Notes on Tan's theorem on similarity between the Mandelbrot set and the Julia sets *

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Abstract

This note gives a simplified proof of the similarity between the Mandelbrot set and the quadratic Julia sets at the Misiurewicz parameters, originally due to Tan Lei [TL]. We also give an alternative proof of the global linearization theorem of repelling fixed points.

The Mandelbrot set and the Julia sets. Let us consider the quadratic family

$$\{f_c(z) = z^2 + c \,:\, c \in \mathbb{C}\}.$$

The *Mandelbrot set* $\mathbb{M}$ is the set of $c \in \mathbb{C}$ such that the sequence $\{f_c^n(c)\}_{n\in\mathbb{N}}$ is bounded. For each $c \in \mathbb{C}$, the *filled Julia set* K_c is the set of $z \in \mathbb{C}$ such that the sequence $\{f_c^n(z)\}_{n\in\mathbb{N}}$ is bounded. One can easily check that

- $c \notin \mathbb{M}$ if and only if $|f_c^n(c)| > 2$ for some $n \in \mathbb{N}$; and
- for each $c \in \mathbb{M}$, $z \notin K_c$ if and only if $|f_c^n(z)| > 2$ for some $n \in \mathbb{N}$.

The *Julia set* J_c is the boundary of K_c. Note that all $\mathbb{M}, K_c$, and J_c are compact, and also non-empty because we can always solve the equations $f_c^n(c) = c$ and $f_c^n(z) = z$.

Tan showed in [TL] that when $c_0 \in \mathbb{M}$ is a Misiurewicz parameter (to be defined below), the "shapes" of $\mathbb{M}$ and the Julia set J_{c_0} are asymptotically similar at the same point c_0. For example, (JM1) of Figure 1 shows $\mathbb{M}$ and J_{c_0} in squares centered at $c_0 = i$, whose widths range from 6.0 to 0.01. We will prove this by finding an entire function that bridges the dynamical and parameter planes (Lemma 1).

Misiurewicz parameters and a key lemma. Following the terminology of [TL], we say $c_0 \in \mathbb{M}$ is a *Misiurewicz parameter* if the forward orbit of c_0 by f_{c_0} eventually lands on a repelling periodic point. More precisely, there exist minimal $l \geq 1$ and $p \geq 1$ such that $a_0 := f_{c_0}^l(c_0)$ satisfies $a_0 = f_{c_0}^p(a_0)$ and $|(f_{c_0}^p)'(a_0)| > 1$. By the implicit function theorem, we can show that the repelling periodic point a_0 is stable: that is, there exists a neighborhood V of c_0 and a holomorphic map $a : c \mapsto a(c)$ on V such that $a(c_0) = a_0$; $a(c) = f_c^p(a(c))$; and $|(f_c^p)'(a(c))| > 1$. We let $\lambda(c) := (f_c^p)'(a(c))$ and $\lambda_0 := \lambda(c_0)$.

Our key lemma is the following.

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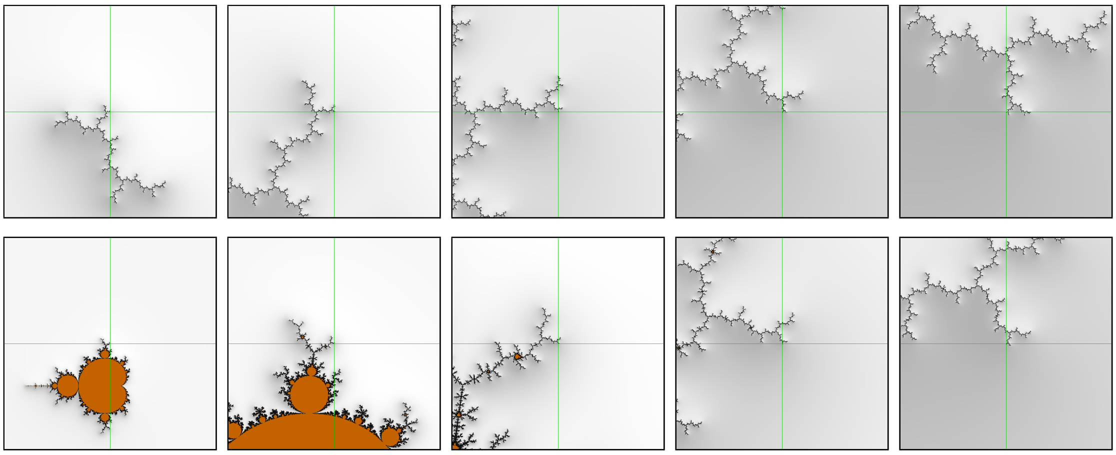
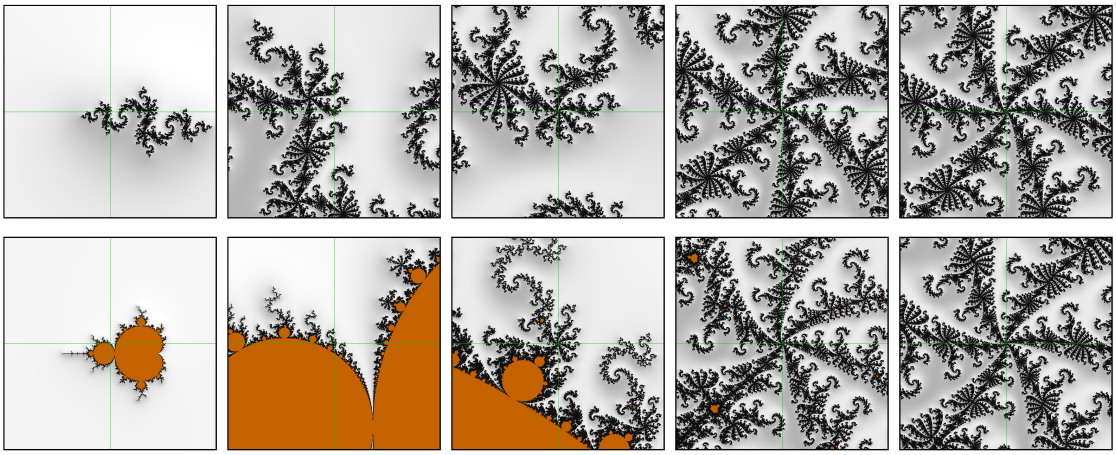
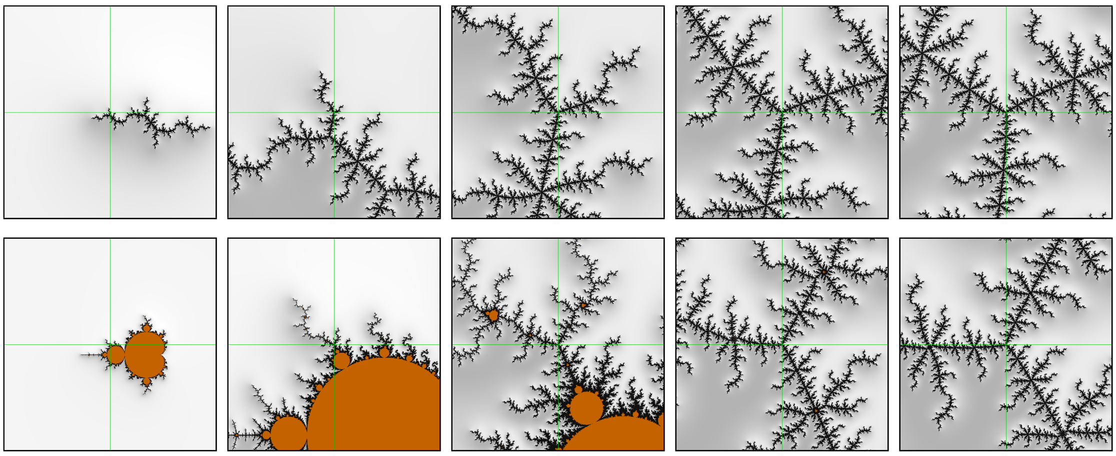


(JM1)



(JM2)



(JM3)

Figure 1: (JM1) Center: $0.0 + 1.0i$, Square width: from 6.0 to 0.01. (JM2) Center: $-0.8597644816892409 + 0.2348792315014578 4i$, Square width: from 5.0 to 0.001. (JM3) Center: $-1.162341599884035 + 0.2923689338965703i$, Square width: from 6.0 to 0.001.

Lemma 1 *Suppose that $c_0 \in \mathbb{M}$ is a Misiurewicz parameter as above. For $k \in \mathbb{N}$, set $\rho_k := 1/(f_{c_0}^{l+kp})'(c_0)$. Then we have the following.*

(1) *The function $\phi_k(w) = f_{c_0}^{l+kp}(c_0 + \rho_k w)$ converges to a non-constant entire function $\phi : \mathbb{C} \to \mathbb{C}$ as $k \to \infty$ uniformly on any compact sets.*

(2) *There exists a constant $Q \neq 0$ such that the function*

$$\Phi_k(w) \ := \ f_{c_0+Q\rho_k w}^{l+kp}(c_0 + Q\rho_k w)$$

converges to the same function $\phi(w)$ as $k \to \infty$ uniformly on compact sets of $\mathbb{C}$.

Proof. It is well-known that the sequence of (polynomial) functions

$$w \ \longmapsto \ f_{c_0}^{kp}\left(a_0 + \frac{w}{\lambda_0^k}\right) \qquad (k \in \mathbb{N})$$

converges to a non-constant entire function $\phi(w)$ uniformly on compact sets of $\mathbb{C}$. (See Theorem 3 in Appendix. Such a ϕ is called a *Poincaré function*. Indeed, ϕ satisfies the functional equation $\phi(\lambda_0 w) = f_{c_0}^p \circ \phi(w)$, but we will not use it.) Note that this function satisfies $\phi(0) = a_0$ and $\phi'(0) = 1$.

Now let us show (1): set $A_0 := (f_{c_0}^l)'(c_0)$, where $A_0 \neq 0$ since otherwise c_0 is strictly periodic. We also have $(f_{c_0}^{l+kp})'(c_0) = A_0\lambda_0^k = 1/\rho_k$. For sufficiently small $t \in \mathbb{C}$, we have the expansion

$$f_{c_0}^l(c_0 + t) \ = \ a_0 + A_0 \cdot t + o(t).$$

Fix an arbitrarily large compact set $E \subset \mathbb{C}$ and take any $w \in E$. Then by setting $t = w/(A_0\lambda_0^k)$,

$$f_{c_0}^{kp}\left(a_0 + \frac{w}{\lambda_0^k}\right) \ \sim \ f_{c_0}^{l+kp}\left(c_0 + \frac{w}{A_0\lambda_0^k} + o(\lambda_0^{-k})\right) \ \sim \ f_{c_0}^{l+kp}(c_0 + \rho_k w) \ = \ \phi_k(w)$$

when $k \to \infty$. (Here by $A_k(w) \sim B_k(w)$ we mean $A_k(w) - B_k(w) \to 0$ uniformly on E as $k \to \infty$.) Hence we have $\phi(w) = \lim_{k\to 0} \phi_k(w)$ on any compact sets. Note that ϕ has no poles, since each ϕ_k is entire.

Next we show (2): suppose that $Q \in \mathbb{C}^*$ is a constant and set $c = c(w) := c_0 + Q\rho_k w$. We also set $\Phi_k(w) := f_c^{l+kp}(c)$ and $b(c) := f_c^l(c)$. Recall that $a(c)$ denotes a repelling periodic point (of f_c) of period p with $a(c_0) = a_0$, and $\lambda(c)$ denotes its multiplier.

Then Theorem 3 in Appendix again implies that the sequence of functions $\phi_k^c(w) := f_c^{kp}(a(c) + w/\lambda(c)^k)$ converges to an entire function $\phi^c(w)$ uniformly on compact sets. In particular, by the proof of Theorem 3, it is not difficult to check that the function $c \mapsto \phi^c(w)$ is holomorphic near $c = c_0$ when we fix a $w \in \mathbb{C}$.

As in Tan's original proof, we employ a theorem on transversality by Douady and Hubbard [DH, Lemma 1, p.333]: There exists a $B_0 \neq 0$ such that

$$b(c) - a(c) = B_0(c - c_0) + o(c - c_0).$$

Hence for $c = c_0 + Q\rho_k w$ (taking w in a compact set), we have

$$b(c) \ = \ a(c) + B_0 Q\rho_k w + o(\rho_k) \ = \ a(c) + \frac{B_0 Q}{A_0} \cdot \frac{\lambda(c)^k}{\lambda_0^k} \cdot \frac{w}{\lambda(c)^k} + o(\rho_k).$$

Set $Q := A_0/B_0$. Since $\lambda(c)$ is a holomorphic function of c and thus $\lambda(c) = \lambda_0 + O(c - c_0)$, we have $|\lambda(c)/\lambda_0 - 1| = O(c - c_0)$. This implies that

$$\log \frac{\lambda(c)^k}{\lambda_0^k} = k \cdot O(c - c_0) = O\left(\frac{k}{\lambda_0^k}\right) \to 0 \quad (k \to \infty)$$

for $c = c_0 + Q\rho_k w = c_0 + O(\lambda_0^{-k})$. Since

$$\Phi_k(w) = f_c^{kp}(b(c)) \;=\; f_c^{kp}\left(a(c) + \frac{w}{\lambda(c)^k} + o(\rho_k)\right)$$

and $\phi^c(w) \to \phi(w)$ as $c \to c_0$ (uniformly for w in a compact set), we conclude that

$$\lim_{k\to\infty} \Phi_k(w) \;=\; \phi(w),$$

where the convergence is uniform on any compact sets. ■

Remark. Lemma 1 implies that $c_0 \in J_{c_0} = \partial K_{c_0}$ and $c_0 \in \partial\mathbb{M}$. Indeed, we can find a $w \in \mathbb{C}$ such that $|\phi(w)| > 2$ and hence $|\phi_k(w)| > 2$ for sufficiently large k. Equivalently, we have $c_0 + \rho_k w \notin K_{c_0}$ for sufficiently large k, where $c_0 + \rho_k w$ tends to c_0 as $k \to \infty$. Since $c_0 \in K_{c_0}$ by definition, we have $c_0 \in J_{c_0}$. The proof for $c_0 \in \partial\mathbb{M}$ is analogous.

The Hausdorff topology. Let us briefly recall the *Hausdorff topology* of the set of non-empty compact sets $\mathrm{Comp}^*(\mathbb{C})$ of $\mathbb{C}$. For a sequence $\{K_k\}_{k\in\mathbb{N}} \subset \mathrm{Comp}^*(\mathbb{C})$, we say K_k converges to $K \in \mathrm{Comp}^*(\mathbb{C})$ as $k \to \infty$ if for any $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $K \subset \mathrm{N}_\epsilon(K_k)$ and $K_k \subset \mathrm{N}_\epsilon(K)$ for any $k \geq k_0$, where $\mathrm{N}_\epsilon(\cdot)$ is the open ϵ neighborhood in $\mathbb{C}$.

Set $\mathbb{D}(r) := \{z \in \mathbb{C} : |z| < r\}$. For a compact set K in $\mathbb{C}$, let $[K]_r$ denote the set $(K \cap \mathbb{D}(r)) \cup \partial\mathbb{D}(r)$. For $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$, let $a(K - b)$ denote the set of $a(z - b)$ with $z \in K$.

Similarity. Let c_0 be a Misiurewicz parameter. Now we state our version of Tan's similarity theorem.

Theorem 2 (Similarity between $\mathbb{M}$ and J) *There exist a non-constant entire function ϕ on $\mathbb{C}$, a sequence $\rho_k \to 0$, and a constant $q \neq 0$ such that if we set $\mathcal{J} := \phi^{-1}(J_{c_0}) \subset \mathbb{C}$, then for any large constant $r > 0$, we have*

(a) $\left[\rho_k^{-1}(J_{c_0} - c_0)\right]_r \to [\mathcal{J}]_r$, *and*

(b) $\left[\rho_k^{-1} q(\mathbb{M} - c_0)\right]_r \to [\mathcal{J}]_r$

as $k \to \infty$ in the Hausdorff topology.

Proof of (a). Let ϕ_k, ϕ, and $\rho_k = 1/(A_0\lambda_0^k)$ be as given in the proof of Lemma 1. Since $f_{c_0}^n(J_{c_0}) = J_{c_0}$, we have $\left[\rho_k^{-1}(J_{c_0} - c_0)\right]_r = \left[\phi_k^{-1}(J_{c_0})\right]_r$. By $[\mathcal{J}]_r = [\phi^{-1}(J_{c_0})]_r$ and uniform convergence of $\phi_k \to \phi$ on $\overline{\mathbb{D}(r)}$, the claim easily follows.

Proof of (b). Set $q := 1/Q$ (where $Q \neq 0$ is defined in the proof of Lemma 1) and $\mathcal{M}_k := \rho_k^{-1} q(\mathbb{M} - c_0)$. Fix any $\epsilon > 0$. Since the set $\overline{\mathbb{D}(r) - \mathrm{N}_\epsilon(\mathcal{J})}$ is compact, there exists an $N = N(\epsilon)$ such that $|f_{c_0}^N \circ \phi(w)| > 2$ for any $w \in \overline{\mathbb{D}(r) - \mathrm{N}_\epsilon(\mathcal{J})}$. By uniform convergence of $\Phi_k(w) = f_{c_0+Q\rho_k w}^{l+kp}(c_0 + Q\rho_k w)$ to $\phi(w)$ on compact sets in $\mathbb{C}$ (Lemma 1), we have

$$|f_{c_0+Q\rho_k w}^{(l+kp)+N}(c_0 + Q\rho_k w)| \; > \; 2$$

for all sufficiently large k. This implies that $c_0 + Q\rho_k w \notin \mathbb{M}$, equivalently, $w \notin \mathcal{M}_k$. Hence we have

$$[\mathcal{M}_k]_r \; \subset \; \mathrm{N}_\epsilon([\mathcal{J}]_r).$$

Next we show the opposite inclusion $[\mathcal{J}]_r \subset \mathrm{N}_\epsilon([\mathcal{M}_k]_r)$ for k large enough. Let us approximate $[\mathcal{J}]_r$ by a finite subset E of $[\mathcal{J}]_r$ such that the $\epsilon/2$ neighborhood of E covers $[\mathcal{J}]_r$. Now it is enough to prove that for any $w_0 \in E$, there exists a sequence $w_k \in [\mathcal{M}_k]_r$ such that $|w_0 - w_k| < \epsilon/2$ for sufficiently large k.

Let Δ be a disk of radius $\epsilon/2$ centered at w_0. When $\Delta \cap \partial\mathbb{D}(r) \neq \emptyset$, we can take such a w_k in $\partial\mathbb{D}(r)$. Hence we may assume that $\Delta \subset \mathbb{D}(r)$.

Since $\phi(w_0) \in J_{c_0}$ and repelling cycles are dense in J_{c_0} (see [Sch] and [Mi]. See also the remark below), we can choose a w_0' such that $\phi(w_0')$ is a repelling periodic point of some period m and $|w_0 - w_0'| < \epsilon/4$. This implies that the function $\chi : w \mapsto f_{c_0}^m(\phi(w)) - \phi(w)$ has a zero at $w = w_0'$.

Let us consider the function $\chi_k : w \mapsto f_{c_0+Q\rho_k w}^m(\Phi_k(w)) - \Phi_k(w)$, where $\Phi_k(w) = f_{c_0+Q\rho_k w}^{l+kp}(c_0 + Q\rho_k w)$ as in Lemma 1. By the Hurwitz theorem and uniform convergence of Φ_k to ϕ on compact sets of $\mathbb{C}$, χ_k has a zero w_k in Δ and $|w_k - w_0'| < \epsilon/4$ for all sufficiently large k. In particular, $c_k := c_0 + Q\rho_k w_k$ satisfies $f_{c_k}^{m+(l+kp)}(c_k) = f_{c_k}^{l+kp}(c_k)$ and thus $c_k \in \mathbb{M}$. Hence we have a desired $w_k \in \mathcal{M}_k$ with $|w_k - w_0| < \epsilon/2$. ■

Example (Calculation of Q). When $c_0 = -2$ (hence $l = p = 1$), $A_0 = f_{c_0}'(-2) = -4$ and $\lambda_0 = f_{c_0}'(f_{c_0}(-2)) = 4$. Since $a(c)^2 + c = a(c)$ we find $da(c)/dc = -1/(2a(c) - 1)$. Moreover, $db(c)/dc = (d/dc)(c^2 + c) = 2c + 1$. Hence for $c = c_0 = -2$, we have $B_0 = -3 - (-1/3) = -8/3$ and the constant Q is $A_0/B_0 = 3/2$.

Remarks.

- Note that in the proof of Theorem 2, $\phi(w_0')$ need not be repelling. We only need the density of periodic points in the Julia set, which is an easy consequence of Montel's theorem. See [Mi, p.157].

- A similar proof can be applied to semi-hyperbolic parameters (i.e., critically non-recurrent parameters) in $\partial\mathbb{M}$ [Ka, Théorème 2.2], and to the unicritical family $\{z \mapsto z^d + c : c \in \mathbb{C}\}$ with $d \geq 2$. This gives an alternative proof of Rivera-Letelier's extension of Tan's theorem [RL].

Appendix. Existence of the Poincaré function

Here we give a proof for the fact used in the proof of Lemma 1 that $w \mapsto f_{c_0}^{kp}\left(a_0 + \dfrac{w}{\lambda_0^k}\right)$ converges as $k \to \infty$. This is originally shown by using a local linearization theorem by Kœnigs. See [Mi, Cor.8.12]. Our proof is based on the normal family argument and the univalent function theory (see [Du] for example).

Theorem 3 *Let $g : \mathbb{C} \to \mathbb{C}$ be an entire function with $g(0) = 0$, $g'(0) = \lambda$, and $|\lambda| > 1$. Then the sequence $\phi_n(w) = g^n(w/\lambda^n)$ converges uniformly on compact sets in $\mathbb{C}$. Moreover, the limit function $\phi : \mathbb{C} \to \mathbb{C}$ satisfies $g \circ \phi(w) = \phi(\lambda w)$ and $\phi'(0) = 1$.*

Proof. Since $g(z) = \lambda z + O(z^2)$ near $z = 0$, there exists a disk $\Delta = \mathbb{D}(\delta) = \{|z| < \delta\}$ such that $g|\Delta$ is univalent and $\Delta \Subset g(\Delta)$. Hence we have a univalent branch g_0^{-1} of g that maps Δ into itself.

First we show that ϕ_n is univalent on $\mathbb{D}(\delta/4)$: Since the map $\phi_n^{-1} : w \mapsto \lambda^n g_0^{-n}(w)$ is well-defined on $\Delta = \mathbb{D}(\delta)$ and univalent, its image contains $\mathbb{D}(\delta/4)$ by the Koebe 1/4 theorem. Hence ϕ_n is univalent on $\mathbb{D}(\delta/4)$, and by the Koebe distortion theorem, the family $\{\phi_n\}_{n\ge 0}$ is locally uniformly bounded on $\mathbb{D}(\delta/4)$ and thus equicontinuous.

Next we show that ϕ_n has a limit on $\mathbb{D}(\delta/4)$: Fix an arbitrarily large $r > 0$ and an integer N such that $r < \delta|\lambda|^N/4$. By using the Koebe 1/4 theorem as above, the function $G_{N,k}(w) := \lambda^N g^k(w/\lambda^{N+k})$ ($k \in \mathbb{N}$) satisfying $\phi_{N+k} = \phi_N \circ G_{N,k}$ is univalent on the disk $\mathbb{D}(\delta|\lambda|^N/4)$. By the Koebe distortion theorem, there exists a constant $C > 0$ independent of N and k such that for any $w \in \mathbb{D}(r)$ and sufficiently large N we have $|G'_{N,k}(w) - 1| \le C|w|/|\lambda|^N$. By integration we have $|G_{N,k}(w) - w| \le Cr^2/(2|\lambda|^N)$ on $\mathbb{D}(r)$. In particular, $G_{N,k} \to \mathrm{id}$ uniformly on $\mathbb{D}(\delta/4)$ as $N \to \infty$. Since the family $\{\phi_n\}$ is equicontinuous on $\mathbb{D}(\delta/4)$, the relation $\phi_{N+k} = \phi_N \circ G_{N,k}$ implies that $\{\phi_n\}_{n\ge 0}$ is Cauchy and has a unique limit ϕ on any compact sets in $\mathbb{D}(\delta/4)$.

Let us check that the convergence extends to $\mathbb{C}$: (We will not use the functional equation $g^n \circ \phi(w) = \phi(\lambda^n w)$. Compare [Mi, Cor.8.12].) Since $|\phi_{N+k}(w) - \phi_N(w)| = |\phi_N(G_{N,k}(w)) - \phi_N(w)|$ and $|G_{N,k}(w) - w| = Cr^2/(2|\lambda|^N)$ on $\mathbb{D}(r)$, it follows that the family $\{\phi_{N+k}\}_{k\ge 0}$ (with fixed N) is uniformly bounded on $\mathbb{D}(r)$. Hence $\{\phi_n\}_{n\ge 0}$ is normal on any compact set in $\mathbb{C}$ and any sequential limit coincides with the local limit ϕ on $\mathbb{D}(\delta/4)$.

The equation $g \circ \phi(w) = \phi(\lambda w)$ and $\phi'(0) = 1$ are immediate from $g \circ \phi_n(w) = \phi_{n+1}(\lambda w)$ and $\phi_n'(0) = 1$. ■

Remark. One can easily extend this proof to the case of meromorphic g by using the spherical metric.

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