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Zalcman functions and similarity between the Mandelbrot set, Julia sets, and the tricorn

Tomoki Kawahira
Tokyo Institute of Technology

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Dedicated to Lawrence Zalcman on the occasion of his 75th birthday

Abstract

We present a simple proof of Tan's theorem on asymptotic similarity between the Mandelbrot set and Julia sets at Misiurewicz parameters. Then we give a new perspective on this phenomenon in terms of Zalcman functions, that is, entire functions generated by applying Zalcman's lemma to complex dynamics. We also show asymptotic similarity between the tricorn and Julia sets at Misiurewicz parameters, which is an antiholomorphic counterpart of Tan's theorem.

1 Similarity between $\mathbb{M}$ and J

The aim of this paper is to give a new perspective on a well-known similarity between the Mandelbrot set and Julia sets (Tan's theorem) in terms of Zalcman's rescaling principle in non-normal families of meromorphic functions. We start with a simplified proof of Tan's theorem [TL] following [Ka], which motivates the whole idea of this paper.

The Mandelbrot set and the Julia sets. Let us consider the quadratic family

$$\big\{f_c(z) = z^2 + c \,:\, c \in \mathbb{C}\big\}.$$

The *Mandelbrot set* $\mathbb{M}$ is the set of $c \in \mathbb{C}$ such that the sequence $\{f_c^n(c)\}_{n\in\mathbb{N}}$ is bounded. For each $c \in \mathbb{C}$, the *filled Julia set* K_c is the set of $z \in \mathbb{C}$ such that the sequence $\{f_c^n(z)\}_{n\in\mathbb{N}}$ is bounded. One can easily check that

- $c \notin \mathbb{M}$ if and only if $|f_c^n(c)| > 2$ for some $n \in \mathbb{N}$; and
- for each $c \in \mathbb{M}$, $z \notin K_c$ if and only if $|f_c^n(z)| > 2$ for some $n \in \mathbb{N}$.

The *Julia set* J_c is the boundary of K_c. Note that all $\mathbb{M}, K_c$, and J_c are compact, and also non-empty because we can always solve the equations $f_c^n(c) = c$ and $f_c^n(z) = z$.

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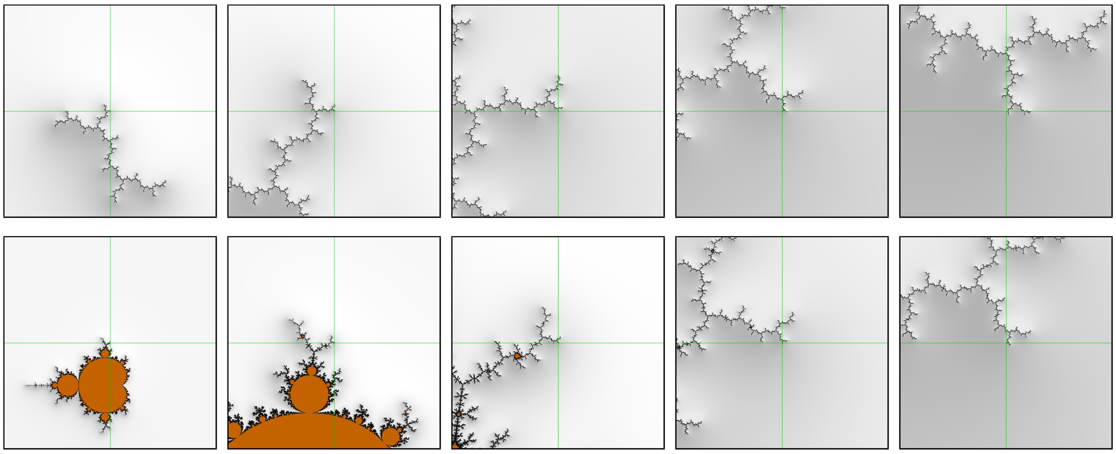
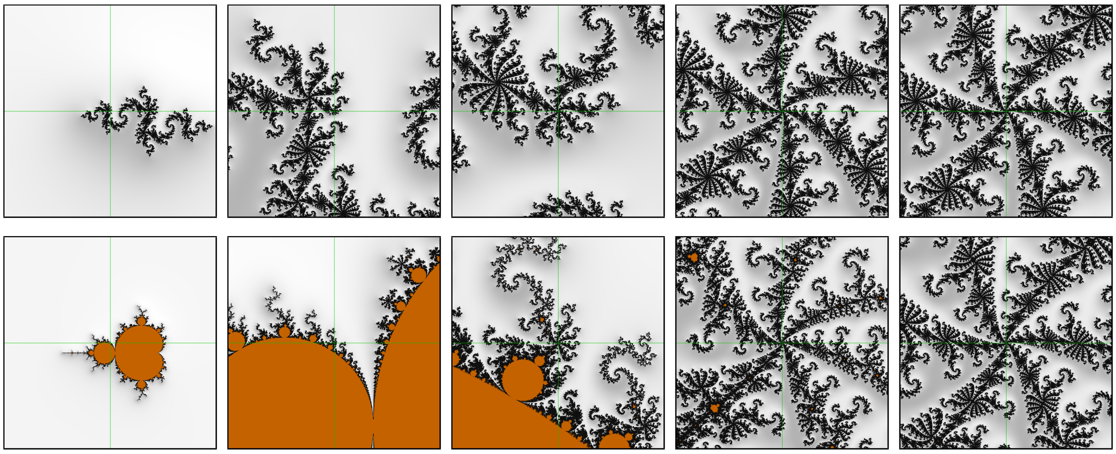
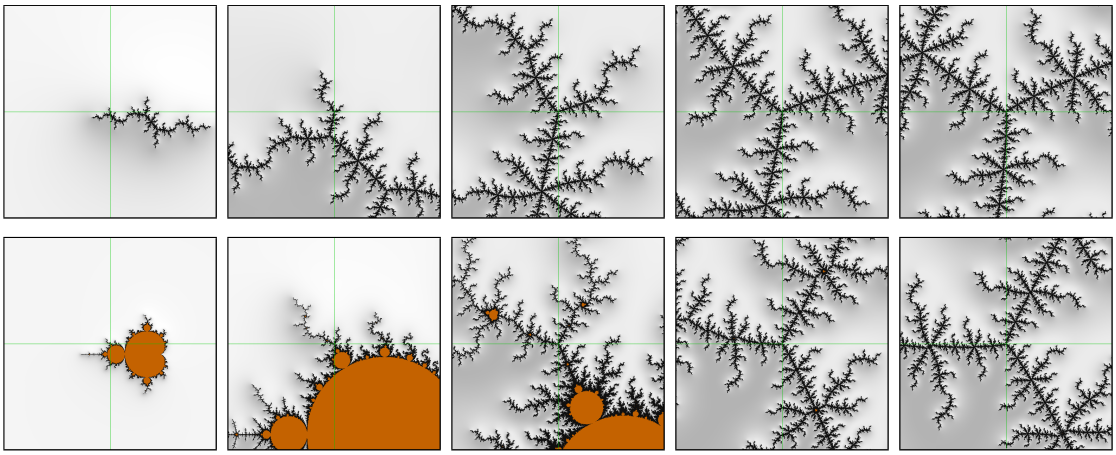


(JM1)



(JM2)



(JM3)

Figure 1: (JM1) Center: $0.0 + 1.0i$, square width: from 6.0 to 0.01. (JM2) Center: $-0.8597644816892409+0.23487923150145784i$, square width: from 5.0 to 0.001. (JM3) Center: $-1.162341599884035 + 0.2923689338965703i$, square width: from 6.0 to 0.001.

Tan showed in [TL] (originally in a chapter of [DH1]) that when $c_0 \in \mathbb{M}$ is a Misiurewicz parameter (to be defined below), the "shapes" of $\mathbb{M}$ and the Julia set J_{c_0} are asymptotically similar at the same point c_0. For example, (JM1) of Figure 1 shows $\mathbb{M}$ and J_{c_0} in squares centered at $c_0 = i$, whose widths range from 6.0 to 0.01. We will prove this by finding an entire function that bridges the dynamical plane and the parameter plane (Lemma 1).

Misiurewicz parameters and a key lemma. Following the terminology of [DH1] and [TL], we say $c_0 \in \mathbb{M}$ is a *Misiurewicz parameter* if the forward orbit of c_0 by f_{c_0} eventually lands on a repelling periodic point. More precisely, there exist minimal $l \geq 1$ and $p \geq 1$ such that $a_0 := f_{c_0}^l(c_0)$ satisfies $a_0 = f_{c_0}^p(a_0)$ and $|(f_{c_0}^p)'(a_0)| > 1$. By the implicit function theorem, we can show that the repelling periodic point a_0 is stable: that is, there exists a neighborhood V of c_0 and a holomorphic map $a : c \mapsto a(c)$ on V such that $a(c_0) = a_0$; $a(c) = f_c^p(a(c))$; and $|(f_c^p)'(a(c))| > 1$. We let $\lambda(c) := (f_c^p)'(a(c))$ and $\lambda_0 := \lambda(c_0)$.

Our key lemma is the following.

Lemma 1 *Suppose that $c_0 \in \mathbb{M}$ is a Misiurewicz parameter as above. For $k \in \mathbb{N}$, set $\rho_k := 1/(f_{c_0}^{l+kp})'(c_0)$. Then we have the following.*

(1) *The function $\phi_k(w) = f_{c_0}^{l+kp}(c_0 + \rho_k w)$ converges to a non-constant entire function $\phi : \mathbb{C} \to \mathbb{C}$ as $k \to \infty$ uniformly on any compact sets.*

(2) *There exists a constant $Q \neq 0$ such that the function*

$$\Phi_k(w) \;:=\; f_{c_0+Q\rho_k w}^{l+kp}(c_0 + Q\rho_k w)$$

converges to the same function $\phi(w)$ as $k \to \infty$ uniformly on compact sets of $\mathbb{C}$.

Proof. It is well-known that the sequence of (polynomial) functions

$$w \;\longmapsto\; f_{c_0}^{kp}\left(a_0 + \frac{w}{\lambda_0^k}\right) \qquad (k \in \mathbb{N})$$

converges to a non-constant entire function $\phi(w)$ uniformly on compact sets of $\mathbb{C}$. (See Theorem 12 in Appendix. Such a ϕ is called a *Poincaré function*. Indeed, ϕ satisfies the functional equation $\phi(\lambda_0 w) = f_{c_0}^p \circ \phi(w)$, but we will not use it.) Note that this function satisfies $\phi(0) = a_0$ and $\phi'(0) = 1$.

Now let us show (1): set $A_0 := (f_{c_0}^l)'(c_0)$, where $A_0 \neq 0$ since otherwise c_0 is strictly periodic. We also have $(f_{c_0}^{l+kp})'(c_0) = A_0\lambda_0^k = 1/\rho_k$. For sufficiently small $t \in \mathbb{C}$, we have the expansion

$$f_{c_0}^l(c_0 + t) \;=\; a_0 + A_0 \cdot t + o(t).$$

Fix an arbitrarily large compact set $E \subset \mathbb{C}$ and take any $w \in E$. Then by setting $t = w/(A_0\lambda_0^k)$,

$$f_{c_0}^{kp}\left(a_0 + \frac{w}{\lambda_0^k}\right) \;\sim\; f_{c_0}^{l+kp}\left(c_0 + \frac{w}{A_0\lambda_0^k} + o(\lambda_0^{-k})\right) \;\sim\; f_{c_0}^{l+kp}(c_0 + \rho_k w) \;=\; \phi_k(w)$$

when $k \to \infty$. (Here by $A_k(w) \sim B_k(w)$ we mean $A_k(w) - B_k(w) \to 0$ uniformly on E as $k \to \infty$.) Hence we have $\phi(w) = \lim_{k\to 0} \phi_k(w)$ on any compact sets. Note that ϕ has no poles, since each ϕ_k is entire.

Next we show (2): suppose that $Q \in \mathbb{C}^*$ is a constant and set $c = c(w) := c_0 + Q\rho_k w$. We also set $\Phi_k(w) := f_c^{l+kp}(c)$ and $b(c) := f_c^l(c)$. Recall that $a(c)$ denotes a repelling periodic point (of f_c) of period p with $a(c_0) = a_0$, and $\lambda(c)$ denotes its multiplier.

Then Theorem 12 in Appendix again implies that the sequence of functions $\phi_k^c(w) := f_c^{kp}(a(c) + w/\lambda(c)^k)$ converges to an entire function $\phi^c(w)$ uniformly on compact sets. In particular, by the proof of Theorem 12, it is not difficult to check that the function $c \mapsto \phi^c(w)$ is holomorphic near $c = c_0$ when we fix a $w \in \mathbb{C}$.

As in Tan's original proof, we employ a theorem on transversality by Douady and Hubbard [DH2, Lemma 1, p.333]: There exists a $B_0 \neq 0$ such that

$$b(c) - a(c) = B_0(c - c_0) + o(c - c_0).$$

Hence for $c = c_0 + Q\rho_k w$ (taking w in a compact set), we have

$$b(c) \;=\; a(c) + B_0 Q\rho_k w + o(\rho_k) \;=\; a(c) + \frac{B_0 Q}{A_0} \cdot \frac{\lambda(c)^k}{\lambda_0^k} \cdot \frac{w}{\lambda(c)^k} + o(\rho_k).$$

Set $Q := A_0/B_0$. Since $\lambda(c)$ is a holomorphic function of c and thus $\lambda(c) = \lambda_0 + O(c - c_0)$, we have $|\lambda(c)/\lambda_0 - 1| = O(c - c_0)$. This implies that

$$\log \frac{\lambda(c)^k}{\lambda_0^k} = k \cdot O(c - c_0) = O\left(\frac{k}{\lambda_0^k}\right) \to 0 \quad (k \to \infty)$$

for $c = c_0 + Q\rho_k w = c_0 + O(\lambda_0^{-k})$. Since

$$\Phi_k(w) = f_c^{kp}(b(c)) \;=\; f_c^{kp}\left(a(c) + \frac{w}{\lambda(c)^k} + o(\rho_k)\right)$$

and $\phi^c(w) \to \phi(w)$ as $c \to c_0$ (uniformly for w in a compact set), we conclude that

$$\lim_{k\to\infty} \Phi_k(w) \;=\; \phi(w),$$

where the convergence is uniform on any compact sets. ■

Remark. Lemma 1 implies that $c_0 \in J_{c_0} = \partial K_{c_0}$ and $c_0 \in \partial\mathbb{M}$. Indeed, we can find a $w \in \mathbb{C}$ such that $|\phi(w)| > 2$ and hence $|\phi_k(w)| > 2$ for sufficiently large k. Equivalently, we have $c_0 + \rho_k w \notin K_{c_0}$ for sufficiently large k, where $c_0 + \rho_k w$ tends to c_0 as $k \to \infty$. Since $c_0 \in K_{c_0}$ by definition, we have $c_0 \in J_{c_0}$. The proof for $c_0 \in \partial\mathbb{M}$ is analogous.

The Hausdorff topology. Let us briefly recall the *Hausdorff topology* of the set of non-empty compact sets $\mathrm{Comp}^*(\mathbb{C})$ of $\mathbb{C}$. For a sequence $\{K_k\}_{k\in\mathbb{N}} \subset \mathrm{Comp}^*(\mathbb{C})$, we say K_k converges to $K \in \mathrm{Comp}^*(\mathbb{C})$ as $k \to \infty$ if for any $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $K \subset \mathrm{N}_\epsilon(K_k)$ and $K_k \subset \mathrm{N}_\epsilon(K)$ for any $k \geq k_0$, where $\mathrm{N}_\epsilon(\cdot)$ is the open ϵ neighborhood in $\mathbb{C}$.

Let $\mathbb{D}(r) := \{z \in \mathbb{C} : |z| < r\}$. For a compact set K in $\mathbb{C}$, let $[K]_r$ denote the set $(K \cap \mathbb{D}(r)) \cup \partial\mathbb{D}(r)$. For $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$, let $a(K - b)$ denote the set of $a(z - b)$ with $z \in K$.

Similarity. Let c_0 be a Misiurewicz parameter. Now we state our version of Tan's similarity theorem.

Theorem 2 (Similarity between $\mathbb{M}$ and J) *There exist a non-constant entire function ϕ on $\mathbb{C}$, a sequence $\rho_k \to 0$, and a constant $q \neq 0$ such that if we set $\mathcal{J} := \phi^{-1}(J_{c_0}) \subset \mathbb{C}$, then for any large constant $r > 0$, we have*

(a) $\left[\rho_k^{-1}(J_{c_0} - c_0)\right]_r \to [\mathcal{J}]_r$, *and*

(b) $\left[\rho_k^{-1} q(\mathbb{M} - c_0)\right]_r \to [\mathcal{J}]_r$

as $k \to \infty$ in the Hausdorff topology.

Proof of (a). Let ϕ_k, ϕ, and $\rho_k = 1/(A_0\lambda_0^k)$ be as given in the proof of Lemma 1. Since $f_{c_0}^n(J_{c_0}) = J_{c_0}$, we have $\left[\rho_k^{-1}(J_{c_0} - c_0)\right]_r = \left[\phi_k^{-1}(J_{c_0})\right]_r$. By $[\mathcal{J}]_r = [\phi^{-1}(J_{c_0})]_r$ and uniform convergence of $\phi_k \to \phi$ on $\overline{\mathbb{D}(r)}$, the claim easily follows.

Proof of (b). Let $q := 1/Q$ (where $Q \neq 0$ is defined in the proof of Lemma 1) and $\mathcal{M}_k := \rho_k^{-1} q(\mathbb{M} - c_0)$. Fix any $\epsilon > 0$. Since the set $\overline{\mathbb{D}(r)} - \mathrm{N}_\epsilon(\mathcal{J})$ is compact, there exists an $N = N(\epsilon)$ such that $|f_{c_0}^N \circ \phi(w)| > 2$ for any $w \in \overline{\mathbb{D}(r)} - \mathrm{N}_\epsilon(\mathcal{J})$. By uniform convergence of $\Phi_k(w) = f_{c_0+Q\rho_k w}^{l+kp}(c_0 + Q\rho_k w)$ to $\phi(w)$ on compact sets in $\mathbb{C}$ (Lemma 1), we have

$$|f_{c_0+Q\rho_k w}^{(l+kp)+N}(c_0 + Q\rho_k w)| \ > \ 2$$

for all sufficiently large k. This implies that $c_0 + Q\rho_k w \notin \mathbb{M}$, equivalently, $w \notin \mathcal{M}_k$. Hence we have

$$[\mathcal{M}_k]_r \ \subset \ \mathrm{N}_\epsilon([\mathcal{J}]_r).$$

Next we show the opposite inclusion $[\mathcal{J}]_r \subset \mathrm{N}_\epsilon([\mathcal{M}_k]_r)$ for k large enough. Let us approximate $[\mathcal{J}]_r$ by a finite subset E of $[\mathcal{J}]_r$ such that the $\epsilon/2$ neighborhood of E covers $[\mathcal{J}]_r$. Now it is enough to prove that for any $w_0 \in E$, there exists a sequence $w_k \in [\mathcal{M}_k]_r$ such that $|w_0 - w_k| < \epsilon/2$ for sufficiently large k.

Let Δ be a disk of radius $\epsilon/2$ centered at w_0. When $\Delta \cap \partial\mathbb{D}(r) \neq \emptyset$, we can take such a w_k in $\partial\mathbb{D}(r)$. Hence we may assume that $\Delta \subset \mathbb{D}(r)$.

Since $\phi(w_0) \in J_{c_0}$ and repelling cycles are dense in J_{c_0} (see [Sch] and [Mi2]. See also the remark below), we can choose a w_0' such that $\phi(w_0')$ is a repelling periodic point of some period m and $|w_0 - w_0'| < \epsilon/4$. This implies that the function $\chi : w \mapsto f_{c_0}^m(\phi(w)) - \phi(w)$ has a zero at $w = w_0'$.

Let us consider the function $\chi_k : w \mapsto f_{c_0+Q\rho_k w}^m(\Phi_k(w)) - \Phi_k(w)$, where $\Phi_k(w) = f_{c_0+Q\rho_k w}^{l+kp}(c_0+Q\rho_k w)$ as in Lemma 1. By the Hurwitz theorem and uniform convergence of Φ_k to ϕ on compact sets of $\mathbb{C}$, χ_k has a zero w_k in Δ and $|w_k - w_0'| < \epsilon/4$ for all sufficiently large k. In particular, $c_k := c_0 + Q\rho_k w_k$ satisfies $f_{c_k}^{m+(l+kp)}(c_k) = f_{c_k}^{l+kp}(c_k)$ and thus $c_k \in \mathbb{M}$. Hence we have a desired $w_k \in \mathcal{M}_k$ with $|w_k - w_0| < \epsilon/2$. ■

Example (Calculation of Q). When $c_0 = -2$ (hence $l = p = 1$), $A_0 = f_{c_0}'(-2) = -4$ and $\lambda_0 = f_{c_0}'(f_{c_0}(-2)) = 4$. Since $a(c)^2 + c = a(c)$ we find $da(c)/dc = -1/(2a(c) - 1)$. Moreover, $db(c)/dc = (d/dc)(c^2 + c) = 2c + 1$. Hence for $c = c_0 = -2$, we have $B_0 = -3 - (-1/3) = -8/3$ and the constant Q is $A_0/B_0 = 3/2$.

Remarks.

- Note that in the proof of Theorem 2, $\phi(w_0')$ need not be repelling. We only need the density of periodic points in the Julia set, which is an easy consequence of Montel's theorem. See [Mi2, p.157].
- A similar proof can be applied to semi-hyperbolic parameters (i.e., critically non-recurrent and no parabolic cycle) in $\partial\mathbb{M}$ [Ka, Théorème 2.2], and to the unicritical family $\{z \mapsto z^d + c : c \in \mathbb{C}\}$ with $d \geq 2$. This gives an alternative proof of Rivera-Letelier's extension of Tan's theorem [RL].

2 Zalcman's lemma and Zalcman functions

The key to the proof above is Lemma 1 that bridges the dynamical and parameter planes by one entire function ϕ. Let us characterize this property in a generalized setting by means of *Zalcman's lemma*, which gives a precise condition for non-normality:

Zalcman's lemma [Za, Za2]. *Let D be a domain in the complex plane $\mathbb{C}$ and $\mathcal{F}$ a family of meromorphic functions on D. The family $\mathcal{F}$ is not normal on any neighborhood of $z_0 \in D$ if and only if there exist sequences $F_k \in \mathcal{F}$, $\rho_k \in \mathbb{C}^*$ with $\rho_k \to 0$; and $z_k \in D$ with $z_k \to z_0$ such that the function $\phi_k(w) = F_k(z_k + \rho_k w)$ converges to a non-constant meromorphic function $\phi : \mathbb{C} \to \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ uniformly on compact subsets in $\mathbb{C}$.*

A universal setting. Let $\overline{\mathcal{U}}$ be the space of meromorphic functions on $\mathbb{C}$ with a topology induced by uniform convergence on compact subsets in the spherical metric. Let $\mathcal{U} \subset \overline{\mathcal{U}}$ be the set of non-constant meromorphic functions on $\mathbb{C}$ and $\mathrm{Aff} \subset \mathcal{U}$ the set of complex affine maps. One can easily check that Zalcman's lemma above can be translated as follows: *A family of meromorphic functions $\mathcal{F}$ on the domain D is not normal in any neighborhood of $z_0 \in D$ if and only if there exist $A_k \in \mathrm{Aff}$ and $f_k \in \mathcal{F}$ ($k \in \mathbb{N}$) such that as $k \to \infty$,*

1. *A_k converges to a constant function z_0 in $\overline{\mathcal{U}}$, and*
2. *$f_k \circ A_k$ converges to some ϕ in $\mathcal{U}$.*

We denote the set of all possible limit function $\phi \in \mathcal{U}$ of this form by $\mathcal{Z}(\mathcal{F}, z_0)$, and we say $\phi \in \mathcal{Z}(\mathcal{F}, z_0)$ is a *Zalcman function of the family $\mathcal{F}$ at z_0*. If $\mathcal{F}$ is normal on a neighborhood of z_0, we formally set $\mathcal{Z}(\mathcal{F}, z_0) := \emptyset$. Now the union

$$\mathcal{Z}(\mathcal{F}) := \bigcup_{z_0 \in D} \mathcal{Z}(\mathcal{F}, z_0) \subset \mathcal{U}$$

is the set of Zalcman functions of the family $\mathcal{F}$.

Dynamical and parametric Zalcman functions. We want to give a new perspective on the asymptotic similarity between the Mandelbrot set $\mathbb{M}$ and the Julia set J_c in terms of Zalcman's lemma. Let us recall the following facts.

Proposition 3 (J and $\partial\mathbb{M}$ as non-normality loci) *For the quadratic family $f_c(z) = z^2 + c$ ($c \in \mathbb{C}$), the Julia sets and the boundary of the Mandelbrot set are characterized as follows:*

(1) *For each $c \in \mathbb{C}$, the Julia set $J_c = \partial K_c$ is the set of points where the family $\{z \mapsto f_c^n(z)\}_{n\geq 0}$ is not normal.*

(2) *The boundary $\partial\mathbb{M}$ of the Mandelbrot set is the set of points where the family $\{c \mapsto f_c^n(c)\}_{n\geq 0}$ is not normal.*

The proof is done by an elementary equicontinuity argument. For the Mandelbrot set case, see [Mc, Theorem 4.6].

Dynamical Zalcman functions. Let $\mathcal{F}_c$ denote the family $\{z \mapsto f_c^n(z)\}_{n\geq 0}$ of polynomial functions on $\mathbb{C}$ for each parameter $c \in \mathbb{C}$. By Proposition 3, we can apply Zalcman's lemma to this family and obtain the sets

$$\mathcal{Z}_c := \mathcal{Z}(\mathcal{F}_c) \quad \text{and} \quad \mathcal{Z}_c(z_0) := \mathcal{Z}(\mathcal{F}_c, z_0)$$

of *dynamical Zalcman functions* of f_c (for each $z_0 \in J_c$). These sets have a good invariance with respect to the operations '$f_c\circ$' and '$\circ\mathrm{Aff}$', as shown by Steinmetz [St, Theorems 1 and 2]:

Proposition 4 (Invariance) *For each $z_0 \in J_c$, the family $\mathcal{Z}_c(z_0)$ satisfies*

$$f_c \circ \mathcal{Z}_c(z_0) \;=\; \mathcal{Z}_c(z_0) \;=\; \mathcal{Z}_c(z_0) \circ \mathrm{Aff}.$$

More precisely,

(1) *If $\phi \in \mathcal{Z}_c(z_0)$ then $f_c \circ \phi \in \mathcal{Z}_c(z_0)$ and $\phi = f_c \circ \phi_1$ for some $\phi_1 \in \mathcal{Z}_c(z_0)$.*

(2) *For any $A \in \mathrm{Aff}$ and $\phi \in \mathcal{Z}_c(z_0)$, we have $\phi \circ A^{\pm 1} \in \mathcal{Z}_c(z_0)$.*

Note that the universal space $\mathcal{U}$ only satisfies $f_c \circ \mathcal{U} \subset \mathcal{U}$ and $\mathcal{U} \circ \mathrm{Aff} = \mathcal{U}$.

Proposition 4 implies that we also have $f_c \circ \mathcal{Z}_c = \mathcal{Z}_c = \mathcal{Z}_c \circ \mathrm{Aff}$. However, the equality $\mathcal{Z}_c = \mathcal{Z}_c(z_0)$ holds for any $z_0 \in J_c$ in most cases. To see this, we introduce some terminology: The *univalent grand orbit* $UGO(z_0)$ of $z_0 \in \mathbb{C}$ is the set of ζ such that $f_c^m(z_0) = f_c^n(\zeta)$ for some $m, n \in \mathbb{N}$ and there is a univalent branch g of $f_c^{-n} \circ f_c^m$ in a neighborhood of z_0 with $g(z_0) = \zeta$. The *postcritical set* P_c of f_c is the closure of the orbit $\{c, f_c(c), f_c^2(c), \cdots\}$. (Note that c is a unique critical value of f_c.) We say f_c *satisfies $(*)$-condition* if

$(*)$: For any $z_0 \in J_c$, there exists a $\zeta \in UGO(z_0) - P_c$.

Now we claim the following:

Theorem 5 *If f_c satisfies $(*)$-condition, then $\mathcal{Z}_c = \mathcal{Z}_c(z_0)$ for any $z_0 \in J_c$. Moreover, the set of such c contains $\mathbb{C} - \partial\mathbb{M}$, which is an open and dense subset of $\mathbb{C}$, and the Misiurewicz parameters in $\partial\mathbb{M}$ except $c_0 = -2$.*

Proof. The first claim of the theorem is an immediate corollary of [Ka, Théorème 3.8]. For the second claim, note that $\partial\mathbb{M}$ consists of the parameters c for which f_c is expanding (hyperbolic) or infinitely renormalizable (see [Mc, §4 and §8]). Since expanding f_c does not contain any critical point in the Julia set J_c, it satisfies $(*)$-condition. Any infinitely renormalizable f_c satisfies $(*)$-condition by [Ka, Proposition 3.7].

Now suppose that c is a Misiurewicz parameter, satisfying $f_c^l(c) = f_c^{l+p}(c)$ with minimal l, $p \geq 1$. Then $z_0 \in P_c$ if and only if $z_0 = f_c^k(c)$ for some $0 \leq k \leq l+p-1$. For such a $z_0 \in P_c$, when $l \geq 2$ or $p \geq 2$, one can easily find some $\zeta \in UGO(z_0) - P_c$. Otherwise $l = p = 1$, equivalent to $c = -2$. In this case we have $P_{-2} = \{-2, 2\} = UGO(-2) = UGO(2)$.

If $z_0 \notin P_c$, z_0 itself is an element of $UGO(z_0) - P_c$. Hence f_c for the Misiurewicz parameter c satisfies $(*)$-condition unless $c = -2$. ■

Remark. When f_c satisfies $(*)$-condition, the dynamical Zalcman function $\mathcal{Z}_c$ can be an alternative ingredient for the Lyubich-Minsky Riemann surface lamination for f_c. See [Ka, §3] for more details, and see [LM] for Lyubich and Minsky's lamination theory in complex dynamics.

Parametric Zalcman functions. Let $\mathcal{Q}$ denote the family $\{c \mapsto f_c^n(c)\}_{n\geq 0}$ of polynomial functions on $\mathbb{C}$. Again by Proposition 3, we can apply Zalcman's lemma to this family and obtain the sets

$$\mathcal{P} := \mathcal{Z}(\mathcal{Q}) \quad \text{and} \quad \mathcal{P}(c_0) := \mathcal{Z}(\mathcal{Q}, c_0)$$

of *parametric Zalcman functions* of the quadratic family $\{f_c(z) = z^2 + c\}_{c\in\mathbb{C}}$ (for each $c_0 \in \partial\mathbb{M}$). These sets have weaker invariance, which is also pointed out by Steinmetz [St, Remark in §1]:

Proposition 6 (Invariance for $\mathcal{P}$) *For each $c_0 \in \partial\mathbb{M}$, the family $\mathcal{P}(c_0)$ satisfies*

$$f_{c_0} \circ \mathcal{P}(c_0) \;=\; \mathcal{P}(c_0) \;=\; \mathcal{P}(c_0) \circ \mathrm{Aff}.$$

Hence we only have $\mathcal{P} = \mathcal{P} \circ \mathrm{Aff}$ for the total space $\mathcal{P}$ of the parametric Zalcman functions.

In a forthcoming paper we will present a general account on dynamical and parametric Zalcman functions for families of rational functions parametrized by Riemann surfaces.

Dynamical-parametric intersection and similarity. By Proposition 4 and Proposition 6 above, if $c_0 \in J_{c_0}$ and $c_0 \in \partial\mathbb{M}$, then $\mathcal{Z}_{c_0}(c_0)$ and $\mathcal{P}(c_0)$ exhibit the same invariance in the universal space $\mathcal{U}$. Hence one might expect that there exists some $\phi \in \mathcal{Z}_{c_0}(c_0) \cap \mathcal{P}(c_0)$ when $c_0 \in J_{c_0} \cap \partial\mathbb{M}$. Indeed, the existence of such an intersection implies asymptotic similarity between J_{c_0} and $\mathbb{M}$ at c_0:

Theorem 7 (Intersection implies similarity) *Suppose that $\mathcal{Z}_{c_0}(c_0) \cap \mathcal{P}(c_0) \neq \emptyset$ for some $c_0 \in \partial\mathbb{M}$. More precisely, there exist sequences of affine maps A_k, $B_k \in \mathrm{Aff}$ and positive integers m_k, $n_k \in \mathbb{N}$ such that as $k \to \infty$ we have*

(1) *Both A_k and B_k converge to the same constant map c_0 in $\overline{\mathcal{U}}$; and*

(2) *Both $f_{c_0}^{n_k}(A_k(w))$ and $f_{B_k(w)}^{n_k}(B_k(w))$ converge to the same entire function $\phi(w)$ in $\mathcal{U}$.*

Suppose in addition that $c_0 \in J_{c_0}$ and let $\mathcal{J} := \phi^{-1}(J_{c_0}) \subset \mathbb{C}$. Then for any large constant $r > 0$, we have

(a) $\left[A_k^{-1}(J_{c_0})\right]_r \to [\mathcal{J}]_r$, *and*

(b) $\left[B_k^{-1}(\mathbb{M})\right]_r \to [\mathcal{J}]_r$

as $k \to \infty$ in the Hausdorff topology.

Proof. Condition (1) implies that if we write $A_k(w) = c_k + \rho_k w$ and $B_k(w) = c'_k + \rho'_k w$ then $c_k, c'_k \to c_0$ and $\rho_k, \rho'_k \to 0$ as $k \to \infty$. Then the proof is only a slight modification of that of Theorem 2. ■

Intersection at Misiurewicz parameters. Suppose that c_0 is a Misiurewicz parameter. (Hence $c_0 \in J_{c_0}$ and $c_0 \in \partial\mathbb{M}$ by the remark below the proof of Lemma 1.) By the construction of the entire function ϕ in Lemma 1, we obviously have $\phi \in \mathcal{Z}_{c_0}(c_0)$ and $\phi \in \mathcal{P}(c_0)$. Thus we conclude the following:

Theorem 8 (Intersection) *For any Misiurewicz parameter c_0, $\mathcal{Z}_{c_0}(c_0)$ and $\mathcal{P}(c_0)$ share at least one element $\phi \in \mathcal{U}$.*

Note that the set of Misiurewicz parameters is a dense subset of $\partial\mathbb{M}$. (This can be shown by a standard normal family argument. See Levin [Le] or [Ka, Théorème 1.1 (1)].)

In [Ka] the author proved Lemma 1 for a wider class of parameters in $\partial\mathbb{M}$, called *semi-hyperbolic parameters*. Shishikura proved in [Sh] that the set of semi-hyperbolic parameters is a dense subset of $\partial\mathbb{M}$ of Hausdorff dimension 2. Hence we also have a generalization of Theorem 8 for semi-hyperbolic parameters.

Question. Besides the example of Lemma 1 and its generalization to semi-hyperbolic parameters, are there any other intersections of the sets of dynamical and parametric Zalcman functions?

3 Similarity between $\mathbb{T}$ and J

We apply the arguments in Section 1 to the antiholomorphic quadratic family and we will show an analogous result to Tan's theorem. For the theory of antiholomorphic quadratic family and the tricorn (as a counterpart of the Mandelbrot set), see [CHRS], [Mi1], [N], [NS], [MNS], [HS], [I], and [IM] for example.

The tricorn and the Julia sets. Let us consider the antiholomorphic quadratic family

$$\left\{g_c(z) = \overline{z}^2 + c \,:\, c \in \mathbb{C}\right\}.$$

The *tricorn* $\mathbb{T}$ is the set of $c \in \mathbb{C}$ such that the sequence $\{g_c^n(c)\}_{n\in\mathbb{N}}$ is bounded. For each $c \in \mathbb{C}$, the *filled Julia set* K_c^* is the set of $z \in \mathbb{C}$ such that the sequence $\{g_c^n(z)\}_{n\in\mathbb{N}}$ is bounded. One can easily check that

- $c \notin \mathbb{T}$ if and only if $|g_c^n(c)| > 2$ for some $n \in \mathbb{N}$; and

- for each $c \in \mathbb{T}$, $z \notin K_c^*$ if and only if $|g_c^n(z)| > 2$ for some $n \in \mathbb{N}$.

The *Julia set* J_c^* is the boundary of K_c^*. Note that all $\mathbb{T}, K_c^*$, and J_c^* are non-empty compact sets.

An intriguing property of $\mathbb{T}$ is that $\mathbb{T} = \omega\mathbb{T} = \omega^2\mathbb{T}$ for $\omega = e^{2\pi i/3}$. One can also check that $\mathbb{T} \cap \mathbb{R} = \mathbb{M} \cap \mathbb{R} = [-2, 1/4]$, and we have $K_c = K_c^*$ and $J_c = J_c^*$ for real c.

Misiurewicz parameters. We say $c_0 \in \partial\mathbb{T}$ is a *Misiurewicz parameter* if the forward orbit of c_0 by g_{c_0} eventually lands on a repelling periodic point. That is, there exist minimal $l \geq 1$ and $p \geq 1$ such that $g_{c_0}^l(c_0) = g_{c_0}^{l+p}(c_0)$ and $|Dg_{c_0}^p(g_{c_0}^l(c_0))| > 1$. (Since $g_{c_0}^p$ or $\overline{g_{c_0}^p}$ is holomorphic, the absolute value $|Dg_{c_0}^p|$ is well-defined.)

Note that both $g_{c_0}^{2l}$ and $g_{c_0}^{2p}$ are holomorphic and that the relation $g_{c_0}^{2l}(c_0) = g_{c_0}^{2l+2p}(c_0)$ is satisfied. We let $a_0 := g_{c_0}^{2l}(c_0)$ and $\tilde{a}_0 := g_{c_0}(a_0)$, which are repelling fixed points of $g_{c_0}^{2p}$ with the same multiplier $\lambda_0 := (g_{c_0}^{2p})'(a_0)$.

Remark. Unlike the Mandelbrot set, the Misiurewicz parameters are not dense in the boundary of $\mathbb{T}$. See [IM, Corollary 5.1].

Similarity. Let us show an antiholomorphic counterpart of Tan's theorem (Theorem 2). That is, the tricorn and the Julia sets are asymptotically similar at the Misiurewicz parameters *up to real linear transformation.* See Figures 2 and 3. In Figure 3, we take a real parameter and compare the Mandelbrot set, the Julia set, and the tricorn.

Let c_0 be a Misiurewicz parameter. Then we have the following:

Theorem 9 (Similarity between $\mathbb{T}$ and J^*) *There exist an entire function ϕ on $\mathbb{C}$, a real linear transformation $h : \mathbb{C} \to \mathbb{C}$, and a sequence $\rho_k \to 0$ such that if we set $\mathcal{J}^* := \phi^{-1}(J_{c_0}^*) \subset \mathbb{C}$, then for any large constant $r > 0$, we have*

(a) $\left[\rho_k^{-1}(J_{c_0}^* - c_0)\right]_r \to [\mathcal{J}^*]_r$, *and*

(b) $\left[\rho_k^{-1}h(\mathbb{T} - c_0)\right]_r \to [\mathcal{J}^*]_r$

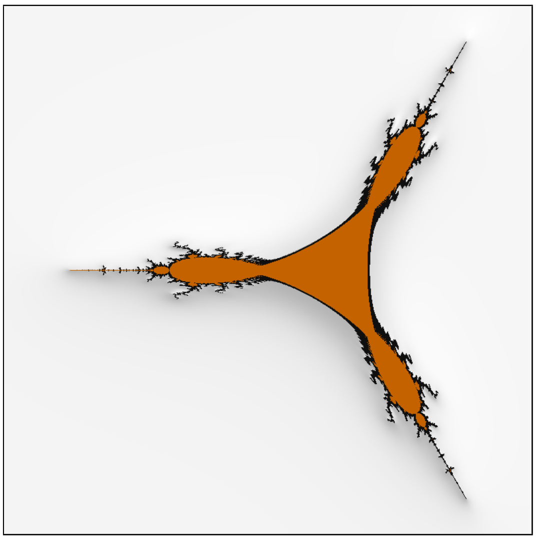
as $k \to \infty$ in the Hausdorff topology.

The proof is analogous to that of Theorem 2: we start with showing an antiholomorphic version of Lemma 1. With the Misiurewicz parameter c_0, integers l and p, and repelling periodic point $a_0 = g_{c_0}^{2l}(c_0)$ taken as above, we have the following:

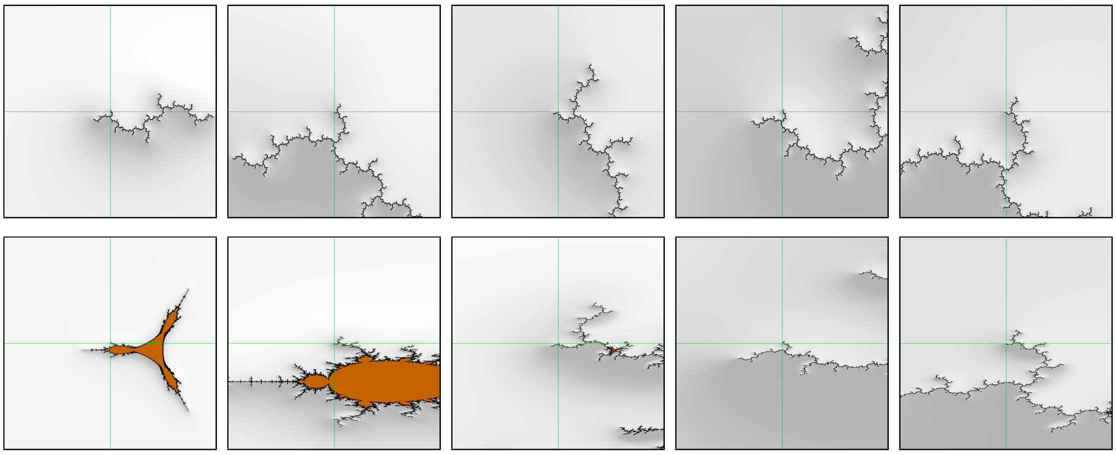
Lemma 10 *Suppose that $c_0 \in \mathbb{T}$ is a Misiurewicz parameter as above. For $k \in \mathbb{N}$, set $\rho_k := 1/(g_{c_0}^{2l+2kp})'(c_0)$. Then we have the following.*

(1) *The function $\phi_k(w) = g_{c_0}^{2l+2kp}(c_0 + \rho_k w)$ converges to a non-constant entire function $\phi : \mathbb{C} \to \mathbb{C}$ as $k \to \infty$ uniformly on any compact sets.*

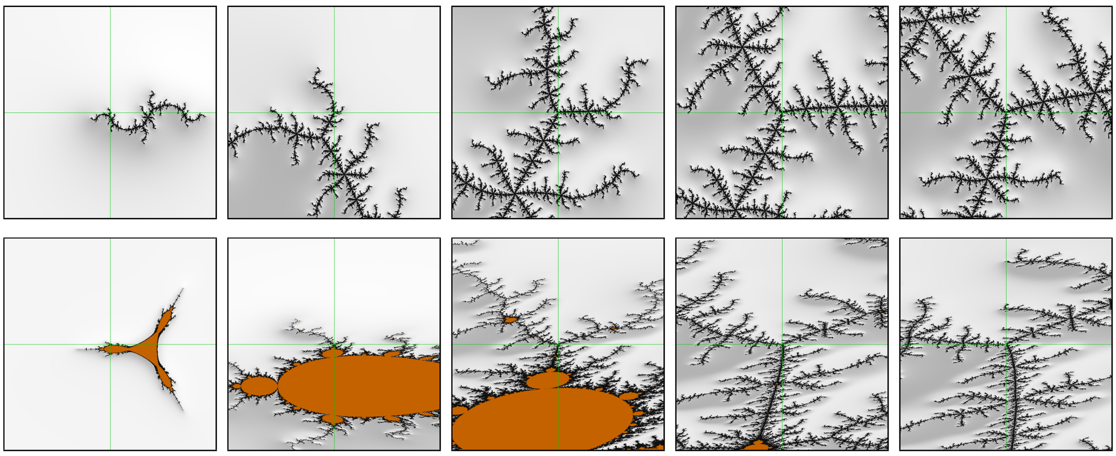
(2) *There exists a real linear transformation $H(w) = Qw + Q'\overline{w}$ with $|Q| \neq |Q'|$ such that the function*
$$\Phi_k(w) \;:=\; g_{c_0+H(\rho_k w)}^{2l+2kp}(c_0 + H(\rho_k w))$$
converges to the same function $\phi(w)$ as $k \to \infty$ uniformly on compact sets of $\mathbb{C}$.



(T)

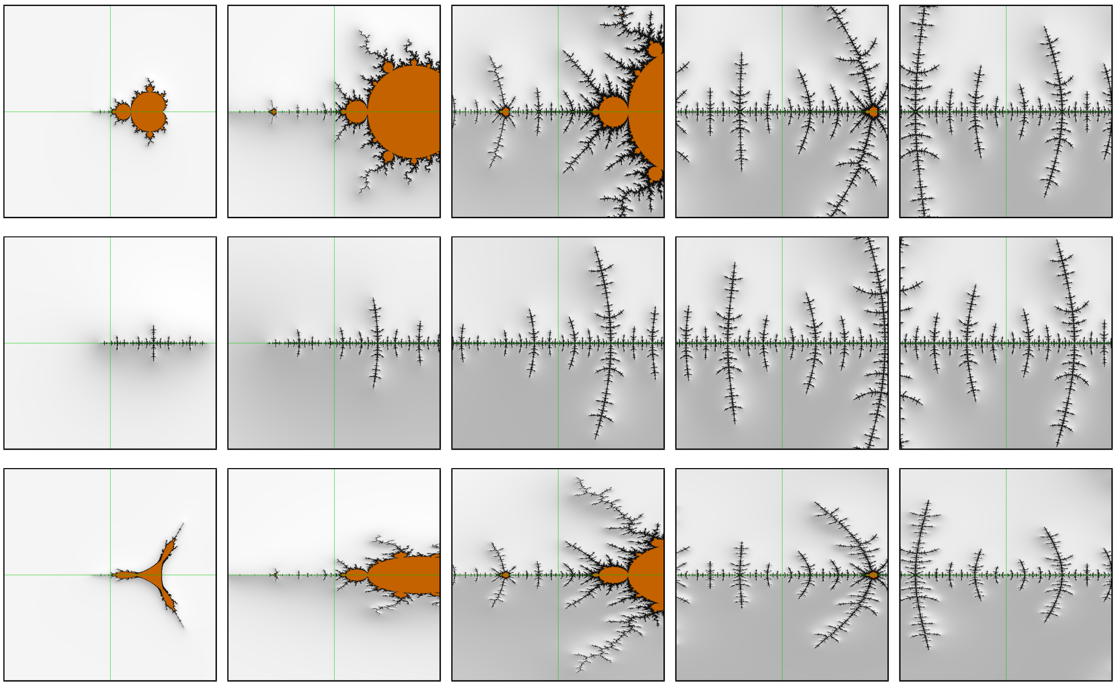


(JT1)



(JT2)

Figure 2: (T) The tricorn $\mathbb{T}$. (JT1) Center: $-1.2222454262925588 + 0.18411010266019595i$, square width: from 6.0 to 0.005. (JT2) Center: $-1.0672232757314006 + 0.13470887783195631i$, square width: from 6.0 to 0.001.



(MJT)

Figure 3: (MJT) Center: -1.4303576324513074, square width: from 7.0 to 0.005.

Note that $\Phi_k(w)$ is a real analytic function of w but it converges to an entire function $\phi(w)$. We prove (1) and (2) separately.

Proof of (1). Both $g_{c_0}^{2l}$ and $g_{c_0}^{2p}$ are holomorphic and thus the usual derivatives $A_0 := (g_{c_0}^{2l})'(c_0) \neq 0$ and $\lambda_0 = (g_{c_0}^{2p})'(a_0)$ make sense. Since $a_0 = g_{c_0}^{2l}(c_0)$ is a repelling fixed point (that is, $|\lambda_0| > 1$) of $g_{c_0}^{2p}$, the Poincaré function

$$\phi(w) := \lim_{n\to\infty} g_{c_0}^{2kp}\left(a_0 + \frac{w}{\lambda_0^k}\right)$$

exists and is an entire function, where the convergence is uniform on compact sets. (See Theorem 12 in Appendix below.) By the expansion $g_{c_0}^{2l}(c_0+t) = a_0 + A_0 t + o(t)$ $(t \to 0)$ we obtain

$$\phi(w) = \lim_{n\to\infty} g_{c_0}^{2l+2kp}\left(c_0 + \frac{w}{A_0\lambda_0^k}\right).$$

Hence we set $\rho_k := (A_0\lambda_0^k)^{-1} = 1/(g_{c_0}^{2l+2kp})'(c_0)$. ■

To show (2), we need an extra lemma about stability and transversality of repelling periodic point a_0:

Lemma 11 (Stability and transversality)

(1) *Stability: There exists a real analytic function $c \mapsto a(c)$ defined near c_0 with $a(c_0) = a_0$ such that $a(c)$ is a repelling fixed point of g_c^{2p}, for which the multiplier $\lambda(c) := (g_c^{2p})'(a(c))$ is also a real analytic function near c_0.*

(2) *Transversality: Let $b(c) := g_c^{2l}(c)$. Then there exist two complex numbers B_0 and B_0' with $|B_0| \neq |B_0'|$ such that*

$$b(c) - a(c) = B_0(c - c_0) + B_0'\overline{(c - c_0)} + o(|c - c_0|)$$

as $c \to c_0$.

Note that B_0 and B_0' do not vanish simultaneously by the condition $|B_0| \neq |B_0'|$.

I learned the idea of using bi-quadratic maps to the antiholomorphic quadratic family from Hiroyuki Inou.

Proof. Let $G_c(z) := g_c^2(z) = (z^2 + \overline{c})^2 + c$, whose iterations are polynomial functions with variable z with coefficients in $\mathbb{Z}[c, \overline{c}]$. Then $a_0 = g_{c_0}^{2l}(c_0) = G_{c_0}^l(c_0)$ is a repelling fixed point of $G_{c_0}^p$ with multiplier $(G_{c_0}^p)'(a_0) = \lambda_0$.

For (1), by a variant of the argument principle, such an $a(c)$ is explicitly given by

$$a(c) := \frac{1}{2\pi i} \int_{|z-a_0|=\epsilon} z \cdot \frac{1 - (G_c^p)'(z)}{z - G_c^p(z)}\, dz$$

where z makes one turn anticlockwise around a_0 with a fixed, sufficiently small radius ϵ. Indeed, one can check that $G_c^p(a(c)) = a(c)$ and real analyticity of $a(c)$ in c comes from that of $G_c^p(z)$. The multiplier $\lambda(c) := (G_c^p)'(a(c))$ is real analytic as well, and it satisfies $|\lambda(c)| > 1$ for c sufficiently close to c_0, since we have $|\lambda(c_0)| > 1$ by assumption.

To show (2) we employ a transversality result by van Strien [vS, Main Theorem 1.1], applied to the bi-quadratic family

$$\left\{F_{s,t}(z) := f_t \circ f_s(z) = (z^2 + s)^2 + t\right\}_{s,\, t \in \mathbb{C}} \simeq \mathbb{C}^2.$$

Since $g_c^2(z) = G_c(z) = F_{\overline{c},c}(z)$, our antiholomorphic family $\{g_c\}_{c\in\mathbb{C}}$ can be regarded as a real analytic curve $\{(s,t) = (\overline{c}, c) \in \mathbb{C}^2 \,:\, c \in \mathbb{C}\}$ in $\mathbb{C}^2$. The critical points of $F_{s,t}$ are $\pm\sqrt{-s}$ and 0, but there is essentially one critical orbit when $(s,t) = (\overline{c}, c)$ since $F_{\overline{c},c}(\pm\sqrt{-\overline{c}}) = g_c(0) = c$. One can also check that $F_{s,t} = A \circ F_{s',t'} \circ A^{-1}$ for some $A \in \mathrm{Aff}$ if and only if $(s',t') = (s,t)$, $(\omega s, \omega^2 t)$ or $(\omega^2 s, \omega t)$. This implies that the family $\{F_{s,t}\}$ is locally normalized near $(s,t) = (\overline{c_0}, c_0) \neq (0,0)$.

When $(s,t) = (\overline{c_0}, c_0)$, we have

$$a_0 = G_{c_0}^{l+1}(\pm\sqrt{-\overline{c_0}}) = G_{c_0}^{l+p+1}(\pm\sqrt{-\overline{c_0}})$$

and

$$\tilde{a}_0 := g_{c_0}(a_0) = G_{c_0}^{l+1}(0) = G_{c_0}^{l+p+1}(0),$$

where both a_0 and $\tilde{a}_0$ are repelling fixed points of $G_{c_0}^p = F_{\overline{c_0},c_0}^p$. Then we have two analytic families

$$a(s,t) := \frac{1}{2\pi i} \int_{|z-a_0|=\epsilon} z \cdot \frac{1 - (F_{s,t}^p)'(z)}{z - F_{s,t}^p(z)}\, dz$$

and

$$\tilde{a}(s,t) := \frac{1}{2\pi i} \int_{|z-\tilde{a}_0|=\epsilon} z \cdot \frac{1 - (F_{s,t}^p)'(z)}{z - F_{s,t}^p(z)}\, dz$$

of repelling fixed points of $F^p_{s,t}$, where (s,t) are sufficiently close to $(\overline{c_0}, c_0)$ in $\mathbb{C}^2$.

By [vS, Main Theorem 1.1], the map

$$(s,t) \mapsto \big(F^{l+1}_{s,t}(-\sqrt{-s}) - a(s,t), F^{l+1}_{s,t}(\sqrt{-s}) - a(s,t), F^{l+1}_{s,t}(0) - \tilde{a}(s,t)\big)$$

is a local immersion near $(s,t) = (\overline{c_0}, c_0)$. In other words, its Jacobian derivative at $(s,t) = (\overline{c_0}, c_0)$ has rank 2. Since the first and the second coordinates of the image always coincide, the derivative of the map

$$(s,t) \mapsto (u,v) := \big(F^{l+1}_{s,t}(\sqrt{-s}) - a(s,t), F^{l+1}_{s,t}(0) - \tilde{a}(s,t)\big)$$

at $(s,t) = (\overline{c_0}, c_0)$ is rank 2 as well. Hence if we write

$$\begin{aligned} u &= B_0'(s - \overline{c_0}) + B_0(t - c_0) + o\Big(\sqrt{(s-\overline{c_0})^2 + (t-c_0)^2}\Big), \quad \text{and} \\ v &= B_1'(s - \overline{c_0}) + B_1(t - c_0) + o\Big(\sqrt{(s-\overline{c_0})^2 + (t-c_0)^2}\Big), \end{aligned}$$

we have $B_0'B_1 - B_0B_1' \neq 0$. Now we let $(s,t) = (\bar{c}, c)$ with c sufficiently close to c_0. Then we have

$$u = G^{l+1}_c(\sqrt{-\bar{c}}) - a(\bar{c}, c) = g^{2l}_c(c) - a(c) = b(c) - a(c)$$

and the expansion above implies

$$u = b(c) - a(c) = B_0'(\bar{c} - \overline{c_0}) + B_0(c - c_0) + o(|c - c_0|)$$

as in the statement. Hence it is enough to show $|B_0| \neq |B_0'|$. Since we have

$$v = G^{l+1}_c(0) - \tilde{a}(\bar{c}, c) = g^{2l+2}_c(0) - g_c(a(c)) = g_c(b(c)) - g_c(a(c)),$$

the expansion above implies

$$v = \overline{b(c)}^2 - \overline{a(c)}^2 = B_1'(\bar{c} - \overline{c_0}) + B_1(c - c_0) + o(|c - c_0|).$$

On the other hand,

$$\begin{aligned} b(c)^2 - a(c)^2 &= (b(c) - a(c))(b(c) + a(c)) \\ &= (B_0'(\bar{c} - \overline{c_0}) + B_0(c - c_0) + o(|c - c_0|))(2a_0 + O(|c - c_0|)) \\ &= 2a_0B_0'(\bar{c} - \overline{c_0}) + 2a_0B_0(c - c_0) + o(|c - c_0|). \end{aligned}$$

Hence $B_1 = \overline{2a_0B_0'}$ and $B_1' = \overline{2a_0B_0}$ (where we have $a_0 \neq 0$, otherwise a_0 cannot be a repelling periodic point of g_c). Since $B_0'B_1 - B_0B_1' \neq 0$, we conclude $|B_0'|^2 - |B_0|^2 \neq 0$. This completes the proof. ■

Remark. In the proof of Lemma 1 for the quadratic family, we used the transversality result [DH2, Lemma 1, p.333], which is proved in a purely algebraic way. On the other hand, [vS, Main Theorem 1.1] used above assumes Thurston's rigidity theorem. Recently Levin, Shen, and van Strien [LSvS] showed a different version of transversality (in the space of rational maps) by a rather elementary method. It would be interesting to show Lemma 11(2) in a purely algebraic way.

Now let us go back to the proof of Lemma 10.

Proof of Lemma 10(2). Let us take a real linear transformation $H : \mathbb{C} \to \mathbb{C}$ of the form $W \mapsto H(W) := QW + Q'\overline{W}$, with some complex constants Q and Q'. For $k \in \mathbb{N}$ and $w \in \mathbb{C}$ we define a real analytic function $\Phi_k(w)$ of the form

$$\Phi_k(w) := g_c^{2l+2kp}(c) = g_{c_0+H(\rho_k w)}^{2l+2kp}(c_0 + H(\rho_k w)),$$

where we let $c = c_0 + H(\rho_k w)$ and $\rho_k = (A_0\lambda_0^k)^{-1}$. We want to determine the constants Q and Q' such that $\Phi_k(w)$ converges to $\phi(w)$ uniformly on compact sets of $\mathbb{C}$.

Let $\lambda(c) = (g_c^{2p})'(a(c))$, as given in Lemma 11(1). Then the function $w \mapsto g_c^{2kp}(a(c)+ w/\lambda(c)^k)$ converges to the Poincaré function for $a(c)$ as $k \to \infty$.

On the other hand, when $c = c_0 + H(\rho_k w)$ with w in a compact set, we have

$$\begin{aligned}
\Phi_k(w) &= g_c^{2kp}(b(c)) \\
&= g_c^{2kp}\Big(a(c) + B_0(c - c_0) + B_0'\overline{(c - c_0)} + o(|c - c_0|)\Big) \\
&\sim g_c^{pk}\Big(a(c) + B_0 H(\rho_k w) + B_0'\overline{H(\rho_k w)} + o(\rho_k)\Big)
\end{aligned}$$

by Lemma 11(2). Hence it is enough to regard $B_0 H(\rho_k w) + B_0'\overline{H(\rho_k w)}$ as $w/\lambda(c)^k$. Since $\lambda_0^k/\lambda(c)^k \to 1$ as $k \to \infty$, we obtain the condition

$$B_0(Q\rho_k w + Q'\overline{\rho_k w}) + B_0'(\overline{Q\rho_k w} + \overline{Q'}\rho_k w) = \frac{w}{\lambda_0^k}.$$

Since $\rho_k = 1/(A_0\lambda_0^k)$, we necessarily have

$$Q = \frac{A_0\overline{B_0}}{|B_0|^2 - |B_0'|^2} \quad \text{and} \quad Q' = -\frac{\overline{A_0}B_0'}{|B_0|^2 - |B_0'|^2}.$$

Note that $|B_0| \neq |B_0'|$ by Lemma 11(2), and this implies $|B_0|^2 - |B_0'|^2 \neq 0$ and $|Q| \neq |Q'|$. Conversely, with these Q and Q' we have $\Phi_k(w) \to \phi(w)$ $(k \to \infty)$ uniformly on compact sets. ■

Proof of Theorem 9. Let $h := H^{-1}$, where H is given in Lemma 10. Then the proof follows the same argument as that of Theorem 2. However, an extra effort should be made when we choose a repelling periodic point $\phi(w_0')$ in the Julia set of g_{c_0}. We should take a w_0' (with the original condition $|w_0 - w_0'| < \epsilon/4$) such that $g_{c_0}^{2m}(\phi(w_0')) = \phi(w_0')$ for some $m \in \mathbb{N}$ and $\phi'(w_0) \neq 0$. Then w_0' is a simple zero of a holomorphic function $F(w) := g_{c_0}^{2m}(\phi(w)) - \phi(w)$. By Lemma 10(2), this function is uniformly approximated by a real analytic function (hence the Hurwitz theorem does not work)

$$G_k(w) := g_{c_0+H(\rho_k w)}^{2m}(\Phi_k(w)) - \Phi_k(w) = g_c^{2m+2l+2kp}(c) - g_c^{2l+2kp}(c)$$

near $w = w_0'$ with sufficiently large k, where $c = c_0 + H(\rho_k w)$. Then one can show that G_k maps a small round disk D centered at w_0' homeomorphically onto a topological disk containing 0 when k is sufficiently large; and that $w_k := (G_k|_D)^{-1}(0)$ tends to w_0' as $k \to 0$. Then $c_k := c_0 + H(\rho_k w_k)$ satisfies $g_{c_k}^{2m+2l+2kp}(c_k) = g_{c_k}^{2l+2kp}(c_k)$ and this implies that $c_k \in \mathbb{T}$. The remaining details are left to the reader. ■

Remark. Theorem 9 can be easily generalized to the unicritical antiholomorphic family $\big\{z \mapsto \overline{z}^d + c \,:\, c \in \mathbb{C}\big\}$ with $d \geq 2$. (In this case the tricorn becomes the *multicorn*, usually denoted by $\mathcal{M}_d^*$.)

Appendix. Existence of the Poincaré function

Here we give a proof of the existence of the Poincaré functions associated with repelling periodic points. This is originally shown by using a local linearization theorem by Koenigs. See [Mi2, Cor.8.12]. Our proof is based on the normal family argument and the univalent function theory (see [Du] for example), which follows the idea of [LM, Lemma 4.7].

Theorem 12 *Let $g : \mathbb{C} \to \mathbb{C}$ be an entire function with $g(0) = 0$, $g'(0) = \lambda$, and $|\lambda| > 1$. Then the sequence $\phi_n(w) = g^n(w/\lambda^n)$ converges uniformly on compact sets in $\mathbb{C}$. Moreover, the limit function $\phi : \mathbb{C} \to \mathbb{C}$ satisfies $g \circ \phi(w) = \phi(\lambda w)$ and $\phi'(0) = 1$.*

Proof. Since $g(z) = \lambda z + O(z^2)$ near $z = 0$, there exists a disk $\Delta = \mathbb{D}(\delta) = \{z \in \mathbb{C} : |z| < \delta\}$ such that $g|\Delta$ is univalent and $\Delta \Subset g(\Delta)$. Hence we have a univalent branch g_0^{-1} of g that maps Δ into itself.

First we show that ϕ_n is univalent on $\mathbb{D}(\delta/4)$: Since the map $\phi_n^{-1} : w \mapsto \lambda^n g_0^{-n}(w)$ is well-defined on $\Delta = \mathbb{D}(\delta)$ and univalent, its image contains $\mathbb{D}(\delta/4)$ by the Koebe 1/4 theorem. Hence ϕ_n is univalent on $\mathbb{D}(\delta/4)$, and by the Koebe distortion theorem, the family $\{\phi_n\}_{n\geq 0}$ is locally uniformly bounded on $\mathbb{D}(\delta/4)$ and thus equicontinuous.

Next we show that ϕ_n has a limit on $\mathbb{D}(\delta/4)$: Fix an arbitrarily large $r > 0$ and an integer N such that $r < \delta|\lambda|^N/4$. By using the Koebe 1/4 theorem as above, the function $G_{N,k}(w) := \lambda^N g^k(w/\lambda^{N+k})$ ($k \in \mathbb{N}$) satisfying $\phi_{N+k} = \phi_N \circ G_{N,k}$ is univalent on the disk $\mathbb{D}(\delta|\lambda|^N/4)$. By the Koebe distortion theorem, there exists a constant $C > 0$ independent of N and k such that for any $w \in \mathbb{D}(r)$ and sufficiently large N we have $|G'_{N,k}(w) - 1| \leq C|w|/|\lambda|^N$. By integration we have $|G_{N,k}(w) - w| \leq Cr^2/(2|\lambda|^N)$ on $\mathbb{D}(r)$. In particular, $G_{N,k} \to \mathrm{id}$ uniformly on $\mathbb{D}(\delta/4)$ as $N \to \infty$. Since the family $\{\phi_n\}$ is equicontinuous on $\mathbb{D}(\delta/4)$, the relation $\phi_{N+k} = \phi_N \circ G_{N,k}$ implies that $\{\phi_n\}_{n\geq 0}$ is Cauchy and has a unique limit ϕ on any compact sets in $\mathbb{D}(\delta/4)$.

Let us check that the convergence extends to $\mathbb{C}$: (We will not use the functional equation $g^n \circ \phi(w) = \phi(\lambda^n w)$. Compare [Mi2, Cor.8.12].) Since $|\phi_{N+k}(w) - \phi_N(w)| = |\phi_N(G_{N,k}(w)) - \phi_N(w)|$ and $|G_{N,k}(w) - w| = Cr^2/(2|\lambda|^N)$ on $\mathbb{D}(r)$, it follows that the family $\{\phi_{N+k}\}_{k\geq 0}$ (with fixed N) is uniformly bounded on $\mathbb{D}(r)$. Hence $\{\phi_n\}_{n\geq 0}$ is normal on any compact set in $\mathbb{C}$ and any sequential limit coincides with the local limit ϕ on $\mathbb{D}(\delta/4)$.

The equation $g \circ \phi(w) = \phi(\lambda w)$ and $\phi'(0) = 1$ are immediate from $g \circ \phi_n(w) = \phi_{n+1}(\lambda w)$ and $\phi_n'(0) = 1$. ■

Remark. One can easily extend this proof to the case of meromorphic g by using the spherical metric.

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Tomoki Kawahira
Department of Mathematics
Tokyo Institute of Technology
Tokyo 152-8551, Japan
kawahira@math.titech.ac.jp

Mathematical Science Team
RIKEN Center for Advanced Intelligence Project (AIP)
1-4-1 Nihonbashi, Chuo-ku
Tokyo 103-0027, Japan