Riemann's zeta function, Newton's method, and holomorphic index

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Abstract. We apply some root finding algorithms to characterize the zeros of Riemann's zeta. We also give an intriguing interpretation of the Riemann Hypothesis in terms of one dimensional dynamical systems.

1 Riemann's zeta and primes

For $s = \sigma + it \in \mathbb{C}$, one can easily see that the series

$$\zeta(s) \ = \ 1 + \frac{1}{2^s} + \frac{1}{3^s} \cdots \ = \ \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^s}\right)^{-1}$$

converges if $\sigma > 1$, where p_n is the nth prime number. Indeed, $\zeta(s)$ is analytic on $\{\mathrm{Re}\, s = \sigma > 1\}$ and by analytic continuation we consider it a meromorphic function $\zeta : \mathbb{C} \to \bar{\mathbb{C}}$ with only one pole at $s = 1$, which is simple.

The Riemann Hypothesis. The most famous conjecture on Riemann's zeta function is: *ζ has non-real(non-trivial) zeros only on the critical line* $\mathrm{Re}\, s = \sigma = 1/2$ *(the Riemann Hypothesis)*. If this conjecture is affirmative, we will have a nice result on the distribution of prime numbers;

$$p_{n+1} - p_n \ = \ O(p_n^{1/2} \log p_n).$$

This is better than any known results, for example;

$$p_{n+1} - p_n \ = \ O(p_n^{0.525+\epsilon})$$

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for any $\epsilon > 0$. To show the hypothesis, it is known that we only have to care the zeros on the critical stripe $\mathcal{S} = \{s \in \mathbb{C} : 0 < \operatorname{Re} s < 1\}$. In particular, *wider zero-free regions imply better estimates of distribution of primes.*

For example, it is known that there exists a constant $A > 0$ such that

$$\left\{ s = \sigma + it \in \mathcal{S} \; : \; \sigma \geq 1 - \frac{A}{(\log(|t|+1))^{2/3}(\log\log(|t|+1))^{1/3}} \right\}$$

is zero-free.

2 Newton's method

There are some root finding algorithms, but the most famous one would be *Newton's method.* From now on, we work with complex variable $z = x + yi$ instead of conventional s for ζ.

For a meromorphic function $f : \mathbb{C} \to \bar{\mathbb{C}}$, we define its *Newton's map* N_f by

$$N_f(z) \;=\; z - \frac{f(z)}{f'(z)},$$

which is again meromorphic. One can easily check that $f(\alpha) = 0$ iff $N_f(\alpha) = \alpha$. The idea of Newton's method is: Start with an initial value z_0 sufficiently close to α. Then the sequence $\{z_n\}$ defined by $z_{n+1} = N_f(z_n)$ converges (rapidly) to α.

More precisely, we have the following property:

> *If α is a simple zero of f, then $N_f(\alpha) = \alpha$ and $N_f'(\alpha) = 0$. Thus*
>
> $$N_f(z) - \alpha \;=\; O((z-\alpha)^2) \quad (z \to \alpha).$$
>
> *If α is a multiple zero, then $N_f(\alpha) = \alpha$ and $|N_f'(\alpha)| < 1$. Thus*
>
> $$|N_f(z) - \alpha| \;\leq\; C|z - \alpha| \quad (z \to \alpha)$$
>
> *for some $0 < C < 1$.*

Hence the precision of z_n as an approximate value of α is exponentially or linearly increasing according to the multiplicity of α.

Newton's method as a dynamical systems. What makes this method more intriguing is the theory of iteration of holomorphic function developed by Fatou and Julia in early 1920s. For given $z_0 \in \mathbb{C}$, convergence of $z_n = N_f^n(z_0)$ (where N_f^n is nth iteration of N_f) is not guaranteed in general. To investigate the behaver of such sequence, we consider the global dynamical systems

$$\bar{\mathbb{C}} \xrightarrow{N_f} \bar{\mathbb{C}} \xrightarrow{N_f} \bar{\mathbb{C}} \xrightarrow{N_f} \cdots$$

given by iteration of Newton's map. (As we will see, we need a spacial care for poles of N_f.) For example, set $f(z) := z^3 - 1$. Then the iteration of its Newton's map gives the following picture (Figure 1):

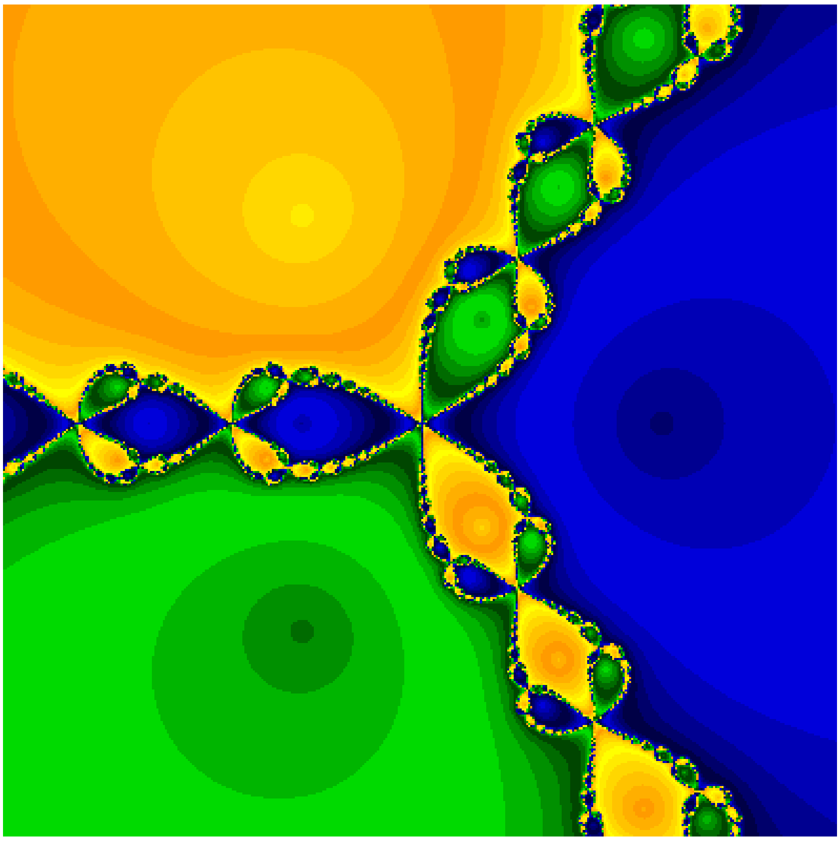

Figure 1: Dynamics of N_f for $f(z) = z^3 - 1$.

Blue, yellow, and green regions are the set of initial values z_0 such that the orbit $z_n = N_f^n(z_0)$ converges to 1, $\frac{-1+\sqrt{3}i}{2}$, and $\frac{-1-\sqrt{3}i}{2}$ respectively. Shades distinguish the number of iteration to trap the orbit in small disks around roots. The boundary of these regions has complicated structure known as *fractal.* It is the *Julia set* of N_f, where the dynamics shows chaotic behavior. In particular, orbits from the Julia set stay within the Julia set and never converge to the roots.

Newton's method for meromorphic functions. If f is a rational function, then so is N_f thus it has no essential singularity. For a meromorphic function f, its Newton's map has an essential singularity at infinity. Since $N_f(\infty)$ is indeterminate, we must stop the iteration when the orbit lands on a pole of N_f. In this particular setting, we define its *Fatou set* $F(N_f)$ by:

$$\begin{aligned} & z_0 \in F(N_f) \\ \iff & \exists U \text{ a nbd of } z_0 \text{ s.t. } \left\{N_f^n|U\right\}_{n\geq 0} \text{ is defined and a normal family} \end{aligned}$$

The *Julia set* $J(N_f)$ is the complement $\mathbb{C} - F(N_f)$.

3 Applying the method to zeta.

Now let us apply Newton's method to Riemann's zeta. For the meromorphic function $\zeta : \mathbb{C} \to \bar{\mathbb{C}}$, we set

$$\nu(z) \ := \ N_\zeta(z) \ = \ z - \frac{\zeta(z)}{\zeta'(z)}.$$

We also apply the method to the functions

$$\eta(z) \ := \ (z-1)\zeta(z)$$

and

$$\xi(z) \ = \ \frac{1}{2}z(1-z)\pi^{z/2}\Gamma(z/2)\zeta(z),$$

where $\xi(z)$ a classical zeta-related function with symmetry $\xi(z) = \xi(1-z)$. Since $\eta(z)$ and $\xi(z)$ are entire functions, we may expect better dynamics for

$$\mu(z) \ := \ z - \frac{\eta(z)}{\eta'(z)} \qquad \text{and} \qquad \lambda(z) \ := \ z - \frac{\xi(z)}{\xi'(z)}.$$

Now let us go to the gallery!

Pictures for ν. The first picture is on the dynamics of ν. The coloring indicates the number of iteration to trap the orbits in attracting fixed points:

$$0 = \text{orange} < \text{yellow} < \text{green} < \text{blue} < \text{purple} < \text{red} = \mathit{maximum}.$$

Probably points colored in red are close to the Julia set.

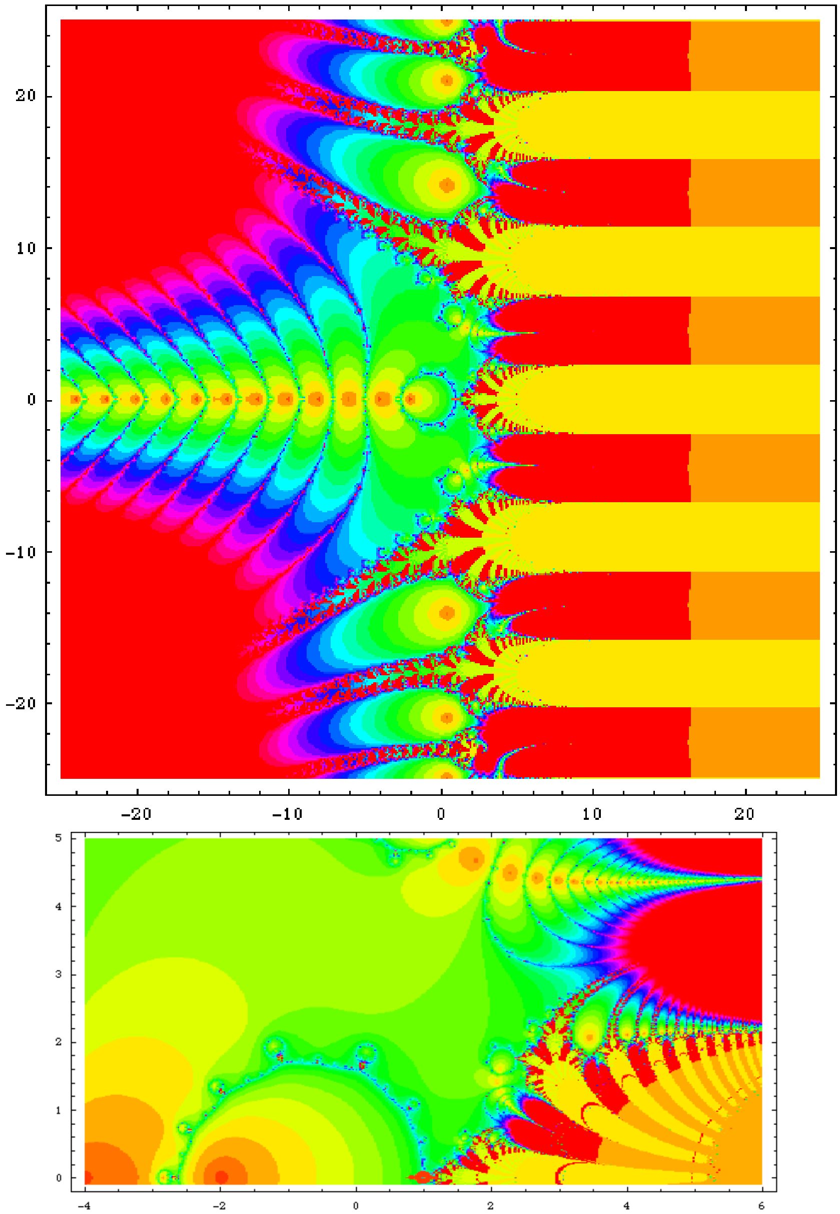

Figure 2: The orange dots are arrayed on $-2\mathbb{N}$ and the critical line. The picture in the bottom is a magnification near the origin. Probably the sequence of orange dots near $\{\operatorname{Im} z = 4.5\}$ are preimages of $-2\mathbb{N}$.

Pictures for μ Next we show the pictures of the dynamics of μ. The Julia set of μ seems much simpler.

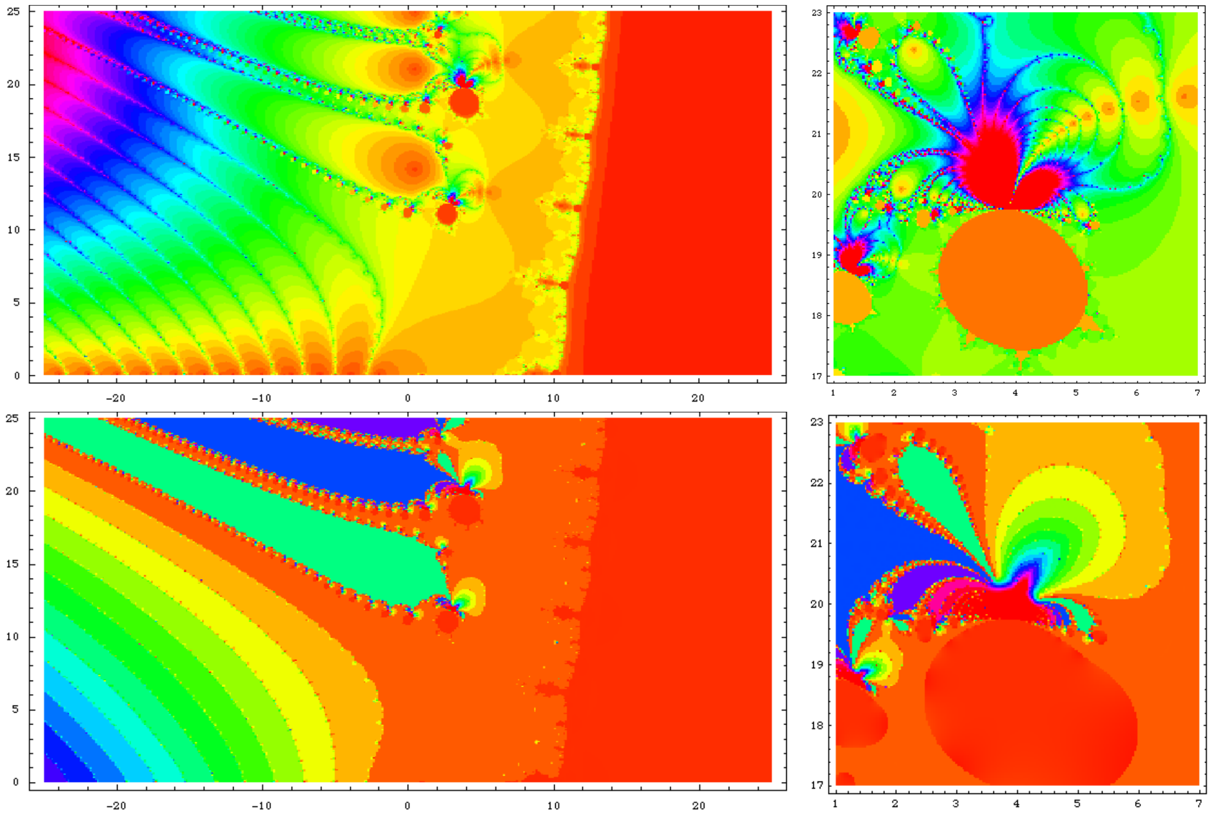

Figure 3: The Julia set of $\mu(z)$. The pictures in the second row are colored to distinguish the fixed points to converge. The pictures on the right shows the details of a prospective pole of $\mu(z)$ ("A head of chicken").

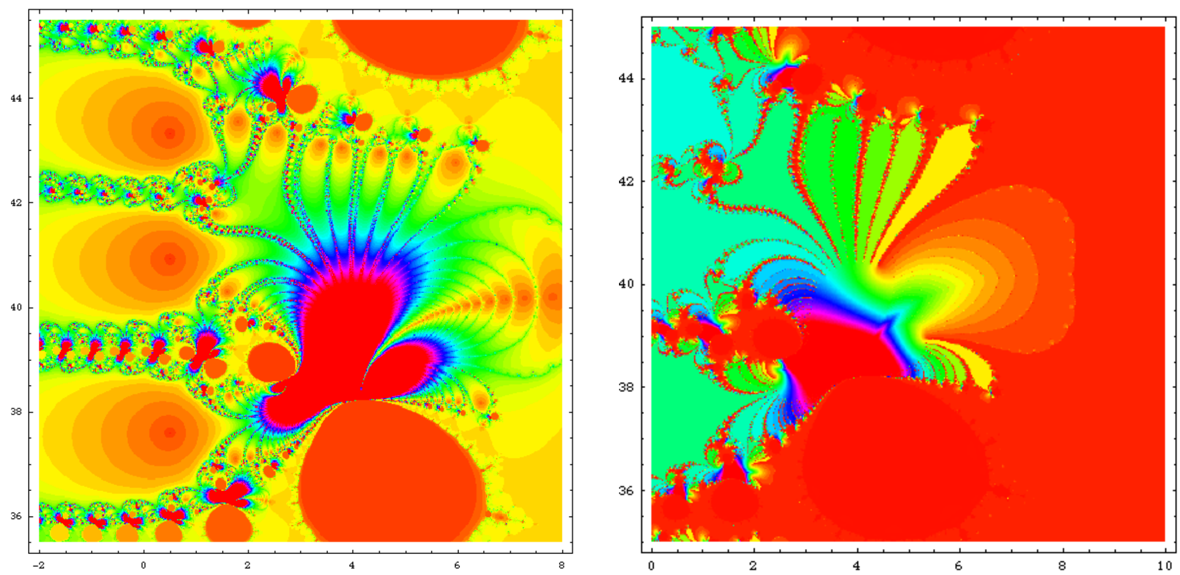

Figure 4: Head of another chicken in different colorings.

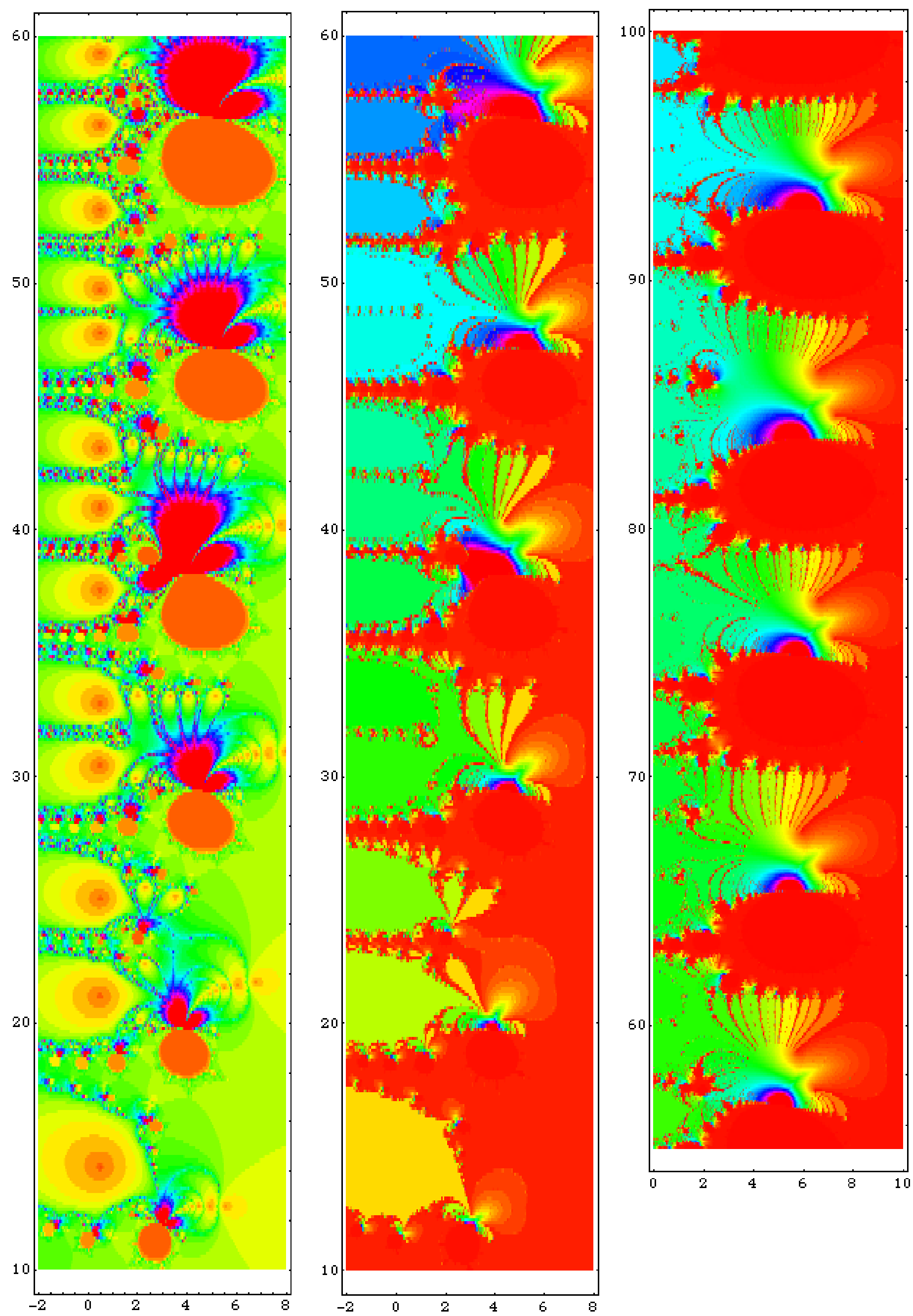

Figure 5: Chickens for $\mu(z)$. Heads appear constantly in this range, though the zeros get denser as their imaginary parts increase.

Pictures for λ. Finally we go to λ. One can easily check that the Newton's map λ has a symmetry with respect to the point $z = 1/2$. The dynamics seems the simplest, but the calculation for λ is the heaviest.

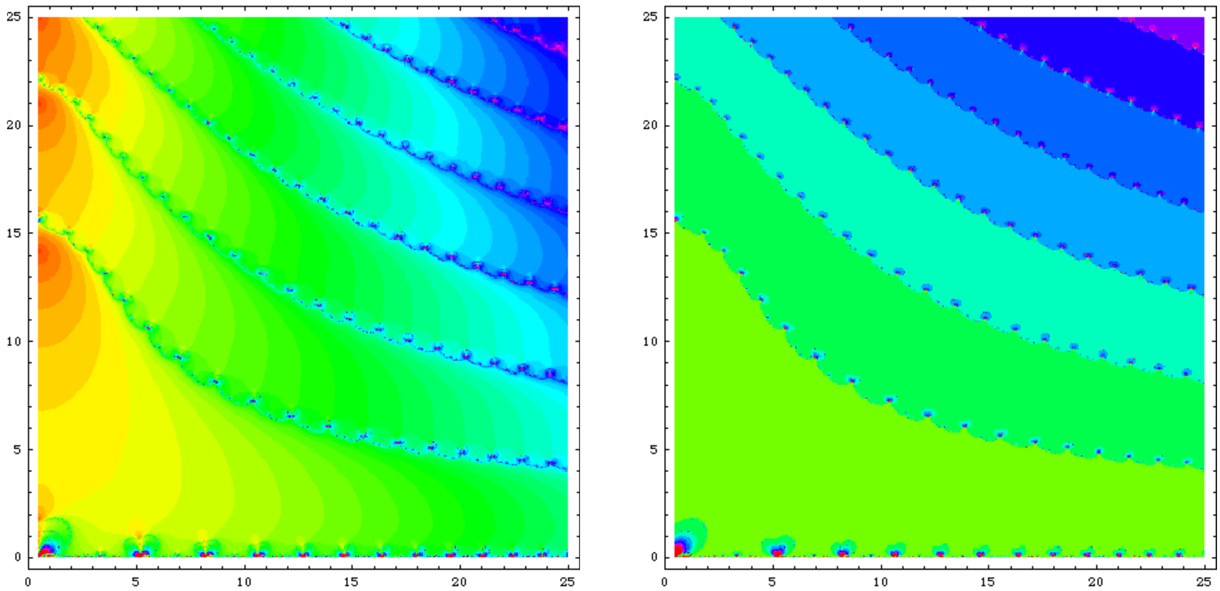

Figure 6: Julia sets for $\lambda(z)$. The dynamics seems very simple: Probably each layer has conformally the same dynamics as $z \mapsto z^2$ on the unit disk.

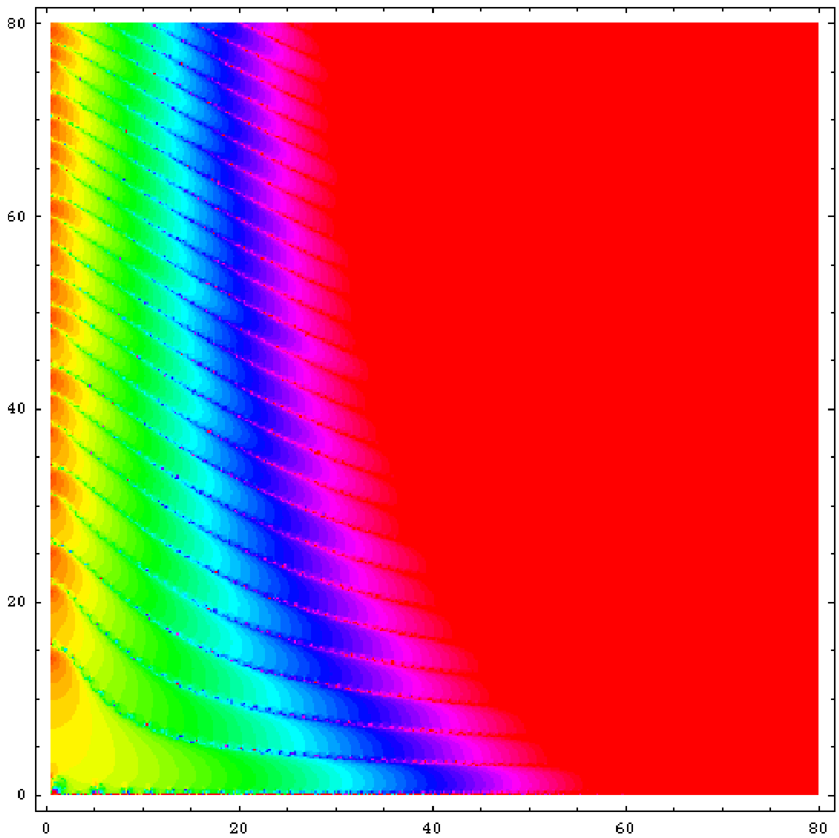

Figure 7: Julia set for $\lambda(z)$ (large scaled).

4 Holomorphic index and the Riemann Hypothesis

Let $g : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function. Suppose $\alpha \in \mathbb{C}$ satisfies $g(\alpha) = \alpha$ (i.e., a fixed point of g) with $g'(\alpha) = \kappa = \kappa_\alpha$. Then the Taylor expansion about α gives a representation of the local action of g near α as follows:

$$g(z) - \alpha \ = \ \kappa(z - \alpha) + O(|z - \alpha|^2)$$

This implies that g is locally approximated by an affine action $z - \alpha \mapsto \kappa(z - \alpha)$. We say κ is the *multiplier* of α.

We say the fixed point α is

- *attracting* if $|\kappa| < 1$,
- *repelling* if $|\kappa| < 1$, and
- *indifferent* if $|\kappa| = 1$.

We define the *holomorphic index* of α by

$$\iota \ = \ \iota(g, \alpha) \ := \ \frac{1}{2\pi i} \int_C \frac{dz}{z - f(z)},$$

where C is a small circle around α with counterclockwise direction. It is not difficult to check

$$\kappa \neq 1 \implies \iota(g, \alpha) \ = \ \frac{1}{1 - \kappa} \qquad \text{—— } (*).$$

Thus α with $\kappa = \kappa_\alpha \neq 1$ is

- attracting $\iff |\kappa| < 1 \iff \operatorname{Re} \iota > \frac{1}{2}$,
- repelling $\iff |\kappa| > 1 \iff \operatorname{Re} \iota < \frac{1}{2}$
- indifferent $\iff |\kappa| = 1 \iff \operatorname{Re} \iota = \frac{1}{2}$ —— $(**)$

Example: Newton's method. For $\lambda(z) = z - \xi(z)/\xi'(z)$, $\zeta(\alpha) = 0$ implies that $\lambda(\alpha) = \alpha$ and $\lambda'(\alpha) = (m-1)/m < 1$, where $m \in \mathbb{N}$ is the multiplicity of α.

Let C be a simple closed path in $\mathbb{C}$. Now we have

$$\frac{1}{2\pi i} \int_C \frac{dz}{z - \lambda(z)} \ = \ \frac{1}{2\pi i} \int_C \frac{\xi'(z) dz}{\xi(z)} \ = \ \frac{1}{2\pi i} \int_C d \log \xi(z).$$

By the argument principle, this integral is the number of zeros inside C counting with multiplicity. (Recall that ξ is entire, thus no pole.) In fact, if $\alpha_1, \ldots, \alpha_p$ are such zeros with multiplicity $m_1, \ldots, m_p$, one can check by $(*)$ that

$$\frac{1}{2\pi i}\int_C \frac{dz}{z-\lambda(z)} \;=\; \sum_{j=1}^{p} \iota(\lambda, \alpha_p) \;=\; \sum_{j=1}^{p} \frac{1}{1-\frac{m_p-1}{m_p}} \;=\; \sum_{j=1}^{p} m_p.$$

The Riemann Hypothesis. Let us give an interpretation of the Riemann Hypothesis in terms of holomorphic index. Set

$$\Lambda(z) \;:=\; z - \frac{\xi(z)}{z\xi'(z)}.$$

Then $\xi(\alpha) = 0$ implies that $\Lambda(\alpha) = \alpha$ and by $(*)$,

$$\Lambda'(\alpha) \;=\; 1 - \frac{1}{\alpha} \;\;(\neq\; 1) \quad\Longleftrightarrow\quad \iota \;=\; \iota(\Lambda, \alpha) \;=\; \alpha.$$

Now we have an interpretation of the Riemann Hypothesis in complex-dynamics context. By $(**)$,

> **The Riemann Hypothesis 1.** *Any fixed point of Λ function is indifferent.*

By the functional equation $\xi(z) = \xi(1-z)$, if α is a fixed point of Λ then so is $1-\alpha$. If α is attracting, then $\mathrm{Re}\,\iota(\Lambda, \alpha) = \alpha < 1/2$ implies that $1-\alpha$ is repelling. This implies that any attracting fixed point has a corresponding repelling fixed point. Thus we can also put the interpretation above as:

> **The Riemann Hypothesis 2.** *There is no attracting fixed point of Λ function.*

If the Hypothesis is true, any non-trivial zero of ζ (or ξ) is of the form $\alpha = 1/2 + \gamma i$ ($\gamma \in \mathbb{R}$). On the other hand, it must be an indifferent fixed point of Λ with multiplier $e^{2\pi i\theta}$ ($\theta \in \mathbb{R}$). The value γ and θ are related by

$$\gamma \;=\; \frac{1}{2\tan \pi\theta} \quad\Longleftrightarrow\quad \theta \;=\; \frac{1}{\pi}\arctan\frac{1}{2\gamma}.$$

Here is a question for people who know the linearization problem of fixed point:

> **Linearization problem.** *Can θ be a rational number? Is Λ linearizable at α? That is, can Λ has an invariant Siegel disk?*

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