0))$ is called the *kneading sequence* for f_c.

Remark 16.1. One can also define an itinerary $\{s_0, s_1, \ldots\}$ in such a way that $s_n = 0$ if $f_c^n(z) \in \overline{W_0} \cap J(f_c)$ and that $s_n = 1$ if $f_c^n(z) \in \overline{W_1} \cap J(f_c)$. When $\hat{c}$ is a semi-hyperbolic parameter, if $f_{\hat{c}}^k(z) = 0$ for some $k \geq 0$, then $f_{\hat{c}}^n(z) \neq 0$ for all $n \neq k$ since the critical point is non-recurrent. Suppose $f_{\hat{c}}^n(z) \in W_{s_n}$ for $n \neq k$, then with the definition of itinerary in this remark, the itinerary of z will have two values $\{s_0, \ldots, s_{k-1}, 0, s_{k+1}, \ldots\}$ and $\{s_0, \ldots, s_{k-1}, 1, s_{k+1}, \ldots\}$. We employ the symbol $*$ in the above definition so as to identify sequences $\{s_0, \ldots, s_{k-1}, 0, s_{k+1}, \ldots\}$ and $\{s_0, \ldots, s_{k-1}, 1, s_{k+1}, \ldots\}$ by the one $\{s_0, \ldots, s_{k-1}, *, s_{k+1}, \ldots\}$.

Lemma Z. Let $\hat{c}$ be a semi-hyperbolic parameter.

(i) $I_{\hat{c}}(z) = I_{\hat{c}}(w)$ if and only if $z = w$.

(ii) If $I_{\hat{c}}(z)_k = *$ and $I_{\hat{c}}(z)_n = I_{\hat{c}}(w)_n$ for all $n \neq k$, then $I_{\hat{c}}(w)_k = *$ and $w = z$.

Proof. Since $\mathbb{C} - \mathcal{R}_{\hat{c}}(\theta) \cup \{\hat{c}\}$ is a simply connected domain without a critical value, there exist inverse branches $f_{\hat{c},i}^{-1} : \mathbb{C} - \mathcal{R}_{\hat{c}}(\theta) \cup \{\hat{c}\} \to W_i$ of $f_{\hat{c}}$, $i = 0$ or 1. Each of these two branches can be extended at the critical value $\hat{c}$, and each extended branch is one-to-one.

(i) If $I_{\hat{c}}(z)_n = I_{\hat{c}}(w)_n = s_n$ for all $n \geq 0$, then for any $N \in \mathbb{N}_0$ both $f_{\hat{c}}^N(z)$ and $f_{\hat{c}}^N(w)$ belong to W_{s_N} provided $s_N \neq *$, or belong to $\{0\}$ provided $s_N = *$. The set $J(f_{\hat{c}}) \cap \overline{W_{s_N}}$ can be convered by a finite number of disks $\mathbb{D}(y_i, \epsilon)$ with $y_i \in J(f_{\hat{c}}) \cap \overline{W_{s_N}}$, $i \in F$, and F is a finite index set. We choose ϵ to be the constant such that the inequality (1) holds. Let $B_N(y_i, \epsilon)$ be the component of $f_{\hat{c}}^{-N}(\mathbb{D}(y_i, \epsilon))$ such that $f_{\hat{c}}^{-N}(y_i) \in \overline{W_{s_0}}$, $f_{\hat{c}}^{-N+1}(y_i) \in \overline{W_{s_1}}$, $\ldots$, $f_{\hat{c}}^{-1}(y_i) \in \overline{W_{s_{N-1}}}$. It is not difficult to see that both z and w are contained in a simply connected domain covered by the union $\bigcup_{i \in F} B_N(y_i, \epsilon)$. It follows that $z = w$ easily from the exponential contraction (1) by taking $N \to \infty$.

(ii) If $I_{\hat{c}}(z)_k = *$ and $I_{\hat{c}}(z)_n = I_{\hat{c}}(w)_n$ for all $n > k$, then $I_{\hat{c}}(f_{\hat{c}}^{k+1}(z)) = I_{\hat{c}}(f_{\hat{c}}^{k+1}(w))$. Thus, $f_{\hat{c}}^{k+1}(z) = f_{\hat{c}}^{k+1}(w) = \hat{c}$ by (i). Since $\hat{c}$ is the critical value, $f_{\hat{c}}^k(w) = 0$, and then $I_{\hat{c}}(w)_k = *$. Therefore, $I_{\hat{c}}(z) = I_{\hat{c}}(w)$, and then $z = w$ by (i). ■

Let $z(c)$ and $\hat{c}$ be as in Theorem 1.1, and let c_0 be $c(2)$ in Theorem 1.2 or be as in Theorem 1.3. The statement (i) of following corollary describes how the itinerary of $z(c)$ retains. The statement (ii) tells that every given point, say w, of $J(f_{\hat{c}})$ is a limiting point $z(\hat{c})$ of some $z(c)$ in $J(f_c)$ where the limit is taken as in Theorem 1.2.

Corollary 16.2.

(i) *Suppose $I_{c_0}(z(c_0)) = \mathbf{s}$, then $I_{\hat{c}}(z(\hat{c})) = \mathbf{s}$ if and only if $f_{\hat{c}}^n(z(\hat{c})) \neq 0$ for all $n \geq 0$, otherwise $I_{\hat{c}}(z(\hat{c})) = \{s_0, \ldots, s_{k-1}, *, s_{k+1}, \ldots\}$ if and only if $f_{\hat{c}}^k(z(\hat{c})) = 0$ for some $k \geq 0$.*

(ii) *Let $w \in J(f_{\hat{c}})$ and $I_{\hat{c}}(w) = \mathbf{s}$. If $f_{\hat{c}}^n(w) \neq 0$ for all $n \geq 0$, there exists a unique $z(c_0)$, with $I_{c_0}(z(c_0)) = \mathbf{s}$, such that $w = z(\hat{c})$. If $f_{\hat{c}}^k(w) = 0$ for some $k \geq 0$, then there exist exactly two $z(c_0)$ and $\tilde{z}(c_0)$, having itineraries $\{s_0, \ldots, s_{k-1}, 0, s_{k+1}, \ldots\}$ and $\{s_0, \ldots, s_{k-1}, 1, s_{k+1}, \ldots\}$ respectively, such that $w = z(\hat{c}) = \tilde{z}(\hat{c})$.*

Proof. (i) For $c \notin \mathbb{M}$, every point $z \in J(f_c)$ of given itinerary is bounded away from $\mathcal{R}_c(\theta/2) \cup \mathcal{R}_c((\theta+1)/2) \cup \{0\}$ and moves holomorphically with c. Thus, $f_{\hat{c}}^n(z(\hat{c})) \in \overline{W_{s_n}(\hat{c})}$ if $f_{c_0}^n(z(c_0)) \in W_{s_n}(c_0)$. Hence, $I_{\hat{c}}(z(\hat{c})) = \mathbf{s}$ if $f_{\hat{c}}^n(z(\hat{c})) \neq 0$ for all $n \geq 0$. If $f_{\hat{c}}^k(z(\hat{c})) = 0 = \overline{W_0(\hat{c})} \cap \overline{W_1(\hat{c})} \cap J(f_{\hat{c}})$, then $0 \neq f_{\hat{c}}^n(z(\hat{c})) \in W_{s_n}(\hat{c})$ for all $n \neq k$ and $I_{\hat{c}}(z(\hat{c})) = \{s_0, \ldots, s_{k-1}, *, s_{n+1}, \ldots\}$.

(ii) For any $w \in J(f_{\hat{c}})$, by Theorem 1.3, there exists $z(c_0) \in J(f_{c_0})$ such that $h_c(z(c_0)) = z(c) \to z(\hat{c}) = w$ as $c \to \hat{c}$ along $\mathcal{R}_{\mathbb{M}}(\theta)$. If $f_{\hat{c}}^n(w) \neq 0$ for all $n \geq 0$, then $z(c_0) \neq 0$ for all $n \geq 0$, and we conclude that $I_{c_0}(z(c_0)) = I_{\hat{c}}(w)$. If there exists another $\tilde{z}(c_0) \in J(f_{c_0})$ such that $h_c(\tilde{z}(c_0)) = \tilde{z}(c) \to \tilde{z}(\hat{c}) = w$ as $c \to \hat{c}$ along $\mathcal{R}_{\mathbb{M}}(\theta)$, then $I_{c_0}(\tilde{z}(c_0)) = I_{c_0}(z(c_0))$, and consequently $\tilde{z}(c_0) = z(c_0)$ by the bijectivity between the itinerary sequences and Julia set $J(f_{c_0})$.

If $f_{\hat{c}}^k(w) = 0$ for some $k \geq 0$, then $f_{c_0}^n(z(c_0)) \in W_{s_n}(c_0)$ for $n \neq k$, and $f_{c_0}^k(z(c_0))$ belongs to $W_0(c_0)$ or $W_1(c_0)$. Without loss of generality, assume $f_{c_0}^k(z(c_0)) \in W_0(c_0)$. Let $\tilde{z}(c_0)$ be such a point that $f_{c_0}^{k+1}(\tilde{z}(c_0)) = f_{c_0}^{k+1}(z(c_0))$, $f_{c_0}^k(\tilde{z}(c_0)) \in W_1(c_0)$, and $f_{c_0}^n(\tilde{z}(c_0)) \in W_{s_n}(c_0)$ for $0 \leq n < k$. It is easy to see that such a point exists. We have $f_{c_0}^{k+1}(\tilde{z}(c)) \to f_{\hat{c}}^{k+1}(w)$ as $c \to \hat{c}$. And, by (i) and Lemma Z, we obtain $I_{\hat{c}}(\tilde{z}(\hat{c})) = I_{\hat{c}}(w)$ and $\tilde{z}(\hat{c}) = w$. If there is another $z'(c_0) \in J(f_{c_0})$ such that $h_c(z'(c_0)) \to w$ as $c \to \hat{c}$ along $\mathcal{R}_{\mathbb{M}}(\theta)$, then either $I_{c_0}(z'(c_0)) = I_{c_0}(z(c_0))$ or $I_{c_0}(z'(c_0)) = I_{c_0}(\tilde{z}(c_0))$. Consequently, by the bijectivity between the itinerary sequences and Julia set $J(f_{c_0})$, we conclude that $z'(c_0) = z(c_0)$ or $\tilde{z}(c_0)$. ■

17 Proofs of Theorems 1.5 and 1.6

Proof of Theorem 1.5. Because $J(f_{\hat{c}})$ is locally connected, it is clear that $\mathcal{E}^\theta(\theta) = I_{\hat{c}}(\hat{c})$, namely the kneading sequence of θ is equal to the kneading sequence for $f_{\hat{c}}$. Hence, it is enough to prove the theorem by using $\mathbf{e} = I_{\hat{c}}(\hat{c})$. Note that $\mathbf{e} \in \Sigma_2$ because $\hat{c}$ is not recurrent under iteration of $f_{\hat{c}}$.

For any $w \in J(f_{\hat{c}})$, we have $\sigma^n(I_{\hat{c}}(w)) \neq \mathbf{e}$ for all $n \geq 0$, or $I_{\hat{c}}(w) = \mathbf{e}$, or $\sigma^k(I_{\hat{c}}(w)) = \mathbf{e}$ for some $k \geq 1$. For any $\mathbf{s} \in \Sigma_2$ satisfying $\sigma^n(\mathbf{s}) \neq \mathbf{e}$ for all $n \geq 0$ or $\mathbf{s} = \mathbf{e}$, from Corollary 16.2, there corresponds a unique $w \in J(f_{\hat{c}})$ with $I_{\hat{c}}(w) = \mathbf{s}$. For such $\mathbf{s} \in \Sigma_2$ that $\sigma^{k+1}(\mathbf{s}) = \mathbf{e}$ for some $k \geq 0$, there is a unique $\mathbf{a} \neq \mathbf{s}$ in Σ_2 satisfying $\mathbf{a} \sim_{\mathbf{e}} \mathbf{s}$ and again from Corollary 16.2 there corresponds a unique $w \in J(f_{\hat{c}})$ with $I_{\hat{c}}(w) = \{a_0, \ldots, a_{k-1}, *, a_{k+1}, \ldots\} = \{s_0, \ldots, s_{k-1}, *, s_{k+1}, \ldots\}$. This shows the bijectivity between $\Sigma_2/\sim_{\mathbf{e}}$ and $J(f_{\hat{c}})$. Let the bijection $\Sigma_2/\sim_{\mathbf{e}} \to J(f_{\hat{c}})$ be h. Since $I_{\hat{c}}(h(\mathbf{s})) = \mathbf{s}$ if $f_{\hat{c}}^n(h(\mathbf{s})) \neq 0$ for all $n \geq 0$ or $I_{\hat{c}}(h(\mathbf{s})) = \{s_0, \ldots, s_{k-1}, *, s_{k+1}, \ldots\}$ if $f_{\hat{c}}^k(h(\mathbf{s})) = 0$ for some $k \geq 0$ (we use $\mathbf{s}$ for an element in both Σ_2 and $\Sigma_2/\sim_{\mathbf{e}}$ if it does not cause any confusion), by a similar argument to the proof of Lemma Z (i), the continuity of h follows easily by virtue of the exponential contraction (1). Compactness of $\Sigma_2/\sim_{\mathbf{e}}$ and $J(f_{\hat{c}})$ leads to h a homeomorphism. To show h acts as a conjugacy, observe from Corollary 16.2 that points $h \circ \sigma(\mathbf{s})$ and $f_{\hat{c}} \circ h(\mathbf{s})$ have the same itinerary under $f_{\hat{c}}$, thus they are the same by Lemma Z (i). ■

Proof of Theorem 1.6. There are exactly two cases: $f_{\hat{c}}^n(w) \neq 0$ for all $n \geq 0$ or $f_{\hat{c}}^n(w) = 0$ for some $n \geq 0$. By Corollary 16.2, $h_{\hat{c}}^{-1}(\{w\})$ is a singleton if and only if $f_{\hat{c}}^n(w)$ is as the first case, whereas it consists of two distinct points if and only if $f_{\hat{c}}^n(w)$ is as the second case. ■

Acknowledgments

The authors would like to thank the referee for his/her comments and suggestions that make the paper more precise and readable. Chen was partly supported by NSC 99-2115-M-001-007, MOST 103-2115-M-001-009, 104-2115-M-001-007, and 105-2115-M-001-003. Kawahira was partly supported by JSPS KAKENHI Grant Number 16K05193. They thank the hospitality of Academia Sinica, Nagoya University, RIMS in Kyoto University, and Tokyo Institute of Technology where parts of this research were carried out.

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