JULIA SETS APPEAR QUASICONFORMALLY IN THE MANDELBROT SET

TOMOKI KAWAHIRA AND MASASHI KISAKA

Abstract. In this paper we prove the following: Take any “small Mandelbrot set” and zoom in a neighborhood of a parabolic or Misiurewicz parameter in it, then we can see a quasiconformal image of a Cantor Julia set which is a perturbation of a parabolic or Misiurewicz Julia set. Furthermore, zoom in its middle part, then we can see a certain nested structure (“decoration”) and finally another “smaller Mandelbrot set” appears. A similar nested structure exists in the Julia set for any parameter in the “smaller Mandelbrot set”. We can also find images of a Julia sets by quasiconformal maps with dilatation arbitrarily close to 1. All the parameters belonging to these images are semihyperbolic and this leads to the fact that the set of semihyperbolic but non-Misiurewicz parameters are dense with Hausdorff dimension 2 in the boundary of the Mandelbrot set.

1. Introduction

Let $P_c(z) := z^2 + c$ and recall that its *filled Julia set* K_c is defined by

$$K_c := \{z \in \mathbb{C} \mid \{P_c^n(z)\}_{n=0}^{\infty} \text{ is bounded}\}$$

and its *Julia set* J_c is the boundary of K_c, that is, $J_c := \partial K_c$. It is known that J_c is connected if and only if the critical orbit $\{P_c^n(0)\}_{n=0}^{\infty}$ is bounded and if J_c is disconnected, it is a Cantor set. The connectedness locus of the quadratic family $\{P_c\}_{c\in\mathbb{C}}$ is the famous *Mandelbrot set* and we denote it by M:

$$M := \{c \mid J_c \text{ is connected}\} = \{c \mid \{P_c^n(0)\}_{n=0}^{\infty} \text{ is bounded}\}.$$

A parameter c is called a *Misiurewicz parameter* if the critical point 0 is strictly preperiodic, that is,

$$P_c^k(P_c^l(0)) = P_c^l(0) \quad \text{and} \quad P_c^k(P_c^{l-1}(0)) \neq P_c^{l-1}(0)$$

for some $k,\ l \in \mathbb{N} = \{1, 2, 3, \cdots\}$. A parameter c is called a *parabolic parameter* if P_c has a parabolic periodic point. Here, a periodic point z_0 with period m is called *parabolic* if $P_c^m(z_0) = z_0$ and its multiplier $(P_c^m)'(z_0)$ is a root of unity. For the basic knowledge of complex dynamics, we refer to [**B**] and [**Mil2**].

In 2000, Douady et al. ([**D-BDS**]) proved the following: At a small neighborhood of the cusp point $c_0 \neq 1/4$ in M, which is in a (primitive) “small Mandelbrot set”, there is a sequence $\{M_n\}_{n\in\mathbb{N}}$ of small quasiconformal copies of M tending to c_0. Moreover each M_n is encaged in a nested sequence of sets which are homeomorphic to the preimage of $J_{\frac{1}{4}+\eta}$ (for $\eta > 0$ small) by $z \mapsto z^{2^m}$ for $m \geq 0$ and accumulate on M_n.

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In this paper, firstly we generalize some parts of their results (Theorem A). Actually this kind of phenomena can be observed not only in a small neighborhood of the cusp of a small Mandelbrot set, that is, the point corresponding to a parabolic parameter $1/4 \in \partial M$, but also in every neighborhood of a point corresponding to any Misiurewicz or parabolic parameters c_0 in a small Mandelbrot set. (For example, $c_0 = 1/4 \in \partial M$ can be replaced by a Misiurewicz parameter $c_0 = i \in \partial M$ or a parabolic parameter $c_0 = -3/4 \in \partial M$.) More precisely, we show the following: Take any small Mandelbrot set M_{s_0} (Figure 1-(1))) and zoom in the neighborhood of $c_0' \in \partial M_{s_0}$ corresponding to $c_0 \in \partial M$ which is a Misiurewicz or a parabolic parameter (Figure 1-(2) to (6))). (Note that c_0' itself is also a Misiurewicz or a parabolic parameter.) Then we can find a subset $J' \subset \partial M$ which looks very similar to J_{c_0} (Figure 1–(6)). Zoom in further, then this J' turns out to be similar to $J_{c_0+\eta}$, where $|\eta|$ is very small and $c_0 + \eta \notin M$ rather than J_{c_0}, because J' looks disconnected (Figure 1–(8), (9)). Furthermore, as we further zoom in the middle part of J', we can see a nested structure which is very similar to the iterated preimages of $J_{c_0+\eta}$ by z^2 (we call these a *decoration*) (Figure 1–(10), (12), (14)) and finally another smaller Mandelbrot set M_{s_1} appears (Figure 1–(15)).

Secondly we show the following result for filled Julia sets (Theorem B): Take a parameter c from the above smaller Mandelbrot set M_{s_1} and look at the filled Julia set K_c and its zooms around the neighborhood of $0 \in K_c$. Then we can observe a very similar nested structure to what we saw as zooming in the middle part of the set $J' \subset \partial M$ (see Figure 2).

Thirdly we show that some of the smaller Mandelbrot sets M_{s_1} and their decorations are images of certain model sets by quasiconformal maps whose dilatations are arbitrarily close to 1 (Theorem C).

Finally we show that all the parameters belonging to the decorations are semihyperbolic and also the set of semihyperbolic but non-Misiurewicz parameters are dense in the boundary of the Mandelbrot set (Corollary D). This together with Theorem C leads to a direct and intuitive explanation for the fact that the Hausdorff dimension of ∂M is equal to 2, which is a famous result by Shishikura ([**S**]).

According to Wolf Jung, a structure in the Mandelbrot set which resembles a whole Julia set in appearance was observed in computer experiments decades ago by Robert Munafo and Jonathan Leavitt. He also claims that he described a general explanation in his website ([**J**]). We believe some other people have already observed these phenomena so far. For example, we note that Morosawa, Nishimura, Taniguchi and Ueda observed this kind of "similarity" in their book ([**MTU**, p.19], [**MNTU**, p.26]). Further, earlier than this observation, Peitgen observed a kind of local similarity between Mandelbrot set and a Julia set by computer experiment ([**PS**, Figure 4.23]).

There are different kinds of known results so far which show that some parts of the Mandelbrot set are similar to some (part of) Julia sets. The first famous result for this kind of phenomena is the one by Tan Lei ([**T1**]). She showed that as we zoom in the neighborhood of any Misiurewicz parameter $c \in \partial M$, it looks like very much the same as the magnification of J_c in the neighborhood of $c \in J_c$. Later this result was generalized to the case where c is a semihyperbolic parameter by Rivera-Letelier ([**Riv**]) and its alternative proof is given by the first author ([**K2**]). On the other hand, some connected

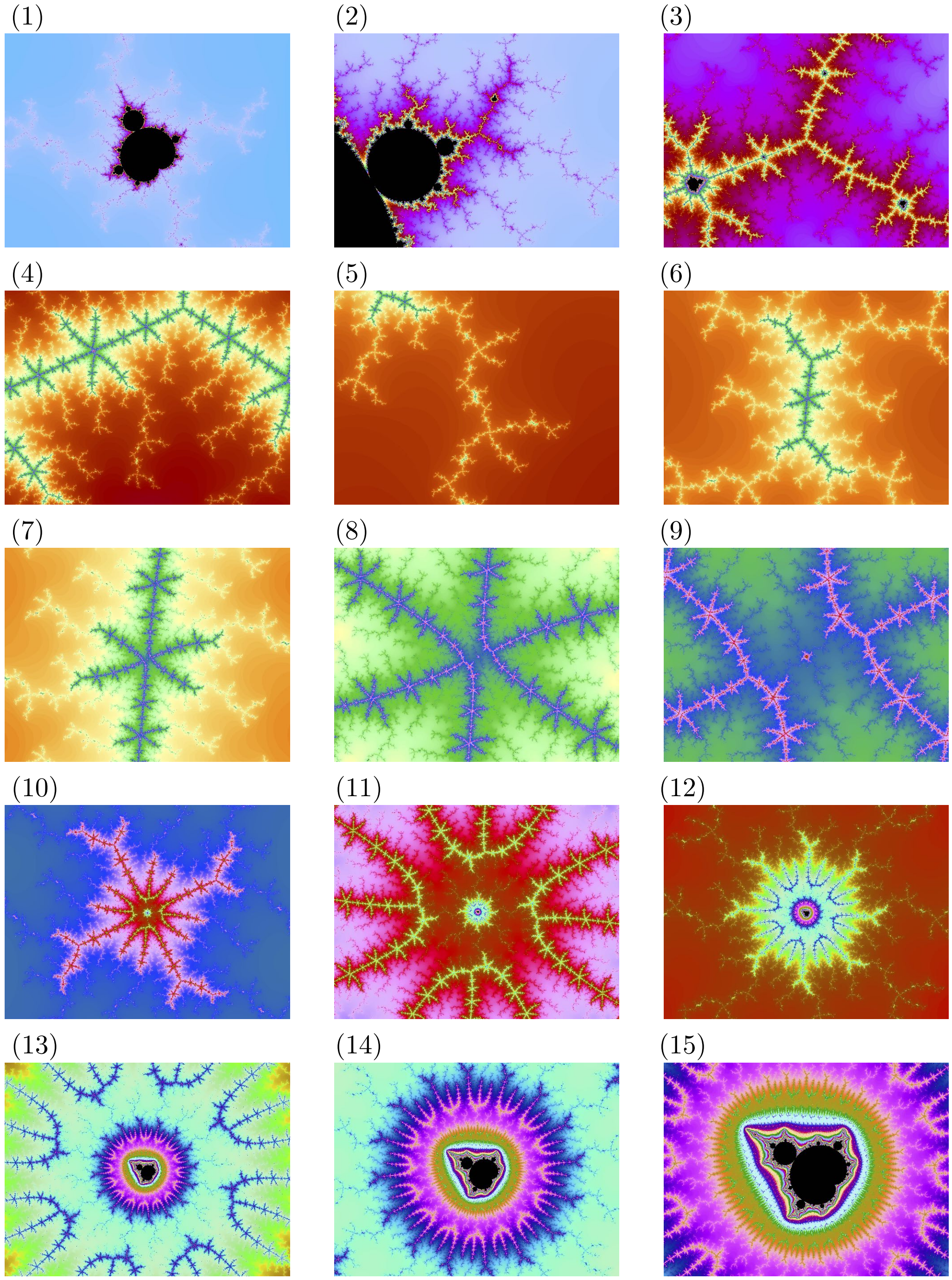

FIGURE 1. Zooms around a Misiurewicz point $c_0' = c_1 = s_0 \perp c_0$ in a primitive small Mandelbrot set M_{s_0}, where c_0 is a Misiurewicz parameter satisfying $P_{c_0}(P_{c_0}^4(0)) = P_{c_0}^4(0)$. After a sequence of nested structures, another smaller Mandelbrot set M_{s_1} appears in (15). Here, $s_0 = 0.3591071125276155 + 0.6423830938166145i$, $c_0 = -0.1010963638456221 + 0.9562865108091415i$, $c_1 = 0.3626697754647427 + 0.6450273437137847i$, $s_1 = 0.3626684938191616 + 0.6450238859863952i$. The widths of the figures (1) and (15) are about $10^{-1.5}$ and $10^{-11.9}$, respectively.

Julia sets of quadratic polynomial can appear quasiconformally in a certain parameter space of a family of cubic polynomials. Buff and Henriksen showed that the bifurcation locus of the family $\{f_b(z) = \lambda z + bz^2 + z^3\}_{b \in \mathbb{C}}$, where $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ contains quasiconformal copies of $J(\lambda z + z^2)$ ([**BH**]).

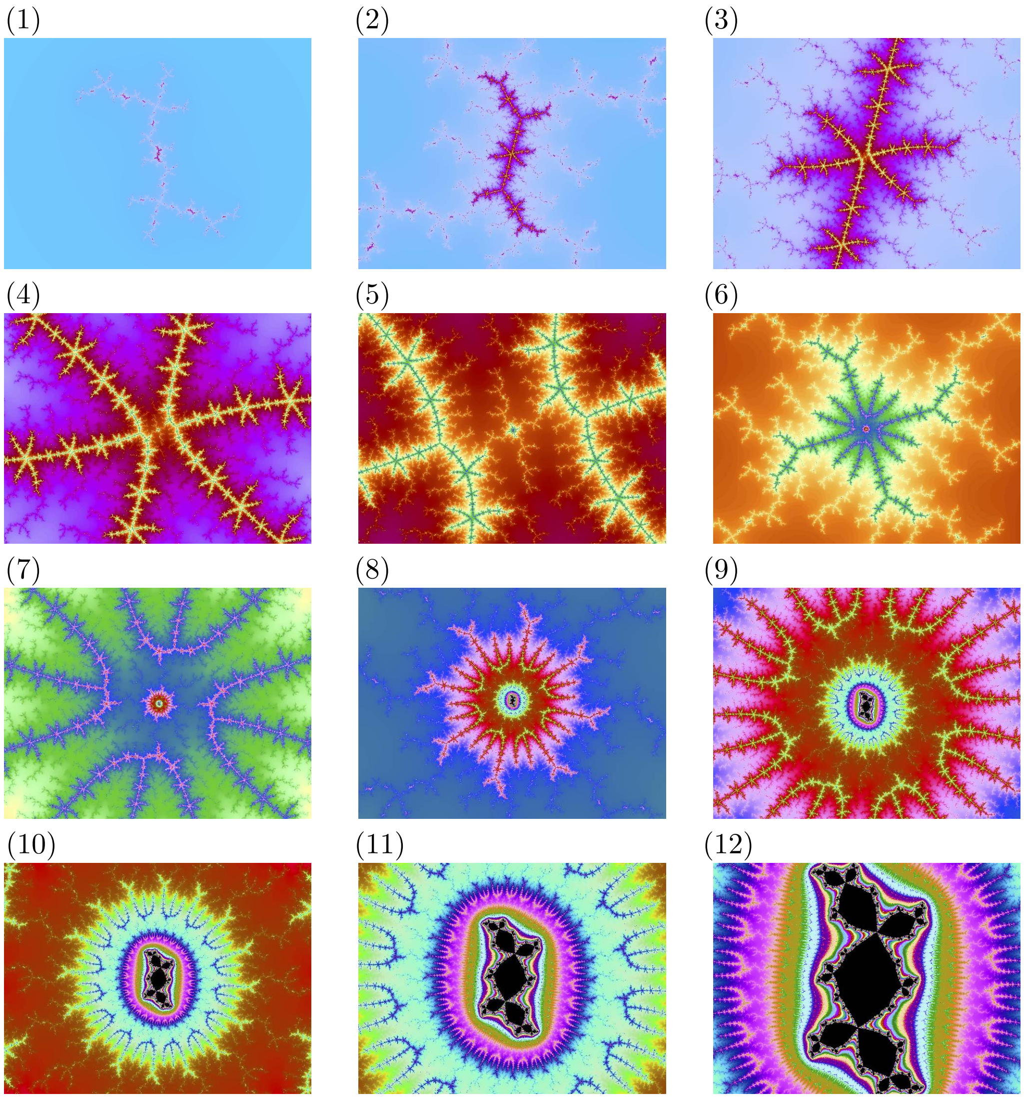

FIGURE 2. Zooms around the critical point 0 in K_c for $c = s_1 \perp c' \in M_{s_1}$, which is the smaller Mandelbrot set in Figure 1–(15). Here $s_1 = 0.3626684938191616 + 0.6450238859863952i$, $c' = -0.12256 + 0.74486i$, $c = 0.3626684938192285 + 0.6450238859865394i$.

The organization of this paper is as follows: In section 2, we construct models for the nested structures mentioned above, define the small Mandelbrot set, and show the precise statements of the results. In section 3 we recall the definitions and basic facts on quadratic-like maps and Mandelbrot-like families. We prove Theorem A for Misiurewicz case in section 4 and for parabolic case in section 5. We prove Theorem B in section 6, Theorem C in section 7 and Corollary D in section 8. We end this paper with some concluding remarks in section 9. Appendices A and B are devoted to the proofs of Lemma 4.3 and Lemma 4.4, respectively.

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2. The Model Sets and the Statements of the Results

Notation. We use the following notation for disks and annuli:

$$D(R) := \{z \in \mathbb{C} \mid |z| < R\}, \quad D(\alpha, R) := \{z \in \mathbb{C} \mid |z - \alpha| < R\},$$
$$A(r, R) := \{z \in \mathbb{C} \mid r < |z| < R\}.$$

We mostly follow Douady's notations in [**D-BDS**] in the following.

Models. Let $c' \notin M$. Then $J_{c'}$ is a Cantor set which does not contain 0. Now take two positive numbers ρ' and ρ such that

$$J_{c'} \subset A(\rho', \rho).$$

We define the *rescaled Julia set* $\Gamma_0(c') = \Gamma_0(c')_{\rho',\rho}$ by

$$\Gamma_0(c') := J_{c'} \times \frac{\rho}{(\rho')^2} = \left\{ \frac{\rho}{(\rho')^2}\, z \mid z \in J_{c'} \right\}$$

such that $\Gamma_0(c')$ is contained in the annulus $A(R, R^2)$ with $R := \rho/\rho'$. (In [**D-BDS**], Douady used the radii of the form $\rho' = R^{-1/2}$ and $\rho = R^{1/2}$ for some $R > 1$ such that $\Gamma(c') = J_{c'} \times R^{3/2}$ is contained in $A(R, R^2)$. In this paper, however, we need more flexibility when we are concerned with the dilatation.)

Let $\Gamma_m(c')$ $(m \in \mathbb{N})$ be the inverse image of $\Gamma_0(c')$ by z^{2^m}. Then $\Gamma_m(c')$ $(m = 0,\ 1,\ 2,\ \cdots)$ are mutually disjoint, because we have

$$\Gamma_0(c') \subset A(R, R^2),\ \Gamma_1(c') \subset A(\sqrt{R}, R),\ \Gamma_2(c') \subset A(\sqrt[4]{R}, \sqrt{R}), \cdots.$$

For another parameter $c \in M$, let $\Phi_c : \mathbb{C} \setminus K_c \to \mathbb{C} \setminus \overline{\mathbb{D}}$ be the Böttcher coordinate (i.e. $\Phi_c(P_c(z)) = (\Phi_c(z))^2$). Let $\Phi_M : \mathbb{C} \setminus M \to \mathbb{C} \setminus \overline{\mathbb{D}}$ be the isomorphism with $\Phi_M(c)/c \to 1$ as $|c| \to \infty$. (Note that $\Phi_M(c) := \Phi_c(c)$.) Now define the *model sets* $\mathcal{M}(c')$ and $\mathcal{K}_c(c')$ as follows (see Figure 3):

$$\mathcal{M}(c') := M \cup \Phi_M^{-1}\Big(\bigcup_{m=0}^{\infty} \Gamma_m(c')\Big), \quad \mathcal{K}_c(c') := K_c \cup \Phi_c^{-1}\Big(\bigcup_{m=0}^{\infty} \Gamma_m(c')\Big).$$

We especially call $\mathcal{M}(c')$ a *decorated Mandelbrot set*, $\mathcal{M}(c') \setminus M = \Phi_M^{-1}\Big(\bigcup_{m=0}^{\infty} \Gamma_m(c')\Big)$ its *decoration* and $M \subset \mathcal{M}(c')$ the *main Mandelbrot set* of $\mathcal{M}(c')$. Also we call $\mathcal{K}_c(c')$ a *decorated filled Julia set* and $\mathcal{K}_c(c') \setminus K_c = \Phi_c^{-1}\Big(\bigcup_{m=0}^{\infty} \Gamma_m(c')\Big)$ its *decoration*. We will apply the same terminologies to the images of $\mathcal{M}(c')$ or $\mathcal{K}_c(c')$ by quasiconformal maps. Note that the sets $\Gamma_m(c')$ $(m \geq 0)$, $\mathcal{M}(c')$ and $\mathcal{K}_c(c')$ depend on the choice of ρ' and ρ. When we want to emphasize the dependence, we denote them by $\Gamma_m(c')_{\rho',\rho}$, $\mathcal{M}(c')_{\rho',\rho}$ and $\mathcal{K}_c(c')_{\rho',\rho}$ respectively.

Small Mandelbrot sets. When we zoom in the boundary of M, a lot of "small Mandelbrot sets" appear and it is known that these sets are obtained as follows: (This is the result by Douady and Hubbard and its proof can be found in [**H**, Théorème 1 du Modulation]. See also [**Mil1**].) Let $s_0 \neq 0$ be a *superattracting parameter*, that is, $P_{s_0}(z) = z^2 + s_0$ has a superattracting periodic point, and denote its period by $p \geq 2$. Then there exists a unique

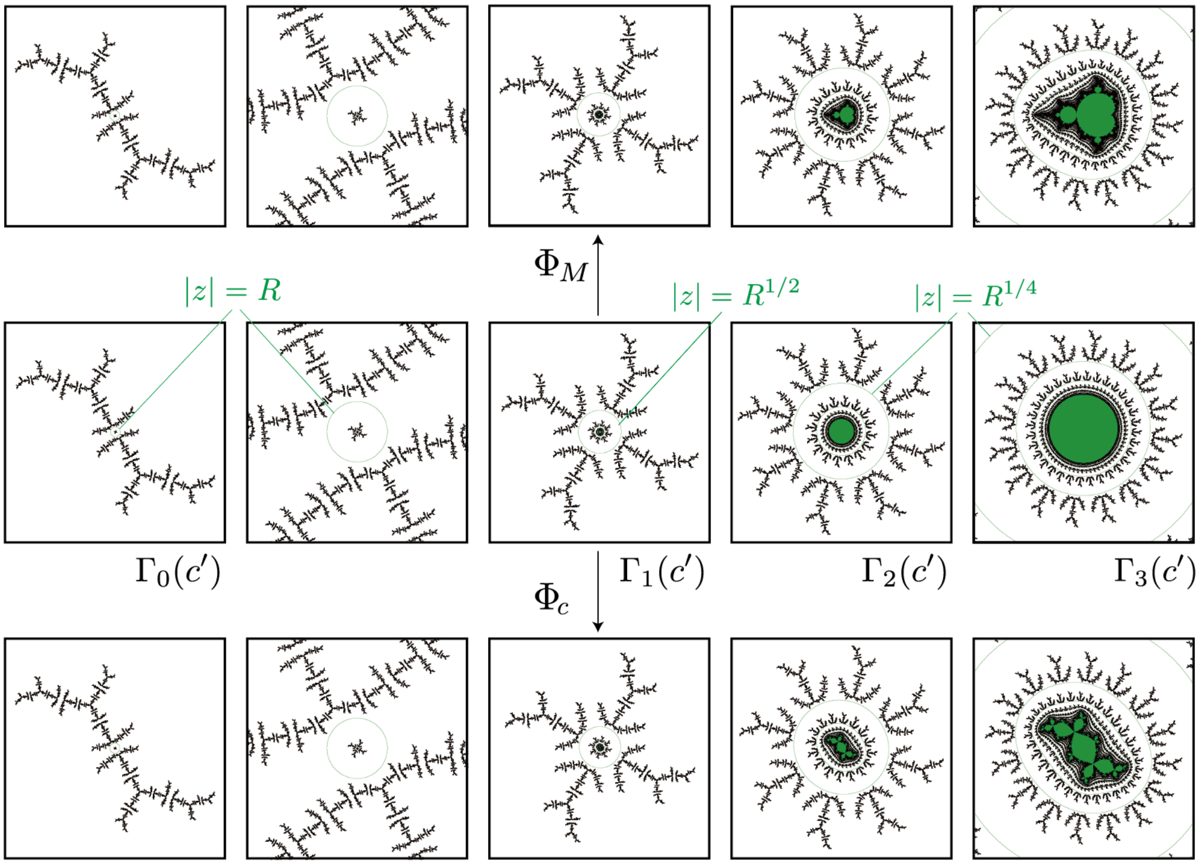

FIGURE 3. The first row depicts the decorated Mandelbrot set $\mathcal{M}(c')$ for $c' = -0.10 + 0.97i$ (close to the Misiurewicz parameter $c_0 \approx -0.1011 + 0.9563i$, the landing point of the external ray of angle $11/56$) and $R = 220$. The second row depicts the set $\bigcup_{m\geq 0} \Gamma_m(c')$. The third row depicts the decorated filled Julia set $\mathcal{K}_c(c')$ for $c \approx -0.123 + 0.745$ (the rabbit).

small Mandelbrot set M_{s_0} containing s_0 and a canonical homeomorphism $\chi : M_{s_0} \to M$ with $\chi(s_0) = 0$. Following Douady and Hubbard we use the notation $s_0 \perp M$ and $s_0 \perp c_0$ to denote $M_{s_0} = \chi^{-1}(M)$ and $\chi^{-1}(c_0)$ ($c_0 \in M$), respectively. The set M_{s_0} is called the *small Mandelbrot set with center* s_0 (see Figure 4). If $c_1 := s_0 \perp c_0$ ($c_0 \in M$), then c_1 is a parameter in M_{s_0} which corresponds to $c_0 \in M$ and it is known that P_{c_1} is renormalizable with period p and $P_{c_1}^p$ is hybrid equivalent (see section 3) to P_{c_0}. We say M_{s_0} is *primitive* if M_{s_0} has a cusp or, in other word, has a cardioid as its hyperbolic component containing s_0. Otherwise, it is called *satellite*, in which case it is attached to some hyperbolic component of $\mathrm{int}(M)$ (see Figure 4).

Definition 2.1. Let X and Y be non-empty compact sets in $\mathbb{C}$. We say X *appears (K-)quasiconformally in* Y or Y *contains a (K-)quasiconformal copy of* X if there is a (K-)quasiconformal map χ on a neighborhood of X such that $\chi(X) \subset Y$ and $\chi(\partial X) \subset \partial Y$. Note that the condition $\chi(\partial X) \subset \partial Y$ is to exclude the case $\chi(X) \subset \mathrm{int}(Y)$.

Now our results are as follows:

Theorem A (Julia sets appear quasiconformally in M). *Let M_{s_0} be any small Mandelbrot set, where $s_0 \neq 0$ is a superattracting parameter and $c_0 \in \partial M$ any Misiurewicz*

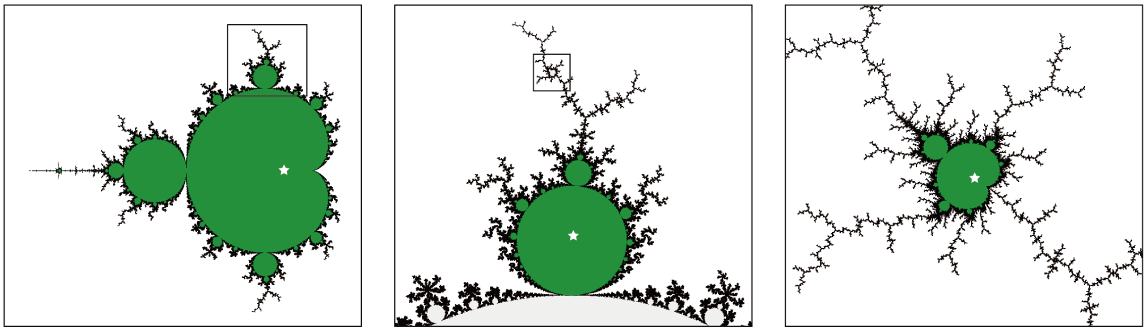

FIGURE 4. The original Mandelbrot set (left), a "satellite" small Mandelbrot set (middle), and a "primitive" small Mandelbrot set (right). The stars indicate the central superattracting parameters.

or parabolic parameter. Then for every small $\varepsilon > 0$ and $\varepsilon' > 0$, there exists an $\eta \in \mathbb{C}$ with $|\eta| < \varepsilon$ and $c_0 + \eta \notin M$ such that $\mathcal{M}(c_0 + \eta)$ appears quasiconformally in M in the neighborhood $D(s_0 \perp c_0, \varepsilon')$ of $s_0 \perp c_0$. In particular, the Cantor Julia set $J_{c_0+\eta}$ appears quasiconformally in M.

Theorem A shows the following: Take any small Mandelbrot set M_{s_0} and zoom in any small neighborhood of $s_0 \perp c_0 \in M_{s_0}$, then we can find a quasiconformal image of $\mathcal{M}(c_0 + \eta)$. That is, as we zoom in, first we observe a quasiconformal image of $J_{c_0+\eta}$, which corresponds to the Φ_M^{-1} image of the rescaled Cantor set $\Gamma_0(c_0 + \eta)$ in $\mathcal{M}(c_0 + \eta)$ and its iterated preimages (decoration) by z^2 and finally the main Mandelbrot set of the quasiconformal image of $\mathcal{M}(c_0 + \eta)$, say M_{s_1}, appears.

Figure 1 shows zooms around a Misiurewicz parameter $c_1 = s_0 \perp c_0$ in a primitive small Mandelbrot set M_{s_0} (Figure 1–(1)). After a sequence of nested structures, a smaller "small Mandelbrot set" M_{s_1} appears (Figure 1–(15)). Here M_{s_0} is the relatively big "small Mandelbrot set" which is located in the upper right part of M. The map P_{s_0} has a superattracting periodic point of period 4 and c_0 is the Misiurewicz parameter which satisfies $P_{c_0}(P_{c_0}^4(0)) = P_{c_0}^4(0)$ and corresponds to the "junction of three roads" as shown in Figure 4 in the middle.

Since Misiurewicz or parabolic parameters are dense in ∂M, we can reformulate Theorem A as follows: *Let M_{s_0} be any small Mandelbrot set, where $s_0 \neq 0$ is a superattracting parameter and $c_0 \in \partial M$ any parameter. Then for every small $\varepsilon > 0$ and $\varepsilon' > 0$, there exists an $\eta \in \mathbb{C}$ with $|\eta| < \varepsilon$ and $c_0 + \eta \notin M$ such that $\mathcal{M}(c_0 + \eta)$ appears quasiconformally in M in the neighborhood $D(s_0 \perp c_0, \varepsilon')$. In particular, the Cantor Julia set $J_{c_0+\eta}$ appears quasiconformally in M.*

Next we show that the same decoration of $\mathcal{M}(c_0 + \eta)$ in Theorem A appears quasiconformally in some filled Julia sets.

Theorem B (Decoration in filled Julia sets). *Let M_{s_1} denote the main Mandelbrot set of the quasiconformal image of $\mathcal{M}(c_0 + \eta)$ in Theorem A. Then for every $c \in M$, $\mathcal{K}_c(c_0 + \eta)$ appears quasiconformally in $K_{s_1 \perp c}$, where $s_1 \perp c \in M_{s_1}$.*

Theorem B shows the following: Choose any parameter from the main Mandelbrot set M_{s_1} in the quasiconformal image of $\mathcal{M}(c_0 + \eta)$, that is, choose any $c \in M$ and consider $s_1 \perp c$

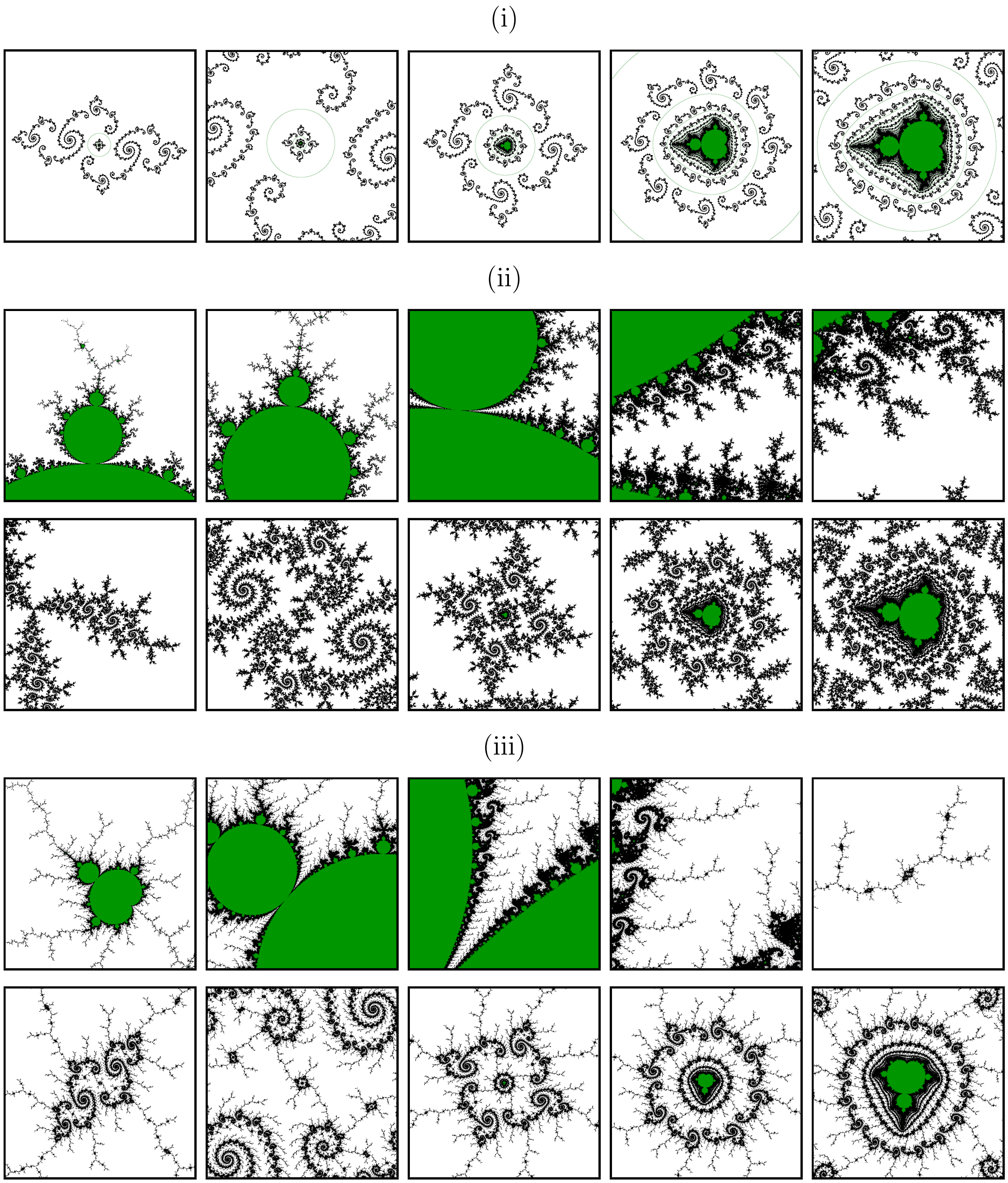

FIGURE 5. (i): The decorated Mandelbrot set $\mathcal{M}(c')$ for $c' = -0.77+0.18i$ (close to the parabolic parameter $c_0 = -0.75$. (ii) and (iii): Embedded quasiconformal copies of $\mathcal{M}(c')$ above near the satellite/primitive small Mandelbrot sets in Figure 4.

$\in M_{s_1}$ and zoom in the neighborhood of $0 \in K_{s_1 \perp c}$. Then we can find a quasiconformal image of $\mathcal{K}_c(c_0+\eta)$, whose decoration is conformally the same as that of $\mathcal{M}(c_0+\eta)$.

Next we show that there are smaller Mandelbrot sets and their decorations which are images of model sets by quasiconformal maps whose dilatations are arbitrarily close to 1.

Theorem C (**Almost conformal copies).** *Let c_0 be any parabolic or Misiurewicz parameter and B any small closed disk whose interior intersects with ∂M. Then for any small $\varepsilon > 0$ and $\kappa > 0$, there exist an $\eta \in \mathbb{C}$ with $|\eta| < \varepsilon$ and two positive numbers ρ and ρ' with $\rho' < \rho$ such that $c_0 + \eta \notin M$ and $\mathcal{M}(c_0 + \eta)_{\rho',\rho}$ appears $(1+\kappa)$-quasiconformally in $B \cap M$. In particular, $B \cap \partial M$ contains a $(1+\kappa)$-quasiconformal copy of the Cantor Julia set $J_{c_0+\eta}$.*

Definition 2.2 (**Semihyperbolicity).** A quadratic polynomial $P_c(z) = z^2 + c$ (or the parameter c) is called *semihyperbolic* if

(1) the critical point 0 is non-recurrent, that is, $0 \notin \omega(0)$, where $\omega(0)$ is the ω-limit set of the critical point 0 and
(2) P_c has no parabolic periodic points.

It is easy to see that if P_c is hyperbolic then it is semihyperbolic. If P_c is semihyperbolic, then it is known that it has no Siegel disks and Cremer points ([**Ma**], [**CJY**]). Also it is not difficult to see that J_c is measure 0 from the result by Lyubich and Shishikura ([**Ly1**]). Thus the semihyperbolic dynamics is relatively understandable. A typical semihyperbolic but non-hyperbolic parameter c is a Misiurewicz parameter. But there seems less concrete examples of semihyperbolic parameter c which is neither hyperbolic nor Misiurewicz. Next corollary shows that we can at least "see" such a parameter everywhere in ∂M.

Corollary D (**Abundance of semihyperbolicity).** *For every parameter c belonging to the quasiconformal image of the decoration of $\mathcal{M}(c_0 + \eta)$ in Theorem A, P_c is semihyperbolic. Also the set of parameters which are not Misiurewicz and non-hyperbolic but semihyperbolic are dense in ∂M.*

Corollary D together with Theorem C explains the following famous result by Shishikura:

Theorem (Shishikura, 1998)**.** *Let*

$$SH := \{c \in \partial M \mid P_c \text{ is semihyperbolic}\},$$

then the Hausdorff dimension of SH is 2. In particular, the Hausdorff dimension of the boundary of M is 2.

Explanation. Since there exist quadratic Cantor Julia sets with Hausdorff dimension arbitrarily close to 2 and we can find such parameters in every neighborhood of a point in ∂M ([**S**, p.231, *proof of Theorem B* and p.232, *Remark* 1.1 (iii)]), we can find an η such that $\dim_H(J_{c_0+\eta})$ is arbitrarily close to 2. Then by Theorem C and Corollary D it follows that we can find a subset of ∂M with Hausdorff dimension arbitrarily close to 2 and consisting of semihyperbolic parameters as a quasiconformal image of the decoration of $\mathcal{M}(c_0 + \eta)$. This implies that $\dim_H(SH) = 2$. ■

Remark. (1) A similar result to Theorem B still holds even when $c \in \mathbb{C} \setminus M$ is sufficiently close to M. Actually the homeomorphism $\chi : M_{s_1} \to M$ can be extended to a homeomorphism between some neighborhoods of M_{s_1} and M and so $s_1 \perp c = \chi^{-1}(c)$ can be still defined for such a $c \in \mathbb{C} \setminus M$. Then in this case, by modifying the definition of $\mathcal{K}_c(c_0 + \eta)$ for this $c \in \mathbb{C} \setminus M$ we can prove that a "$\mathcal{K}_c(c_0 + \eta)$" appears quasiconformally in $K_{s_1 \perp c}$, where $s_1 \perp c \in \mathbb{C} \setminus M_{s_1}$ is a point which is sufficiently close to M_{s_1}. We omit the details.

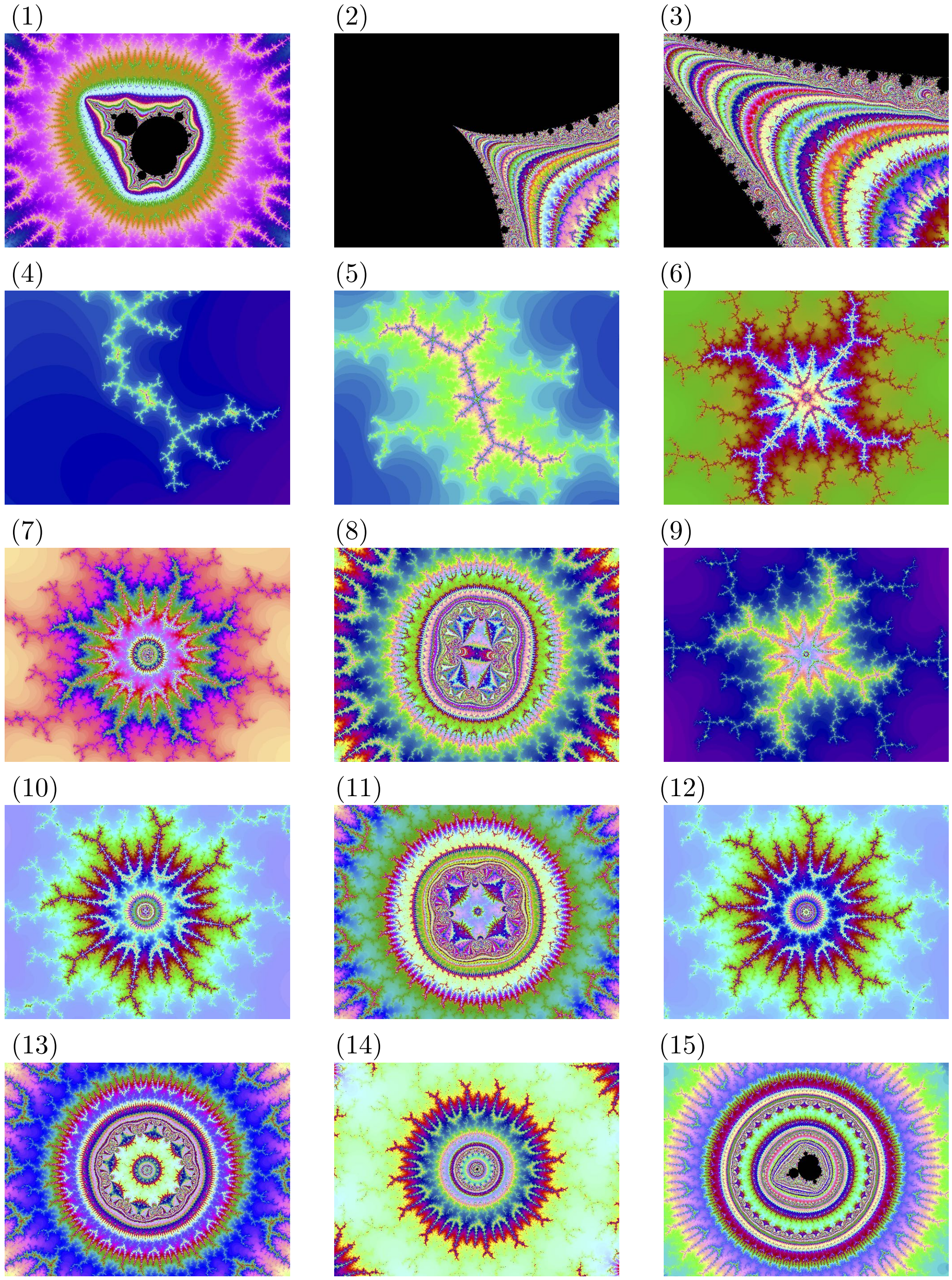

FIGURE 6. Zooms around a parabolic point $s_1 \perp c_1$ in a primitive small Mandelbrot set M_{s_1}. After a sequence of complicated nested structures, another smaller Mandelbrot set M_{s_2} appears ((15)).

(2) Corollary D and the above Theorem (Shishikura, 1998) means that relatively understandable dynamics is abundant in ∂M (provided that Lebesgue measure of ∂M is 0). In [**S**, p.225, THEOREM A], Shishikura actually proved $\dim_H(SH) = 2$, which immediately implies $\dim_H(\partial M) = 2$. A new point of our "explanation" is that we constructed a decoration in M which contains a quasiconformal image of a whole Cantor Julia set and consists of semihyperbolic parameters. So now we can say that "$\dim_H(\partial M) = 2$ holds,

because we can see a lot of almost conformal images of Cantor quadratic Julia sets whose Hausdorff dimension are arbitrarily close to 2".

(3) Take a small Mandelbrot set M_{s_1} (e.g. Figure 1–(15) = Figure 6–(1)) and another Misiurewicz or parabolic parameter c_* (e.g. $c_* = \frac{1}{4}$) and zoom in the neighborhood of $s_1 \perp c_*$. Then we see much more complicated structure than we expected as follows: According to Theorem A, by replacing s_0 with s_1 and c_0 with c_*, it says that $\mathcal{M}(c_* + \eta)$ appears quasiconformally in $D(s_1 \perp c_*, \varepsilon')$. This means that as we zoom in, we first see a quasiconformal image of $J_{c_*+\eta_*}$, say $\widetilde{J}_{c_*+\eta_*}$ (e.g. "broken cauliflower", when $c_* = \frac{1}{4}$). But in reality as we zoom in, what we first see is a $\widetilde{J}_{c_0+\eta_0}$ (e.g. "broken dendrite". See Figure 6–(5)). This seems to contradict with Theorem A, but actually it does not. As we zoom in further in the middle part of $\widetilde{J}_{c_0+\eta_0}$, we see iterated preimages of $\widetilde{J}_{c_0+\eta_0}$ by z^2 (Figure 6–(6), (7)) and then $\widetilde{J}_{c_*+\eta_*}$ appears (Figure 6–(8)). After that we see again iterated preimages of $\widetilde{J}_{c_0+\eta_0}$ by z^2 (Figure 6–(9), (10)) and then a once iterated preimage of $\widetilde{J}_{c_*+\eta_*}$ appears (Figure 6–(11)). This complicated structure continues and finally, we see a smaller Mandelbrot set, say M_{s_2} (Figure 6–(15)). We can explain this complicated phenomena as follows: What we see in the series of magnifications above is a quasiconformal image of $\mathcal{M}(\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0))$, where

$$\begin{aligned}\mathcal{M}(\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)) &:= M \cup \Phi_M^{-1}\Big(\bigcup_{m=0}^{\infty} \Gamma_m(\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0))\Big), \\ \Gamma_m(\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)) &:= \text{inverse image of } \Gamma_0(\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)) \text{ by } z^{2^m}.\end{aligned}$$

Here $\mathcal{M}(\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0))$ is obtained just by replacing $J_{c'}$ with $\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)$ in the definition of $\mathcal{M}(c')$. Here, although $c_* + \eta_* \notin M$, $\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)$ can be defined in the similar manner. See the Remark (1) above. So what we first see as we zoom in the neighborhood of $s_1 \perp c_*$ is a quasiconformal image of $\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)$, whose outer most part is $\widetilde{J}_{c_0+\eta_0}$ (= broken dendrite) and inner most part is $\widetilde{J}_{c_*+\eta_*}$ (= broken cauliflower). As we zoom in further, we see quasiconformal image of the preimage of $\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)$ by z^2, whose inner most part is a once iterated preimage of $\widetilde{J}_{c_0+\eta_0}$. After we see successive preimages of $\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)$ by z^2, a much more smaller Mandelbrot set M_{s_2} finally appears. Since $\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)$ itself has a nested structure, the total picture has this very complicated structure. The proof is completely the same as for the Theorem A.

3. The quadratic-like maps and the Mandelbrot-like family

In this section, we briefly recall the definitions of the quadratic-like map and the Mandelbrot-like family and explain the key Proposition 3.1 which is crucial for the proof of Theorem A.

A map $h : U' \to U$ is called *a polynomial-like map* if U', $U \subset \mathbb{C}$ are topological disks with $U' \Subset U$ (which means $\overline{U'} \subset U$) and h is holomorphic and proper map of degree d with respect to z. It is called a *quadratic-like map* when $d = 2$. The *filled Julia set* $K(h)$ and the *Julia set* $J(h)$ of a polynomial-like map h are defined by

$$\begin{aligned}K(h) &:= \{z \in U' \mid h^n(z) \text{ are defined for every } n \in \mathbb{N}\} = \bigcap_{n=0}^{\infty} h^{-n}(U'), \\ J(h) &:= \partial K(h).\end{aligned}$$

The famous Straightening Theorem by Douady and Hubbard ([**DH2**, p.296, THEOREM 1]) says that every polynomial-like map $h : U' \to U$ of degree d is quasiconformally conjugate to a polynomial P of degree d. More precisely h is *hybrid equivalent* to P, that is, there exists a quasiconformal map ϕ sending a neighborhood of $K(h)$ to a neighborhood of $K(P)$ such that $\phi \circ h = P \circ \phi$ and $\overline{\partial}\phi = 0$ a.e. on $K(h)$. Also if $K(h)$ is connected, then P is unique up to conjugacy by an affine map.

A family of holomorphic maps $\boldsymbol{h} = (h_\lambda)_{\lambda \in W}$ is called a *Mandelbrot-like family* if the following (1)–(8) hold:

(1) $W \subset \mathbb{C}$ is a Jordan domain with C^1 boundary ∂W.

(2) There exists a family of maps $\Theta = \{\Theta_\lambda\}_{\lambda \in W}$ such that for every $\lambda \in W$, $\Theta_\lambda : \overline{A(R, R^2)} \to \mathbb{C}$ is a quasiconformal embedding and that $\Theta_\lambda(Z)$ is holomorphic in λ for every $Z \in \overline{A(R, R^2)}$.

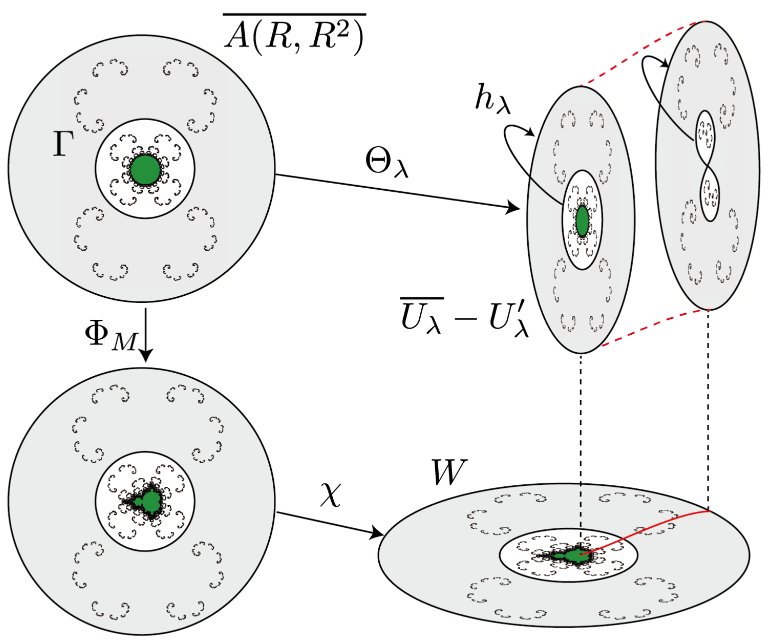

FIGURE 7. Tubing $\Theta = \{\Theta_\lambda\}_{\lambda \in W}$.

(3) Define $C_\lambda := \Theta_\lambda(\partial D(R^2))$, $C'_\lambda := \Theta_\lambda(\partial D(R))$ and let U_λ (resp. U'_λ) be the Jordan domain bounded by C_λ (resp. C'_λ). Then $h_\lambda : U'_\lambda \to U_\lambda$ is a quadratic-like map with a critical point ω_λ. Also let

$$\mathcal{U} := \{(\lambda, z) \mid \lambda \in W,\ z \in U_\lambda\}, \quad \mathcal{U}' := \{(\lambda, z) \mid \lambda \in W,\ z \in U'_\lambda\}$$

then $\boldsymbol{h} : \mathcal{U} \to \mathcal{U}'$, $(\lambda, z) \mapsto (\lambda, h_\lambda(z))$ is analytic and proper.

(4) $\Theta_\lambda(Z^2) = h_\lambda(\Theta_\lambda(Z))$ for $Z \in \partial D(R)$.

The family of maps $\Theta = \{\Theta_\lambda\}_{\lambda \in W}$ satisfying the above conditions (1)–(4) is called a *tubing*.

(5) $\boldsymbol{h}$ extends continuously to a map $\overline{\mathcal{U}'} \to \overline{\mathcal{U}}$ and $\Theta_\lambda : (\lambda, z) \mapsto (\lambda, \Theta_\lambda(Z))$ extends continuously to a map $\overline{W} \times \overline{A(R, R^2)} \to \overline{\mathcal{U}}$ such that Θ_λ is injective on $A(R, R^2)$ for $\lambda \in \partial W$.

(6) The map $\lambda \mapsto \omega_\lambda$ extends continuously to $\overline{W}$.

(7) $h_\lambda(\omega_\lambda) \in C_\lambda$ for $\lambda \in \partial W$.

(8) *The one turn condition*: When λ ranges over ∂W making one turn, then the vector $h_\lambda(\omega_\lambda) - \omega_\lambda$ makes one turn around 0.

Now let $M_{\boldsymbol{h}}$ be the *connectedness locus* of the family $\boldsymbol{h} = (h_\lambda)_{\lambda \in W}$:

$$M_{\boldsymbol{h}} := \{\lambda \in W \mid K(h_\lambda) \text{ is connected}\} = \{\lambda \in W \mid \omega_\lambda \in K(h_\lambda)\}.$$

Douady and Hubbard ([**DH2**, Chapter IV]) showed that there exists a homeomorphism

$$\chi : M_{\boldsymbol{h}} \to M.$$

This is just a correspondence by the Straightening Theorem, that is, for every $\lambda \in M_{\boldsymbol{h}}$ there exist a unique $c = \chi(\lambda) \in M$ such that h_λ is hybrid equivalent to $P_c(z) = z^2 + c$. Furthermore they showed that this χ can be extended to a homeomorphism $\chi_\Theta : W \to W_M$ by using $\Theta = \{\Theta_\lambda\}_{\lambda \in W}$, where

$$W_M := \{c \in \mathbb{C} \mid \mathcal{G}_M(c) < 2 \log R\}, \quad \mathcal{G}_M := \text{the Green function of } M$$

is a neighborhood of M. Also Lyubich showed that χ_Θ is quasiconformal on any W' with $W' \Subset W$ ([**Ly2**, p.366, THEOREM 5.5 (The QC Theorem)]).

Then Douady et al. showed the following:

Proposition 3.1. [**D-BDS**, p.29, PROPOSITION 3] *For any* $\Gamma \subset A(R, R^2)$, *let* Γ_m *be the preimage of* Γ *by* z^{2^m}. *Then*

$$\chi_\Theta^{-1}(\Phi_M^{-1}(\Gamma_m)) = \{\lambda \in W \mid h_\lambda^{m+1}(\omega_\lambda) \in \Theta_\lambda(\Gamma)\}$$

and therefore

$$M_{\boldsymbol{h}} \cup \{\lambda \mid h_\lambda^k(\omega_\lambda) \in \Theta_\lambda(\Gamma) \text{ for some } k \in \mathbb{N}\} = \chi_\Theta^{-1}\left(M \cup \left(\bigcup_{m=0}^{\infty} \Phi_M^{-1}(\Gamma_m)\right)\right).$$

We apply this proposition to the rescaled Julia set $\Gamma = \Gamma_0(c')(:= J_{c'} \times (\rho/(\rho')^2))$ contained in $A(R, R^2)$, where $c' \notin M$, $J_{c'} \subset A(\rho', \rho)$ and $R := \rho/\rho'$. Then we have

$$\chi_\Theta^{-1}\left(M \cup \left(\bigcup_{m=0}^{\infty} \Phi_M^{-1}(\Gamma_m)\right)\right) = \chi_\Theta^{-1}\left(M \cup \Phi_M^{-1}\left(\bigcup_{m=0}^{\infty} \Gamma_m(c')\right)\right) = \chi_\Theta^{-1}(\mathcal{M}(c')).$$

4. Proof of Theorem A for Misiurewicz case

Let M_{s_0} be any small Mandelbrot set, where $s_0 \neq 0$ is a superattracting parameter (i.e., the critical point 0 is a periodic point of period $p \geq 2$ for P_{s_0}), $c_0 \in \partial M$ any Misiurewicz parameter and $c_1 := s_0 \perp c_0 \in M_{s_0}$. By the tuning theorem by Douady and Hubbard [**H**, Théorème 1 du Modulation], there exists a simply connected domain $\Lambda = \Lambda_{s_0}$ in the parameter plane with the following properties:

- If M_{s_0} is a primitive small Mandelbrot set, then $M_{s_0} \subset \Lambda$.
- If M_{s_0} is a satellite small Mandelbrot set, then $M_{s_0} \smallsetminus \{s_0 \perp (1/4)\} \subset \Lambda$.
- For any $c \in \Lambda$, P_c is renormalizable with period p.

This means that P_c^p is a quadratic-like map on a suitable domain U_c' onto U_c which contains the critical point 0. In what follows we denote

$$f_c := P_c^p|_{U_c'} : U_c' \to U_c.$$

We shall first choose U_{c_1} and U_{c_1}' carefully in Lemma 4.1 and then give a definition of U_c and U_c' at the end of Step (M1) below.

Step (M1): Definition of U_c and U_c'

In this case f_{c_1} is Misiurewicz, that is,

$$q_{c_1} := f_{c_1}^l(0) \quad \text{for some } l \in \mathbb{N}$$

is a repelling periodic point of period k with multiplier

$$\mu_{c_1} := (f_{c_1}^k)'(q_{c_1}).$$

Then the Julia set $J(f_{c_1})$ of the quadratic-like map f_{c_1} is a dendrite and $K(f_{c_1}) = J(f_{c_1})$. Since q_{c_1} is repelling, a repelling periodic point q_c of period k for f_c persists by the implicit function theorem. But note that $q_c \neq f_c^l(0)$, if $c \neq c_1$. Take a linearizing coordinate $\phi_c : \Omega_c \to \mathbb{C}$ with $\phi_c(q_c) = 0$, where Ω_c is a neighborhood of q_c such that

$$\phi_c(f_c^k(z)) = T_c(\phi_c(z)), \quad T_c(z) := \mu(c)z, \quad \mu(c) := (f_c^k)'(q_c).$$

Note that $\phi_c(z)$ depends holomorphically on c.

Lemma 4.1. *There exist Jordan domains U_{c_1}, U_{c_1}' and V_{c_1} with C^1 boundaries and integers $N,\ j \in \mathbb{N}$ which satisfy the following:*

(1) $f_{c_1} = P_{c_1}^p|_{U_{c_1}'} : U_{c_1}' \to U_{c_1}$ *is a quadratic-like map.*
(2) $g_{c_1} := P_{c_1}^N|_{V_{c_1}} : V_{c_1} \to U_{c_1}$ *is an isomorphism and* $f_{c_1}^j(V_{c_1}) \subset U_{c_1} \setminus \overline{U_{c_1}'}$.
(3) $V_{c_1} \subset \Omega_{c_1}$. *Also we can take V_{c_1} arbitrarily close to $q_{c_1} \in \Omega_{c_1}$.*

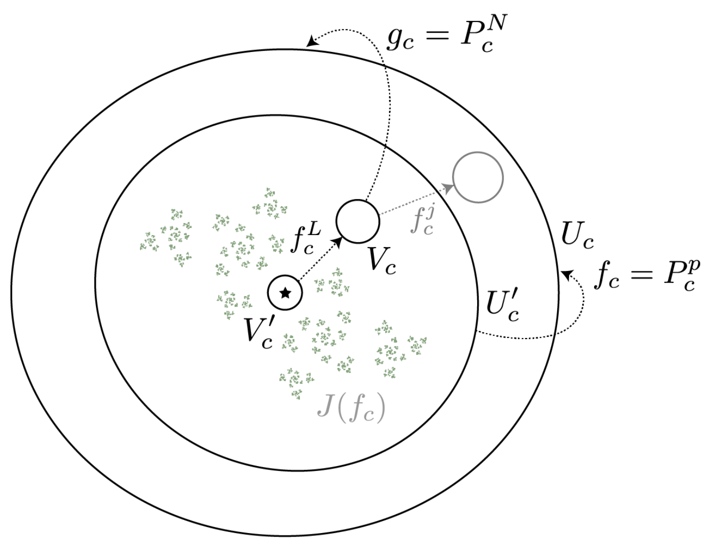

FIGURE 8. The relation between U_c, U_c', V_c, and V_c', which are defined as a holomorphic perturbation of those for $c = c_1$ given in Lemma 4.1.

Remark. There are infinitely many different choices of V_{c_1} and each choice of V_{c_1} gives a different "decorated small Mandelbrot set".

Proof. We can take Jordan domains U_{c_1} and U'_{c_1} with C^1 boundaries which are neighborhoods of $J(f_{c_1})$ and $f_{c_1} : U'_{c_1} \to U_{c_1}$ is a quadratic-like map. Note that we can take these Jordan domains in any neighborhood of $J(f_{c_1})$. Now for $j \in \mathbb{N}$ let

$$A_j := f_{c_1}^{-j}(U_{c_1} \smallsetminus \overline{U'_{c_1}}).$$

Since f_{c_1} is conjugate to z^2 on $U'_{c_1} \smallsetminus J_{f_{c_1}}$, $A_j \cap \Omega_{c_1} \neq \emptyset$ for every sufficiently large j. Also since $(U_{c_1} \smallsetminus \overline{U'_{c_1}}) \cap J_{c_1} \neq \emptyset$, we have $A_j \cap J_{c_1} \neq \emptyset$. From these facts and the shrinking lemma ([**LyMin**, p.86]), it follows that there exist sufficiently large N and $j \in \mathbb{N}$ and V_{c_1} which is a component of $P_{c_1}^{-N}(U_{c_1})$ such that $g_{c_1} := P_{c_1}^N|_{V_{c_1}} : V_{c_1} \to U_{c_1}$ is an isomorphism and $V_{c_1} \subset \Omega_{c_1} \cap A_j$. In particular we have $f_{c_1}^j(V_{c_1}) \subset U_{c_1} \smallsetminus \overline{U'_{c_1}}$. ■ (Lemma 4.1)

Now define $U_c := U_{c_1}$ for every c in a neighborhood S of c_1 contained in $D(c_1, \varepsilon')$. Let U'_c be the Jordan domain bounded by the component of $P_c^{-p}(U_c)$ which is contained in $U_c(\equiv U_{c_1})$ and V_c the Jordan domain bounded by the component of $P_c^{-N}(U_c)$ which is close to V_{c_1}. Then we have

$$\overline{U'_c} \subset U_c, \quad \text{and} \quad f_c^j(V_c) \subset U_c \smallsetminus \overline{U'_c} \quad \text{for } c \in S$$

if S is sufficiently small. Define

$$f_c := P_c^p|_{U'_c} \quad \text{and} \quad g_c := P_c^N|_{V_c}.$$

Then it follows that $f_c : U'_c \to U_c$ is a quadratic-like map with a critical point 0 and $g_c : V_c \to U_c(\equiv U_{c_1})$ is an isomorphism. See Figure 8. Let $b_c := g_c^{-1}(0) \in V_c$ and call it a *pre-critical point.*

Step (M2): Construction of the Mandelbrot-like family $\boldsymbol{G} = \boldsymbol{G}_n$

We construct a Mandelbrot-like family

$$\boldsymbol{G} = \boldsymbol{G}_n := \{G_c : V'_c \to U_c\}_{c \in W_n}$$

such that $V'_c \subset U'_c$ and $W = W_n \subset S$ for every sufficiently large $n \in \mathbb{N}$ as follows: Recall that the quadratic-like map $f_c : U'_c \to U_c$ has a repelling periodic point q_c of period k, since the repelling periodic point q_{c_1} persists under the perturbation. So we take a linearizing coordinate $\phi_c : \Omega_c \to \mathbb{C}$ around q_c with $\phi_c(q_c) = 0$ such that

$$\phi_c(f_c^k(z)) = T_c(\phi_c(z)), \quad T_c(z) := \mu(c)z, \quad \mu(c) := (f_c^k)'(q_c).$$

For later use, here we normalize ϕ_c so that

$$\phi_c(b_c) = 1, \quad \text{for the pre-critical point } b_c = g_c^{-1}(0).$$

Define

$$a_c := f_c^l(0)$$

then note that $a_{c_1} = q_{c_1}$ and $a_c \neq q_c$ when $c \neq c_1$ is very close to c_1. Hence for such a c, a_c is repelled from q_c by the dynamics of f_c^k. By taking a sufficiently small S, we may assume that $a_c = f_c^l(0) \in S$ for each $c \in S$. Now we define

$$W = W_n := \{c \in \Lambda \mid f_c^{kn}(a_c)(= f_c^{l+kn}(0)) \in V_c\}.$$

Lemma 4.2. *The set $W = W_n$ is non-empty for every sufficiently large n. Moreover there exists an $s_n \in W_n$ such that $f_{s_n}^{kn}(a_{s_n}) = b_{s_n}$, that is, $g_{s_n} \circ f_{s_n}^{l+kn}(0) = 0$ and hence P_{s_n} has a superattracting periodic point.*

Proof. We observe the dynamics near q_c through the linearizing coordinate $\phi_c : \Omega_c \to \mathbb{C}$ of q_c. Let

$$\widetilde{c} = \tau(c) := \phi_c(a_c), \quad \widetilde{V}_c := \phi_c(V_c)$$

and define

$$\begin{aligned} \widetilde{W}_n &:= \{\widetilde{c} \mid c \in W_n\} \\ &= \{\tau(c) \mid c \text{ satisfies } f_c^{kn}(a_c) \in V_c\} \\ &= \{\widetilde{c} \mid \widetilde{c} \text{ satisfies } \mu(c)^n \cdot \widetilde{c} \; (= \mu(c)^n \cdot \tau(c)) \in \widetilde{V}_c\}. \end{aligned}$$

Then $c \in W_n$ if and only if $\widetilde{c} = \tau(c) \in \widetilde{W}_n$. Since $\mu(c)$ and $\tau(c)$ depend holomorphically on c, there exist $\alpha \in \mathbb{N}$, $M_0 \neq 0$ and $K_0 \neq 0$ such that

$$\begin{aligned} \mu(c) &= (f_c^k)'(q_c) = \mu_{c_1} + M_0(c-c_1)^\alpha + o(|c-c_1|^\alpha), \\ \tau(c) &= K_0(c-c_1) + o(|c-c_1|). \end{aligned}$$

The fact that $K_0 \neq 0$ for the expansion of $\tau(c)$ follows from the result by Douady and Hubbard ([**DH2**, p.333, Lemma 1]. See also [**T1**, p.609, Lemma 5.4]). In order to show $W_n \neq \emptyset$, it suffices to show that for $1 = \phi_c(b_c) \in \widetilde{V}_c$ the equation $\mu(c)^n\tau(c) = 1$ has a solution. Now we have

$$\begin{aligned} \mu(c)^n\tau(c) &= (\mu_{c_1} + M_0(c-c_1)^\alpha + o(|c-c_1|^\alpha))^n(K_0(c-c_1) + o(|c-c_1|)) \\ &= \left\{\mu_{c_1}\left(1 + \frac{M_0}{\mu_{c_1}}(c-c_1)^\alpha + o(|c-c_1|^\alpha\right)\right\}^n \cdot K_0(c-c_1)(1+o(1)) \\ &= \mu_{c_1}^n K_0(c-c_1)\left(1 + \frac{nM_0}{\mu_{c_1}}(c-c_1)^\alpha + o(|c-c_1|^\alpha)\right)(1+o(1)). \end{aligned}$$

On the other hand, from the above expression the equation $\mu(c)^n\tau(c) = 1$ with respect to c can be written as

$$\mu_{c_1}^n K_0(c-c_1) + h(c) = 1,$$

that is,

$$(\mu_{c_1}^n K_0(c-c_1) - 1) + h(c) = 0, \quad h(c) = O(|c-c_1|^{\alpha+1}). \tag{4.1}$$

Then the equation $\mu_{c_1}^n K_0(c-c_1) = 1$ has a unique solution $c = c_1 + \frac{1}{\mu_{c_1}^n K_0}$. Consider the equation (4.1) on the disk $D(c_1, 2/|\mu_{c_1}^n K_0|)$. Then on the boundary of this disk by putting $c = c_1 + |2/(\mu_{c_1}^n K_0)|e^{i\theta}$, we have

$$|\mu_{c_1}^n K_0(c-c_1) - 1| = |2e^{i\theta} - 1| \geq 1,$$

meanwhile when n is sufficiently large we have

$$|h(c)| = \left|\frac{2}{\mu_{c_1}^n K_0}\right|^{1+\alpha}(1+o(1)) < 1.$$

Since we have

$$|\mu_{c_1}^n K_0(c-c_1) - 1| > |h(c)|$$

on the boundary of the disk, by Rouché's theorem the equation $\mu(c)^n\tau(c) = 1$ has the same number of solutions as the equation $\mu_{c_1}^n K_0(c-c_1) - 1 = 0$, that is, it has a unique solution. This shows that $W_n \neq \emptyset$ if n is sufficiently large. Moreover denote this solution $c := s_n$, then we have $f_{s_n}^{kn}(a_{s_n}) = b_{s_n}$, that is, $g_{s_n} \circ f_{s_n}^{l+kn}(0) = 0$. Hence P_{s_n} has a super-attracting periodic point. This completes the proof. ■ (Lemma 4.2)

Remark. In the proof above, even if we replace $1 \in \widetilde{V}_c$ with $w \neq 1$ which is close to 1, we can still do the same argument.

Note that larger the n is, closer the W_n is to c_1. Hence $W_n \subset S \subset \Lambda$ for every sufficiently large n. We call $s_n \in W_n$ the *center* of W_n. Now let $L = L_n := l + kn$ and V_c' be the Jordan domain bounded by the component of $f_c^{-L}(V_c)$ containing 0 and define

$$G_c := g_c \circ f_c^L : V_c' \to U_c \ \text{ and } \ \boldsymbol{G} = \boldsymbol{G}_n := \{G_c\}_{c \in W_n},$$

where $W_n = \{c \in \Lambda \mid f_c^L(0) \in V_c\}$. See Figure 8.

Step (M3): Proof for $\boldsymbol{G} = \boldsymbol{G}_n$ being a Mandelbrot-like family

The map $f_c^L : V_c' \to V_c$ is a branched covering of degree 2 and $g_c : V_c \to U_c$ is a holomorphic isomorphism. Hence $G_c := g_c \circ f_c^L : V_c' \to U_c$ is a quadratic-like map.

Next we construct a tubing $\Theta = \Theta_n = \{\Theta_c\}_{c \in W_n}$ for $\boldsymbol{G}_n$ as follows: For $s_n \in W_n$, since $f_{s_n}^L(0) \in V_{s_n}$ and $f_{s_n}^j(V_{s_n}) \subset U_{s_n} \smallsetminus \overline{U_{s_n}'}$, from Lemma 4.1, we have $f_{s_n}^{L+j}(0) \notin U_{s_n}'$. It follows that $J(f_{s_n})$ is a Cantor set which is quasiconformally homeomorphic to a quadratic Cantor Julia set $J_{c_0+\eta_n}$ for some $\eta = \eta_n$ with $c_0 + \eta_n \notin M$. By continuity of the straightening of f_c for $c \in \Lambda$, we have $|\eta_n| < \varepsilon$ for sufficiently large n. We denote the homeomorphism from $J(f_{s_n})$ to $J_{c_0+\eta_n}$ by $\Psi_{s_n} : J(f_{s_n}) \to J_{c_0+\eta_n}$, which is indeuced by the straightening theorem. Take an $R > 1$ and let $\rho' := R^{-1/2}$ and $\rho := R^{1/2}$ such that $J_{c_0+\eta_n} \subset A(\rho', \rho)$. Define the rescaled Julia set

$$\Gamma := \Gamma_0(c_0 + \eta_n) = \Gamma_0(c_0 + \eta_n)_{\rho', \rho} := J_{c_0+\eta_n} \times R^{3/2} \subset A(R, R^2)$$

and a homeomorphism

$$\Theta_n^0 : \overline{A(R, R^2)} \to \overline{U}_{s_n} \smallsetminus V_{s_n}'$$

for s_n appropriately so that

- Θ_n^0 is quasiconformal,
- $\Theta_n^0(Z^2) = G_{s_n}(\Theta_n^0(Z))$ for $|Z| = R$,
- $\Theta_n^0(Z) = \Psi_{s_n}^{-1}(R^{-3/2} Z)$ for $Z \in \Gamma_0(c_0 + \eta_n)$,
- $\Theta_n^0(\Gamma_0(c_0 + \eta_n)) = J(f_{s_n})$.

Then the Julia set $J(f_c) \subset U_c' \smallsetminus \overline{V_c'}$ is a Cantor set for every $c \in W_n$ for the same reason for $J(f_{s_n})$ and this, as well as ∂U_c and $\partial V_c'$ undergo holomorphic motion (see [**S**, p.229]). By Słodkowski's theorem there exists a holomorphic motion ι_c on $\mathbb{C}$ which induces these motions. Finally define $\Theta_c := \iota_c \circ \Theta_n^0$, then $\Theta = \Theta_n := \{\Theta_c\}_{c \in W_n}$ is a tubing for $\boldsymbol{G}_n$.

Now we have to check that $\boldsymbol{G}_n$ with Θ_n satisfies the conditions (1)–(8) for a Mandelbrot-like family. It is easy to check that (2)–(7) are satisfied for $\boldsymbol{G}_n$ with Θ_n.

First we prove the condition (1), that is, W_n is a Jordan domain with C^1 boundary. In what follows, we work with the linearizing coordinate again. Let $z_c(t)$ ($t \in [0, 1]$) be a parametrization of the Jordan curve $\partial \widetilde{V}_c$. This is obtained by the holomorphic motion of $\partial \widetilde{V}_{c_1}$. Now we have to solve the equation with respect to the variable c

$$\mu(c)^n \tau(c) = z_c(t) \tag{4.2}$$

to get a parametrization of $\partial \widetilde{W}_n$. As with the equation (4.1), this equation (4.2) can be written as

$$\{\mu_{c_1}^n K_0(c - c_1) - z_c(t)\} + h(c) = 0, \quad h(c) = O(|c - c_1|^{\alpha+1}).$$

Note that this $h(c)$ is, of course, different from the one in (4.1). Rewrite this as

$$F(c) + G(c) = 0, \tag{4.3}$$

where

$$F(c) := \mu_{c_1}^n K_0(c - c_1) - z_{c_1}(t), \quad G(c) := h(c) - (z_c(t) - z_{c_1}(t)).$$

The equation $F(c) = 0$ has a unique solution

$$c = c_n(t) := c_1 + d_n(t), \quad d_n(t) := \frac{z_{c_1}(t)}{\mu_{c_1}^n K_0}.$$

Consider (4.3) in the disk $D(c_n(t), |d_n(t)|^{1+\beta})$ for a small $\beta > 0$. Since we have the estimates

$$|F(c)| = O(|d_n(t)|^\beta), \quad |G(c)| = O(|d_n(t)|)$$

on the boundary of this disk, we have $|F(c)| > |G(c)|$ for sufficiently large n. By Rouché's theorem (4.3) has a unique solution $\check{c}_n(t)$ in the disk $D(c_n(t), |d_n(t)|^{1+\beta})$, so it satisfies

$$\check{c}_n(t) = c_n(t) + O(|d_n(t)|^{1+\beta}) = c_1 + d_n(t)(1 + O(|d_n(t)|^\beta)).$$

By using this solution we can parametrize $\partial\widetilde{W}_n$ as

$$z_{\check{c}_n(t)}(t) = \mu(\check{c}_n(t))^n \tau(\check{c}_n(t))$$

and since $z_c(t)$ is holomorphic in c, in particular it is continuous in c and we have

$$z_{\check{c}_n(t)}(t) \to z_{c_1}(t) \quad \text{i.e.} \quad \partial\widetilde{W}_n \to \partial\widetilde{V}_{c_1} \quad (n \to \infty).$$

Note that this convergence is uniform with respect to t.

Moreover the following holds:

Lemma 4.3. *The function $\check{c}_n(t)$ is of class C^1 and*

$$z_{\check{c}_n(t)}(t) \to z_{c_1}(t) \quad (n \to \infty)$$

with respect to C^1 norm.

We show the proof of this lemma in Appendix A.

Now since $z_{c_1}(t)$ is a Jordan curve of class C^1 and the parametrization $z_{\check{c}_n(t)}(t)$ of $\partial\widetilde{W}_n$ is C^1 close to $z_{c_1}(t)$, it follows that $z_{\check{c}_n(t)}(t)$ is also a Jordan curve for every sufficiently large n by the following lemma (See Appendix B for the proof):

Lemma 4.4. *Let $z_n(t)$ and $z(t)$ $(t \in [0,1])$ be closed curves of class C^1 in $\mathbb{C}$ and assume that $z(t)$ is a Jordan curve. If $z_n(t) \to z(t)$ $(n \to \infty)$ with respect to C^1 norm, then $z_n(t)$ is also a Jordan curve for every sufficiently large $n \in \mathbb{N}$.*

So $\widetilde{W}_n$ is a Jordan domain with C^1 boundary and this implies that W_n is also a Jordan domain with C^1 boundary.

Next the one turn condition (8) is proved as follows: When c ranges over ∂W_n making one turn, the variable t for the function $z_{\check{c}_n(t)}(t) = \mu(\check{c}_n(t))^n \tau(\check{c}_n(t))$ varies from $t = 0$ to $t = 1$. Since $z_{\check{c}_n(t)}(t)$ is C^1 close to $z_{c_1}(t)$, this implies that $f_c^{kn}(a_c) = f_c^{l+kn}(0) = f^L(0)$ makes one turn in a very thin tubular neighborhood of ∂V_{c_1}. Hence $G_c(0) = g_c \circ f_c^{l+kn}(0) = g_c \circ f_c^L(0)$ makes one turn in a very thin tubular neighborhood of ∂U_{c_1}.

Step (M4): End of the proof of Theorem A for Misiurewicz case

For every $\varepsilon > 0$ and $\varepsilon' > 0$, take an $n \in \mathbb{N}$ so sufficiently large that $c_0 + \eta = c_0 + \eta_n \in D(c_0, \varepsilon) \smallsetminus M$. We conclude that the model $\mathcal{M}(c_0 + \eta)$ appears quasiconformally in M in the neighborhood $D(c_1, \varepsilon') = D(s_0 \perp c_0, \varepsilon')$ of $c_1 = s_0 \perp c_0$ by applying Proposition 3.1 to the Mandelbrot-like family $\boldsymbol{G} = \boldsymbol{G_n}$ with $\Theta = \Theta_n$. Indeed from Proposition 3.1, the set

$$\mathcal{N} := M_{\boldsymbol{G}} \cup \{c \mid G_c^k(0) \in \Theta_c(\Gamma_0(c_0 + \eta)) \quad \text{for some } k \in \mathbb{N}\}$$

is the image of $\mathcal{M}(c_0 + \eta)$ by the quasiconformal map $\chi_\Theta^{-1} = \chi_{\Theta_n}^{-1}$, where $M_{\boldsymbol{G}}$ is the connectedness locus of $\boldsymbol{G}$. On the other hand, for $c \in M_{\boldsymbol{G}}$, the orbit of the critical point 0 by $G_c = g_c \circ f_c^{l+kn} = P_c^{Lp+N}$ is bounded, which implies that the orbit of 0 by P_c is also bounded and hence $c \in M$. If $G_c^k(0) \in \Theta_c(\Gamma_0(c_0+\eta))$ for some $k \in \mathbb{N}$, then $c \in M$ as well. So the set $\mathcal{N}$ is a subset of M. In particular, since a conformal image $\Phi_M^{-1}(J_{c_0+\eta} \times R^{3/2})$ of $J_{c_0+\eta}$ is a subset of $\mathcal{M}(c_0 + \eta)$, we conclude that $J_{c_0+\eta}$ appears quasiconformally in M. This completes the proof of Theorem A for Misiurewicz case. ■

Remark. In [**D-BDS**], there is no proof for W_n being C^1 Jordan domain and also the proof for the one turn condition (8) is intuitive.

5. Proof of Theorem A for parabolic case

The proof is a mild generalization of Douady's original proof for the cauliflower. We divide it into four steps (Steps (P1)–(P4)) that are parallel to the Misiurewicz case (Steps (M1)–(M4)).

Step (P1): Definition of U_c, U_c', and V_c

Let us recall the settings: Let M_{s_0} be the small Mandelbrot set with center $s_0 \neq 0$. Now we take any parabolic parameter $c_0 \in \partial M$ and set $c_1 := s_0 \perp c_0$, which is parabolic as well.

Let p be the period of the superattracting cycle in the dynamics of P_{s_0}. As in the previous section, we take the simply connected domain $\Lambda = \Lambda_{s_0}$ in the parameter place associated with the small Mandelbrot set M_{s_0}. Note that the parabolic parameter $c_1 = s_0 \perp c_0$ belongs to Λ unless M_{s_0} is a satellite small Mandelbrot set and $c_0 = 1/4$. For a technical reason, *if M_{s_0} is a satellite small Mandelbrot set, then we assume in addition that* $c_0 \neq 1/4$. (The case we excluded will be discussed at the end of Step (P1).) Then there exist Jordan domains U_{c_1}' and U_{c_1} (with C^1 boundaries) containing 0 such that the restriction $f_{c_1} = P_{c_1}^p|_{U_{c_1}'} : U_{c_1}' \to U_{c_1}$ is a quadratic-like map.

A pair of petals. Let Δ be the Fatou component in $K(f_{c_1})$ containing 0. (We call it the *critical Fatou component*). The boundary $\partial\Delta$ contains a unique parabolic periodic point q_{c_1} of f_{c_1} (resp. P_{c_1}) of period k. (resp. kp.) The multiplier $(f_{c_1}^k)'(q_{c_1})$ is of the form

$$\mu_0 := e^{2\pi i\nu'/\nu},$$

where ν' and ν are coprime integers. Since P_{c_1} has only one critical point, q_{c_1} has ν-petals. That is, by choosing an appropriate local coordinate $w = \psi(z)$ near q_{c_1} with $\psi(q_{c_1}) = 0$, we have

$$\psi \circ f_{c_1}^{k\nu} \circ \psi^{-1}(w) = w(1 + w^\nu + O(w^{2\nu})).$$

See [**B**, Proof of Theorem 6.5.7] and [**K1**, Appendix A.2]. The set of w's with $\arg w^\nu = 0$ (resp. $\arg w^\nu = \pi$) determines *the repelling directions* (resp. *the attracting directions*) of this parabolic point. Note that the critical Fatou component Δ is invariant under $f_{c_1}^{k\nu}$, and it contains a unique attracting direction. In particular, the sequence $f_{c_1}^{k\nu m}(0)$ $(m \in \mathbb{N})$ converges to q_{c_1} within Δ tangentially to the attracting direction.

Set

$$\Omega^+ := \left\{ z = \psi^{-1}(w) \in \mathbb{C} \mid -\frac{2\pi}{3\nu} \le \arg w \le \frac{2\pi}{3\nu},\ 0 < |w| < r \right\},$$
$$\Omega^- := \left\{ z = \psi^{-1}(w) \in \mathbb{C} \mid -\frac{5\pi}{3\nu} \le \arg w \le -\frac{\pi}{3\nu},\ 0 < |w| < r \right\}$$

for some $r \ll 1$ such that Ω^+ and Ω^- are a pair of repelling and attracting petals with $\Omega^+ \cap \Omega^- \neq \emptyset$. (See Figure 9.) By multiplying a ν-th root of unity to the local coordinate $w = \psi(z)$ if necessary, we may assume that the attracting petal Ω^- is contained in Δ.

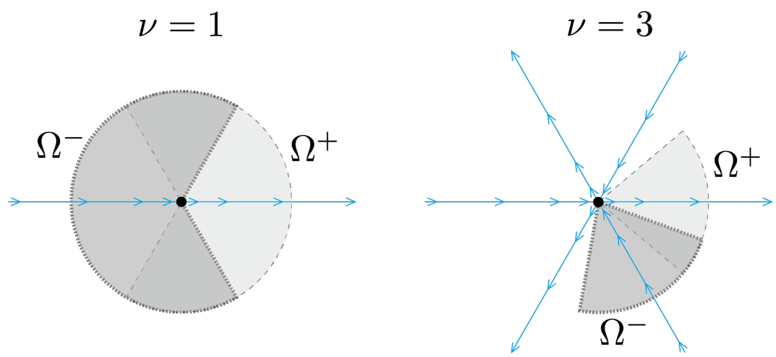

FIGURE 9. We choose a pair of repelling and attracting petals. Their intersection has two components when $\nu = 1$.

Fatou coordinates. For the coordinate $w = \psi(z)$, we consider an additional coordinate change $w \mapsto W = -1/(\nu w^\nu)$. In this W-coordinate, the action of $f_{c_1}^{k\nu}$ on each petal is

$$W \mapsto W + 1 + O(W^{-1}).$$

By taking a sufficiently small r if necessary, there exist conformal mappings $\phi^+ : \Omega^+ \to \mathbb{C}$ and $\phi^- : \Omega^- \to \mathbb{C}$ such that $\phi^\pm(f_{c_1}^{k\nu}(z)) = \phi^\pm(z) + 1$ which are unique up to adding constants. We call $\phi^\pm$ the *Fatou coordinates*. Since the "critical orbit" $\{f_{c_1}^{k\nu m}(0)\}_{m \ge 0}$ lands on Ω^- and never escapes from it, we normalize ϕ^- such that $\phi^-(f_{c_1}^{k\nu m}(0)) = m$ for $m \gg 0$. We will also normalize ϕ^+ after Lemma 4.1' below.

Parabolic counterpart of Lemma 4.1. Now we choose Jordan domains U_{c_1}, U'_{c_1} and V_{c_1} as in the Misiurewicz case:

Lemma 4.1'. *There exist Jordan domains U_{c_1}, U'_{c_1} and V_{c_1} with C^1 boundaries and integers N, $j \in \mathbb{N}$ which satisfy the following:*

(1) $f_{c_1} = P_{c_1}^p|_{U'_{c_1}} : U'_{c_1} \to U_{c_1}$ *is a quadratic-like map.*
(2) $g_{c_1} := P_{c_1}^N|_{V_{c_1}} : V_{c_1} \to U_{c_1}$ *is an isomorphism and* $f_{c_1}^j(V_{c_1}) \subset U_{c_1} \setminus \overline{U'_{c_1}}$.
(3) $V_{c_1} \subset \Omega^+$. *Also we can take V_{c_1} arbitrarily close to $q_{c_1} \in \partial\Omega^+$.*

The proof is the same as that of Lemma 4.1. See Figure 10.

Let $b_{c_1} := g_{c_1}^{-1}(0) \in V_{c_1}$. We normalize the Fatou coordinate ϕ^+ such that $\phi^+(b_{c_1}) = 0$.

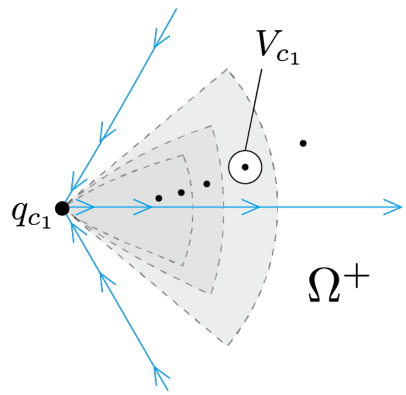

FIGURE 10. We choose V_{c_1} in the repelling petal Ω^+.

For c sufficiently close to c_1, let $U_c := U_{c_1}$ and define U_c' by the connected component of $P_c^{-p}(U_c)$ with $U_c' \Subset U_c$, and V_c by the connected component of $P_c^{-N}(U_c)$ whose boundary is close to that of V_{c_1}. Indeed, if we choose a sufficiently small disk $D(c_1, \varepsilon_1) \subset \Lambda \cap D(c_1, \varepsilon')$, then

- $f_c := P_c^p|_{U_c'}$ gives a quadratic-like map $f_c : U_c' \to U_c$;
- $V_c \subset U_c$; and
- $f_c^j(V_c) \Subset U_c \smallsetminus \overline{U_c'}$ and $g_c := P_c^N|_{V_c} : V_c \to U_c$ is an isomorphism

for each $c \in D(c_1, \varepsilon_1)$. Set $b_c := g_c^{-1}(0) \in V_c$. We regard the family $\{f_c : U_c' \to U_c\}_{c \in D(c_1,\varepsilon_1)}$ as holomorphic perturbations of the quadratic-like map f_{c_1}.

Perturbation of the parabolic periodic point. Let us define a "sector" S in $D(c_1, \varepsilon_1)$ such that the perturbed Fatou coordinates are defined for each parameter $c \in S$. (Cf. [**DH1**, Exposé XI – XIV]. See also [**T2**, Theorems 1.1 and A.1].)

When $\nu = 1$ (equivalently, $\mu_0 = 1$), the parabolic fixed point q_{c_1} of $f_{c_1}^k$ splits into two distinct fixed points q_c and q_c' of f_c^k for each $c \neq c_1$. In local coordinates centered at each of these two points, the action of f_c^k is of the form $w \mapsto \mu_c w + O(w^2)$ and $w \mapsto \mu_c' w + O(w^2)$, where

$$\begin{aligned} \mu_c &= 1 + A_0\sqrt{c - c_1} + O(c - c_1), \\ \mu_c' &= 1 - A_0\sqrt{c - c_1} + O(c - c_1) \end{aligned} \tag{5.1}$$

for some constant $A_0 \neq 0$. (They swap their roles when c makes one turn around c_1.) Hence the parameter c can be locally regarded as a two-to-one function $c = c_1 + A_0^{-2}(\mu_c - 1)^2 + \cdots$ of the multiplier $\mu = \mu_c$ close to 1. We choose a small $r_0 > 0$ and define a "sector" S in $D(c_1, \varepsilon_1)$ by

$$S := \left\{ c \in D(c_1, \varepsilon_1) \mid 0 < |\mu_c - 1| < r_0 \text{ and } \left| \arg(\mu_c - 1) - \frac{\pi}{2} \right| < \frac{\pi}{8} \right\}.$$

See Figure 11 (left). (The conditions for μ_c ensure $\arg \mu_c < 0$ and $\arg \mu_c' > 0$ for $c \neq c_1$.)

When $\nu \geq 2$ (equivalently, $\mu_0 \neq 1$), the parabolic fixed point q_{c_1} of $f_{c_1}^k$ splits into one fixed point q_c and a cycle of period ν of f_c^k for each $c \neq c_1$. In a local coordinate centered at q_c, the action of f_c^k is of the form $w \mapsto \mu_c w + O(w^2)$ where

$$\mu_c = (f_c^k)'(q_c) = \mu_0 \left(1 + B_0(c - c_1) + O((c - c_1)^2)\right) \tag{5.2}$$

for some constant $B_0 \neq 0$. Hence the parameter c can be locally regarded as a one-to-one function $c = c_1 + B_0^{-1}(\mu_c - \mu_0) + \cdots$ of the multiplier $\mu = \mu_c$ close to μ_0. We choose a small $r_0 > 0$ and define a "sector" S in $D(c_1, \varepsilon_1)$ by

$$S := \left\{ c \in D(c_1, \varepsilon_1) \mid 0 < \left| \frac{\mu_c}{\mu_0} - 1 \right| < r_0 \text{ and } \left| \arg\left(\frac{\mu_c}{\mu_0} - 1 \right) - \frac{\pi}{2} \right| < \frac{\pi}{8} \right\}.$$

See Figure 11 (right).

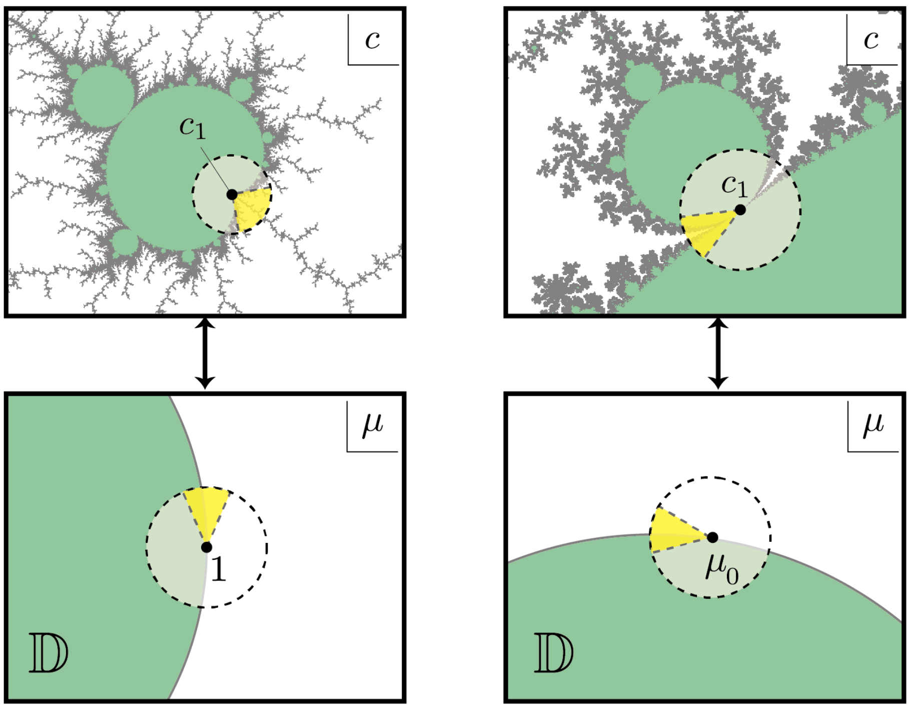

FIGURE 11. The sector S for $\nu = 1$ (left) and $\nu \geq 2$ (right).

Perturbed Fatou coordinates and the phase. For any c sufficiently close to c_1, there exists a local coordinate $w = \psi_c(z)$ near q_c with $\psi_c(q_c) = 0$ such that

$$\psi_c \circ f_c^{k\nu} \circ \psi_c^{-1}(w) = \mu_c^\nu w \, (1 + w^\nu + O(w^{2\nu})),$$

where $\mu_c^\nu \to 1$ and $\psi_c \to \psi$ uniformly as $c \to c_1$. See [**K1**, Appendix A.2]. By a further ν-fold coordinate change $W = -\mu_c^{\nu^2}/(\nu w^\nu)$, the action of $f_c^{k\nu}$ is

$$W \mapsto \mu_c^{-\nu^2} W + 1 + O(W^{-1}),$$

where $W = \infty$ is a fixed point with multiplier exactly $\mu_c^{\nu^2}$ that corresponds to the fixed point q_c of $f_c^{k\nu}$. There is a fixed point of the form $W = 1/(1 - \mu^{-\nu^2}) + O(1)$ with multiplier close to $\mu_c^{-\nu^2}$ on each branch of the ν-fold coordinate. Note that

$$\mu_c^{\pm\nu^2} = 1 \pm \nu^2 A_0 \sqrt{c - c_1} + O(c - c_1)$$

or

$$\mu_c^{\pm\nu^2} = 1 \pm \nu^2 B_0 (c - c_1) + O((c - c_1)^2)$$

according to $\nu = 1$ or $\nu \geq 2$ by (5.1) and (5.2).

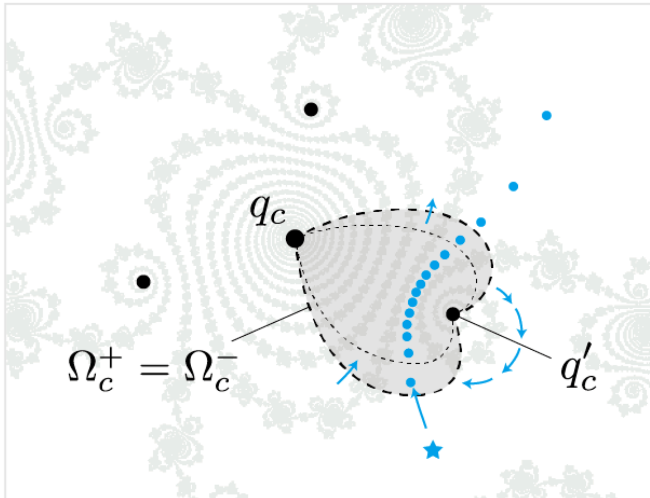

FIGURE 12. A typical behavior of the critical orbit by $f_c^{k\nu}$ near q_c for $\nu = 3$.

It is known that for each c in the sector S, there exist unique *(perturbed) Fatou coordinates* $\phi_c^+ : \Omega_c^+ \to \mathbb{C}$ and $\phi_c^- : \Omega_c^- \to \mathbb{C}$ satisfying the following conditions (See [**La**], [**DSZ**] and [**S**, Proposition A.2.1]):

- Each $\partial\Omega_c^\pm$ contains two fixed points q_c and q_c' of $f_c^{k\nu}$ that converge to q_{c_1} as $c \to c_1$ in S.
- $\phi_c^\pm$ is a conformal map from $\Omega_c^\pm$ onto the image in $\mathbb{C}$.
- $\phi_c^\pm(f_c^{k\nu}(z)) = \phi_c^\pm(z) + 1$ if both z and $f_c^{k\nu}(z)$ are contained in $\Omega_c^\pm$.
- Every compact set E in $\Omega^\pm$ is contained in $\Omega_c^\pm$ if $c \in S$ is sufficiently close to c_1. In particular, for such a c,
  - if $f_{c_1}^{k\nu m}(0) \in \Omega^-$ for some $m \gg 1$, then $f_c^{k\nu m}(0) \in \Omega_c^-$; and
  - $V_c \Subset \Omega_c^+$.
- The maps $\phi_c^\pm$ are normalized as follows: $\phi_c^-(f_c^{k\nu m}(0)) = m$ if $f_c^{k\nu m}(0) \in \Omega_c^-$; and $\phi_c^+(b_c) = 0$, where $b_c = g_c^{-1}(0)$.
- For any compact set E in $\Omega^\pm$, $\phi_c^\pm|_E$ converges uniformly to $\phi^\pm|_E$ as $c \to c_1$ in S.

We can arrange the domains $\Omega_c^\pm$ such that $\Omega_c^+ = \Omega_c^- =: \Omega_c^*$ for each c (Figure 12). Hence for each $z \in \Omega_c^*$,

$$\tau(c) := \phi_c^+(z) - \phi_c^-(z) \in \mathbb{C}$$

is defined and independent of z. The function $\tau : S \to \mathbb{C}$ is called the *lifted phase*. Note that the value $\tau(c)$ is determined by the unique normalized Fatou coordinates associated with the analytic germ $f_c^{k\nu}$. It does not depend on the choice of the parametrization [1]. If $\nu = 1$, then $\tau(c)$ is of the form

$$\tau(c) = -\frac{2\pi i}{A_0\sqrt{c - c_1}} + O(1) \quad (c \to c_1),$$

[1] In [**D-BDS**], the lifted phase for $\nu = 1$ is described in terms of the normalized germ $f_\mu(z) = z + z^2 + \mu + \cdots$ $(\mu \to 0)$. In this case the multiplier for two fixed points of f_μ are $1 \pm 2\sqrt{\mu} i\,(1 + O(\mu))$ and $\tau = -\pi/\sqrt{\mu} + O(1)$ as $\mu \to 0$. In [**DSZ**], they use $\alpha = (\nu/2\pi i)\log(\mu_c/\mu_0 - 1)$ (so that $\mu_c = \exp(2\pi i(\nu' + \alpha)/\nu))$ to parametrize the germs. Any parameterizations are analytically equivalent and they determine the same value τ as long as it represent the same analytic germ near the fixed points.

where A_0 is given in (5.1) and we choose the branch of $\sqrt{c-c_1}$ such that the corresponding multiplier $\mu_c = 1 + A_0\sqrt{c-c_1} + \cdots$ satisfies $\mathrm{Im}\,(\mu_c - 1) > 0$. If $\nu \geq 2$, then

$$\tau(c) = -\frac{2\pi i}{\nu^2 B_0(c-c_1)} + O(1) \quad (c \to c_1),$$

where B_0 is given in (5.2).

Satellite roots. Now we deal with the remaining case: Suppose that M_{s_0} is a satellite small Mandelbrot set of M with renormalization period p and $c_0 = 1/4$. That is, $c_1 = s_0 \perp c_0$ is the root of M_{s_0}. Then P_{c_1} has a parabolic periodic point q_{c_1} of period p' with $\nu \geq 2$ petals such that $p = p'\nu$. In this case, we slightly modify the construction above and get a "degenerate quadratic-like mapping" $f_{c_1} := P_{c_1}^p|_{U'_{c_1}} : U'_{c_1} \to U_{c_1}$, which satisfies:

- U'_{c_1} and U_{c_1} are topological disks with $U'_{c_1} \subset U_{c_1}$ such that $\partial U'_{c_1} \cap \partial U_{c_1} = \{q_{c_1}\}$.
- $f_{c_1} : U'_{c_1} \to U_{c_1}$ is a proper branched covering of degree two.

(This is part of the construction of $\Lambda = \Lambda_{s_0}$ for the satellite M_{s_0}. See [**H**] for more details.) Since q_{c_1} is a fixed point of $P_{c_1}^p$ with multiplier 1 and ν petals, we can find a pair of attracting and repelling petals Ω^- and Ω^+ as in Figure 9 (right). Moreover, we choose a Jordan domain $V_{c_1} \Subset \Omega^+ \cap U'_{c_1}$ satisfying the conditions (1) – (3) of Lemma 4.1'. When c is close to c_1, there exists a fixed point q_c of P_c^p with $q_c \to q_{c_1}$ as $c \to c_1$. The bifurcation of the local dynamics is described in terms of the multiplier $\mu_c = (P_c^p)'(q_c)$. We define the "sector" S as in the case of $\nu \geq 2$. Then perturbed Fatou coordinates make sense for any $c \in S \cap \Lambda$. We may also assume that q_c is repelling, and define a Jordan domain U_c by adding a small disk centered at q_c to U_{c_1}. (We slightly modify U_c such that ∂U_c is a C^1 Jordan curve that moves holomorphically with respect to c.) Then we have a quadratic-like family $f_c : U'_c \to U_c$, where U'_c is a connected component of $P_c^{-p}(U_c)$ with $U'_c \Subset U_c$. We can also choose the Jordan domain V_c such that V_c is a connected component of $P_c^{-N}(U_c)$ that is close to V_{c_1}.

Step (P2): Construction of the Mandelbrot-like family $\boldsymbol{G} = \boldsymbol{G}_n$

We construct a Mandelbrot-like family

$$\boldsymbol{G} = \boldsymbol{G}_n := \{G_c : V'_c \to U_c\}_{c \in W_n}$$

such that $V'_c \subset U'_c$ and $W = W_n \subset S \cap \Lambda$ for every sufficiently large $n \in \mathbb{N}$ as follows: Recall that the quadratic-like map $f_c : U'_c \to U_c$ has a (repelling) periodic point q_c of period k which bifurcates from the parabolic periodic point q_{c_1} of f_{c_1} and we take perturbed Fatou coordinates $\phi_c^{\pm} : \Omega_c^{\pm} = \Omega_c^* \to \mathbb{C}$ such that

$$\phi_c^{\pm}(f_c^{k\nu}(z)) = \phi_c^{\pm}(z) + 1$$

with normalization $\phi_c^-(f_c^{k\nu m}(0)) = m$ if $f_c^{k\nu m}(0) \in \Omega_c^*$ for sufficiently large m; and $\phi_c^+(b_c) = 0$, where $b_c := g_c^{-1}(0)$ is the pre-critical point. For $m \in \mathbb{N}$ with $f_{c_1}^{k\nu m}(0) \in \Omega_{c_1}^-$, we may assume that S is sufficiently small such that $f_c^{k\nu m}(0) \in \Omega_c^*$ for each $c \in S \cap \Lambda$. Now for each $n \geq m$, define

$$W = W_n := \{c \in S \mid f_c^{k\nu n}(0) \in V_c\}.$$

Lemma 4.2'. *The set $W = W_n$ is not empty for every sufficiently large n. Moreover there exists an $s_n \in W_n$ such that $f_{s_n}^{k\nu n}(0) = b_{s_n}$, that is, $g_{s_n} \circ f_{s_n}^{k\nu n}(0) = 0$ and hence P_{s_n} has a super-attracting periodic point.*

Proof. In what follows, we work with the perturbed Fatou coordinate $\phi_c^+ : \Omega_c^+ = \Omega_c^* \to \mathbb{C}$ of q_c. Let

$$\widetilde{c} := \phi_c^+(0), \quad \widetilde{V}_c := \phi_c(V_c)$$

and noting that $\tau(c) = \phi_c^+(z) - \phi_c^-(z)$ is independent of z, we have

$$\widetilde{c} = \phi_c^+(0) = \tau(c).$$

Define

$$\begin{aligned} \widetilde{W}_n &:= \{\widetilde{c} \mid c \in W_n\} \\ &= \{\tau(c) \mid c \text{ satisfies } \ f_c^{k\nu n}(0) \in V_c\} \\ &= \{\widetilde{c} \mid \widetilde{c} \text{ satisfies } \ \widetilde{c} + n \ (= \tau(c) + n) \in \widetilde{V}_c\}. \end{aligned}$$

Then $c \in W_n$ if and only if $\widetilde{c} = \tau(c) \in \widetilde{W}_n$. In order to show $W_n \neq \emptyset$, it suffices to show that for $0 = \phi_c^+(b_c) \in \widetilde{V}_c$ the equation $\tau(c) + n = 0$ has a solution.

Case 1 : $\nu = 1$. Since

$$\tau(c) = -\frac{2\pi i}{A_0\sqrt{c - c_1}} + O(1) \quad (c \to c_1),$$

the equation $\tau(c) + n = 0$ can be written as

$$-\frac{2\pi i}{A_0\sqrt{c - c_1}} + n + h(c) = 0, \quad h(c) = O(1). \tag{5.3}$$

Since the equation $-\frac{2\pi i}{A_0\sqrt{c-c_1}} + n = 0$ has a unique solution

$$c = c_1 + c(n), \quad c(n) := -\Big(\frac{2\pi}{A_0 n}\Big)^2,$$

we consider this equation (5.3) on the disk $D(c_1 + c(n), r)$ for $r > 0$. Then on the boundary of this disk, by putting $c = (c_1 + c(n)) + re^{i\theta}$, we have

$$\begin{aligned} -\frac{2\pi i}{A_0\sqrt{c - c_1}} + n &= -\frac{2\pi i}{A_0\sqrt{c(n) + re^{i\theta}}} + n \\ &= -\frac{2\pi i}{A_0\sqrt{c(n)}}\frac{1}{\sqrt{1 + \frac{re^{i\theta}}{c(n)}}} + n \\ &= -n\bigg(1 - \frac{1}{2}\cdot\frac{re^{i\theta}}{c(n)} + O\Big(\Big|\frac{r}{c(n)}\Big|^2\Big)\bigg) + n \\ &= n\bigg(\frac{1}{2}\cdot\frac{re^{i\theta}}{c(n)} + O\Big(\Big|\frac{r}{c(n)}\Big|^2\Big)\bigg). \end{aligned}$$

Taking $r := |c(n)|^{1+\beta}$ for any small $\beta > 0$, we have

$$\bigg| - \frac{2\pi i}{A_0\sqrt{c - c_1}} + n\bigg| = O(n^{1-2\beta}),$$

when n is sufficiently large. Meanwhile we have $|h(c)| = O(1)$, so $\big| - \frac{2\pi i}{A_0\sqrt{c-c_1}} + n\big| > |h(c)|$. Hence by Rouché's theorem the equation has a unique solution.

Case 2 : $\nu \geq 2$. In this case

$$\tau(c) = -\frac{2\pi i}{\nu^2 B_0(c - c_1)} + O(1) \quad (c \to c_1),$$

the equation $\tau(c)+n=0$ can be written as

$$-\frac{2\pi i}{\nu^2 B_0(c-c_1)}+n+h(c)=0, \quad h(c)=O(1).$$

Since the equation $-\frac{2\pi i}{\nu^2 B_0(c-c_1)}+n=0$ has a unique solution

$$c=c_1+c(n), \quad c(n):=\frac{2\pi i}{\nu^2 B_0 n},$$

we consider this equation on the disk $D(c_1+c(n),r)$ for $r>0$. Then on the boundary of this disk, by putting $c=(c_1+c(n))+re^{i\theta}$, we have

$$\begin{aligned}
-\frac{2\pi i}{\nu^2 B_0(c-c_1)}+n &= -\frac{2\pi i}{\nu^2 B_0(c(n)+re^{i\theta})}+n \\
&= -\frac{2\pi i}{\nu^2 B_0 c(n)}\cdot\frac{1}{1+\frac{re^{i\theta}}{c(n)}}+n \\
&= -n\left(1-\frac{re^{i\theta}}{c(n)}+O\left(\left|\frac{r}{c(n)}\right|^2\right)\right)+n \\
&= n\left(\frac{re^{i\theta}}{c(n)}+O\left(\left|\frac{r}{c(n)}\right|^2\right)\right).
\end{aligned}$$

Taking $r:=|c(n)|^{1+\beta}$, we have

$$\left|-\frac{2\pi i}{\nu^2 B_0(c-c_1)}+n\right|=O(n^{1-\beta})$$

when n is sufficiently large. Meanwhile we have $|h(c)|=O(1)$, so by Rouché's theorem the equation has a unique solution. This shows that $W_n\neq\emptyset$ if n is sufficiently large. Moreover denoting this solution by $c=s_n$, we have

$$f_{s_n}^{k\nu n}(0)=b_{s_n}, \quad \text{that is} \quad g_{s_n}\circ f_{s_n}^{k\nu n}(0)=0,$$

which means that P_{s_n} has a superattracting periodic point. ■ (Lemma 4.2.')

We call $s_n\in W_n$ the *center* of W_n. Now let $L=L_n:=k\nu n$ and V_c' be the component of $f_c^{-L}(V_c)$ containing 0 and define

$$G_c:=g_c\circ f_c^L:V_c'\to U_c \quad \text{and} \quad \boldsymbol{G}=\boldsymbol{G}_n:=\{G_c:V_c'\to U_c\}_{c\in W_n},$$

where $W_n=\{c\in S\mid f_c^L(0)\in V_c\}$.

Step (P3): Proof for $\boldsymbol{G}=\boldsymbol{G}_n$ being a Mandelbrot-like family

The map $f_c^L:V_c'\to V_c$ is a branched covering of degree 2 and $g_c:V_c\to U_c$ is a holomorphic isomorphism. Hence $G_c:=g_c\circ f_c^L:V_c'\to U_c$ is a quadratic-like map. Then a tubing $\Theta=\Theta_n=\{\Theta_c\}_{c\in W_n}$ for $\boldsymbol{G}_n$ can be constructed by exactly the same method as in the Misiurewicz case.

Now we have to check that $\boldsymbol{G}_n$ with $\Theta=\Theta_n:=\{\Theta_c\}_{c\in W_n}$ satisfies the conditions (1)–(8) for a Mandelbrot-like family. It is easy to check that (2)–(7) are satisfied.

First we prove the condition (1), that is, W_n is a Jordan domain with C^1 boundary. In what follows, we work with the perturbed Fatou coordinate again. Let $z_c(t)$ $(t\in[0,1])$ be a parametrization of the Jordan curve $\partial\widetilde{V}_c$. This is obtained by the holomorphic motion of $\partial\widetilde{V}_{c_1}$. Now we have to solve the equation with respect to the variable c

$$\tau(c) + n = z_c(t) \tag{5.4}$$

to get a parametrization of $\partial\widetilde{W}_n$.

Case 1 : $\nu = 1$. As with the equation (5.3), this equation (5.4) can be written as

$$-\frac{2\pi i}{A_0\sqrt{c-c_1}} + n - z_c(t) + h(c) = 0, \quad h(c) = O(1).$$

Note that this $h(c)$ is, of course, different from the one in (5.3). Rewrite this as

$$F(c) + G(c) = 0, \tag{5.5}$$

where

$$F(c) := -\frac{2\pi i}{A_0\sqrt{c-c_1}} + n - z_{c_1}(t) \quad \text{and} \quad G(c) := h(c) - (z_c(t) - z_{c_1}(t)).$$

The equation $F(c) = 0$ has a unique solution

$$c = c_n(t) := c_1 + d_n(t) \quad \text{with} \quad d_n(t) := -\frac{4\pi^2}{A_0^2(n - z_{c_1}(t))^2}.$$

Consider (5.5) in the disk $D(c_n(t), |d_n(t)|^{1+\beta})$ for a small $\beta > 0$. Since we have the estimate

$$|F(c)| = O(|d_n(t)|^{\beta - \frac{1}{2}}) = O(n^{1-2\beta}), \quad |G(c)| = O(1)$$

on the boundary of this disk, we have $|F(c)| > |G(c)|$ for sufficiently large n. By Rouché's theorem (5.5) has a unique solution $\check{c}_n(t)$ in $D(c_n(t), |d_n(t)|^{1+\beta})$, so it satisfies

$$\check{c}_n(t) = c_n(t) + O(|d_n(t)|^{1+\beta}) = c_1 + d_n(t)(1 + O(|d_n(t)|^{\beta})).$$

By using this solution we can parametrize $\partial(\widetilde{W}_n + n)$ as

$$z_{\check{c}_n(t)}(t) = \tau(\check{c}_n(t)) + n$$

and since $z_c(t)$ is holomorphic in c, it is continuous in c in particular and we have

$$z_{\check{c}_n(t)}(t) \to z_{c_1}(t) \quad \text{i.e.} \quad \partial(\widetilde{W}_n + n) \to \partial\widetilde{V}_{c_1} \quad (n \to \infty).$$

Note that this convergence is uniform with respect to t.

Case 2 : $\nu \geq 2$. The argument is completely parallel to **Case 1**. By replacing $d_n(t)$ with

$$\frac{2\pi i}{\nu^2 B_0(n - z_{c_1}(t))},$$

we have the estimates

$$F(c) = O(|d_n(t)|^{-1+\beta}) = O(n^{1-\beta}) \quad \text{and} \quad G(c) = O(1)$$

on $\partial D(c_n(t), |d_n(t)|^{1+\beta})$ for any small $\beta > 0$. So the same conclusion as in **Case 1** follows. The rest of the argument is completely the same as in **Case 1**.

Next the Lemma 4.3 also holds in the parabolic case, but the calculation is different. We show the proof in Appendix A. The rest of the argument is completely the same as in **Step (M3)**, which completes the proof of **Step (P3)**.

Step (P4): End of the proof of Theorem A for Parabolic case

This part is also completely the same as in the Misiurewicz case and hence this completes the proof of Theorem A for Parabolic case. ■

Remark. Theorem A is a kind of generalization of the Douady's result but the statements of the results of ours and his are not quite parallel. Actually Douady considered not only the case of the quadratic family but also more general situation and proved a theorem ([**D-BDS**], p.23, THEOREM 2) and then showed the theorem for Mandelbrot set ([**D-BDS**], p.22, THEOREM 1) by using it. Douady's result also shows that a sequence of quasiconformal images of $\mathcal{M}(\frac{1}{4}+\varepsilon)$ appears in $D(s_0 \perp \frac{1}{4}, \varepsilon')$. It is possible to state our result like Douady's. But in order to do this, it is necessary to assume several conditions which are almost obvious for the quadratic family case and this would make the argument more complicated. So we just concentrated on the case of the quadratic family. We avoided stating our result like "a sequence of quasiconformal images of $\mathcal{M}(c_0+\eta)$ appears" for the same reason.

In what follows, we summarize the general situation under which a result similar to THEOREM 2 in [**D-BDS**] (that is, Theorem A' below) hold and this implies our Theorem A. These are the essential assumptions for more general and abstract settings, which leads to the general result Theorem A'.

- $\{f_c : U'_c \to U_c\}_{c\in\Lambda}$ is an analytic family of quadratic-like maps with a critical point ω_c, where $\Lambda \subset \mathbb{C}$ is an open set. The parameter $c_1 \in \Lambda$ is either Misiurewicz or parabolic.
- $\{g_c : V_c \to U_c\}_{c\in\Lambda}$ is an analytic family of analytic isomorphism, where V_c satisfies $f_c^j(V_c) \subset U_c \setminus \overline{U'_c}$ for some $j \in \mathbb{N}$.
- The open sets U_c, U'_c and V_c are Jordan domains with C^1 boundary and move by a holomorphic motion. Let $z_c(t)$ be a parametrization of ∂V_c. Then $z_c(t)$ is holomorphic in c and C^1 in t and $\frac{\partial^2}{\partial c \partial t} z_c(t)$ exists and continuous.
- (1) When c_1 is Misiurewicz, for some $l \in \mathbb{N}$, $f_{c_1}^l(\omega_{c_1})$ is a repelling periodic point of period k and we let $a_c := f_c^l(\omega_c)$. Let q_c be the repelling periodic point persisting when c is perturbed from c_1. Then assume that $a_c \neq q_c$ for $c(\neq c_1)$ which is sufficiently close to c_1.

  (2) When c_1 is parabolic, f_{c_1} has a parabolic periodic point q_{c_1} of period k with multiplier λ_1. Then assume that q_{c_1} bifurcates such that f_c has an appropriate normal form as in **Step (P1)** for c which is sufficiently close to c_1. (Douady gives a sufficient condition for this condition when $k=1$ and $\lambda_1 = 1$ in [**D-BDS**, p.23].)

Note that there exists a c_0 such that f_{c_1} is hybrid equivalent to P_{c_0}. Now define the map $F_c : U'_c \cup V_c \to U_c$ so that $F_c := f_c$ on U'_c and $F_c := g_c$ on V_c. Also define

$$K(F_c) := \{z \mid F_c^n(z) \text{ is defined for all } n \text{ and } F_c^n(z) \in U'_c \cup V_c\},$$
$$M_F := \{c \in \Lambda \mid \omega_c \in K(F_c)\}.$$

Under the above assumptions, we can show the following theorem which implies our Theorem A:

Theorem A'. *For every small $\varepsilon > 0$ and $\varepsilon' > 0$, there exists an $\eta \in \mathbb{C}$ with $|\eta| < \varepsilon$ and $c_0 + \eta \notin M$ such that the decorated Mandelbrot set $\mathcal{M}(c_0+\eta)$ appears quasiconformally in M_F.*

6. Proof of Theorem B

Let M_{s_1} be the main Mandelbrot set of the quasiconformal copy of the decorated Mandelbrot set $\mathcal{M}(c_0+\eta)$ given in Theorem A. Choose any $c \in M$ and set $\sigma := s_1 \perp c \in M_{s_1}$.

Let $\Phi_c : \mathbb{C} \smallsetminus K_c \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ be the Böttcher coordinate. For any $R > 1$ with $J_{c_0+\eta} \subset A(R^{-1/2}, R^{1/2})$, we take the Jordan domains Ω_1' and Ω_1 in $\mathbb{C}$ with $\Omega_1' \Subset \Omega_1$ whose boundaries are the inner and the outer boundaries of $\Phi_c^{-1}(A(R, R^2))$. (That is, we take Douady's radii $\rho' = R^{-1/2}$ and $\rho = R^{1/2}$ in the definition of the model.) Then $P_c : \Omega_1' \to \Omega_1$ is a quadratic-like restriction of P_c, and the decorated filled Julia set $\mathcal{K}_c(c_0+\eta) = \mathcal{K}_c(c_0+\eta)_{R^{-1/2},R^{1/2}}$ is a compact set in Ω_1.

Now we want to show that for $\sigma = s_1 \perp c \in M_{s_1}$ the filled Julia set K_σ contains a quasiconformal copy of the model set $\mathcal{K}_c(c_0+\eta)$. Let us specify the copy first: Consider the quadratic-like maps $f_\sigma : U_\sigma' \to U_\sigma$ and $G_\sigma : V_\sigma' \to U_\sigma$ given in the proof of Theorem A. Since we have $J(f_\sigma) \Subset U_\sigma \smallsetminus \overline{V_\sigma'}$, the set

$$\Gamma := \bigcup_{m \geq 0} G_\sigma^{-m}(J(f_\sigma))$$

is a "decoration" of the filled Julia set $K(G_\sigma) \subset V_\sigma'$. Then the union $\mathcal{K} := K(G_\sigma) \cup \Gamma$ is a compact subset of U_σ. In particular, the boundary $\partial\mathcal{K}$ is contained in ∂K_σ, since the set of points that eventually lands on a repelling cycle of f_σ or G_σ is dense in $\partial\mathcal{K}$. Hence it is enough to show that there exists a quasiconformal map on a domain that maps the model set to $\mathcal{K}$.

Let $h = h_\sigma : U_\sigma \to \mathbb{C}$ be a straightening map of $G_\sigma : V_\sigma' \to U_\sigma$. By setting $\Omega_2 := h(U_\sigma)$ and $\Omega_2' := h(V_\sigma')$, the map $P_c = h \circ G_\sigma \circ h^{-1} : \Omega_2' \to \Omega_2$ is also a quadratic-like restriction of P_c such that $h(K(G_\sigma)) = K_c$. By slightly shrinking Ω_2, we may assume that the boundaries of Ω_2 and Ω_2' are smooth Jordan curves. Since h is quasiconformal, it suffices to show that there exists a quasiconformal map $H : \Omega_1 \to \Omega_2$ that maps $\mathcal{K}_c(c_0+\eta)$ onto $h(\mathcal{K})$.

Now we claim:

Lemma 6.1. *There exists a quasiconformal homeomorphism $H : \overline{\Omega_1} \smallsetminus \Omega_1' \to \overline{\Omega_2} \smallsetminus \Omega_2'$ such that*

- *H is equivariant. That is, $P_c(H(z)) = H(P_c(z))$ for any $z \in \partial\Omega_1'$.*
- *H maps $J^* := \Phi_c^{-1}(J_{c_0+\eta} \times R^{3/2})$ in the model set onto $h(J(f_\sigma))$.*

Proof. Since the boundary components of each annuli are smooth, we can take a smooth homeomorphism between $\partial\Omega_1$ and $\partial\Omega_2$. By pulling it back by the action of P_c, we have a smooth, equivariant homeomorphism ψ_0 between the boundaries of the closed annuli $\overline{\Omega_1} \smallsetminus \Omega_1'$ and $\overline{\Omega_2} \smallsetminus \Omega_2'$.

Next we consider the Julia sets: Consider a sequence of homeomorphisms

$$J^* = \Phi_c^{-1}(J_{c_0+\eta} \times R^{3/2}) \xrightarrow{(1)} J_{c_0+\eta} \xrightarrow{(2)} J_{\chi_f(\sigma)} \xrightarrow{(3)} J(f_\sigma) \xrightarrow{(4)} h(J(f_\sigma)),$$

where (1) is just a conformal map; (2) is achieved by a holomorphic motion in $\mathbb{C} \smallsetminus M$; (3) is the inverse of a restriction of the quasiconformal straightening of f_σ; and (4) is a restriction of the quasiconformal straightening of G_σ. Note that each step can be connected by a holomorphic motion over a domain isomorphic to the unit disk. Hence for any neighborhood D^* of J^*, we have a quasiconformal map $\psi_1 : D^* \to \mathbb{C}$ that maps J^* onto $h(J(f_\sigma))$. Since J^* is a Cantor set, we choose D^* such that D^* is a finite union of smooth Jordan domains satisfying $D^* \Subset \Omega_1 \smallsetminus \overline{\Omega_1'}$ and $\psi_1(D^*) \Subset \Omega_2 \smallsetminus \overline{\Omega_2'}$. Now the sets $\Omega_1 \smallsetminus \overline{\Omega_1' \cup D^*}$ and $\Omega_2 \smallsetminus \overline{\Omega_2' \cup \psi_1(D^*)}$ are multiply connected domains with the same connectivity. By a standard argument in complex analysis, they are conformally equivalent to round annuli with concentric circular slits, and there is a quasiconformal homeomorphism ψ_2 between these domains. Since each component of $\psi_1(D^*)$ is a quasidisk, we can modify ψ_2 such that the boundary correspondence agree with ψ_0 and ψ_1. Hence we obtain a desired quasiconformal homeomorphism H by gluing ψ_0, ψ_1, and this modified ψ_2. (Cf. Bers' lemma below.) ■(Lemma 6.1)

By pulling back the map H given in Lemma 6.1 by the dynamics of P_c, we have a unique homeomorphic extension $H : \overline{\Omega_1} \smallsetminus K_c \to \overline{\Omega_2} \smallsetminus K_c$ such that $P_c(H(z)) = H(P_c(z))$ for any $\Omega_1' \smallsetminus K_c$ and that H maps the decoration of $\mathcal{K}_c(c_0 + \eta)$ onto $h(\Gamma)$.

We employ the following lemmas: (See Lyubich's book in preparation [**Ly3**].)

Lemma 6.2. *Let $f : U' \to U$ be a quadratic-like map with connected Julia set. Let $W_1 \subset U$ and $W_2 \subset U$ be two open annuli whose inner boundary is $J(f)$. Let $H : W_1 \to W_2$ be an automorphism of f, that is, $f(H(z)) = H(f(z))$ on $f^{-1}(W_1)$. Then H admits a continuous extension to a map $H : W_1 \cup J(f) \to W_2 \cup J(f)$ identical on the Julia set.*

Lemma 6.3 (**Bers' Lemma**)**.** *Let K be a compact set in $\mathbb{C}$ and let Ω_1 and Ω_2 be neighborhoods of K such that there exists two quasiconformal maps $H_1 : \Omega_1 \smallsetminus K \to \mathbb{C}$ and $H_2 : \Omega_2 \to \mathbb{C}$ that match on ∂K, i.e., the map $H : \Omega_1 \to \mathbb{C}$ defined by $H(z) := H_1(z)$ for $z \in \Omega_1 \smallsetminus K$ and $H(z) := H_2(z)$ for $z \in K$ is continuous. Then H is quasiconformal and $\mu_H = \mu_{H_2}$ for almost every $z \in K$.*

Now we apply Lemma 6.2 by regarding P_c and $\Omega_j \smallsetminus K_c$ as f and W_j for each $j = 1, 2$. It follows that the restriction $H_1 := H|_{\Omega_1 \smallsetminus K_c}$ of the map $H : \overline{\Omega_1} \smallsetminus K_c \to \overline{\Omega_2} \smallsetminus K_c$ admits a continuous extension to $H_1 : \Omega_1 \smallsetminus \mathrm{int}(K_c) \to \Omega_2 \smallsetminus \mathrm{int}(K_c)$ that agrees with the identity map $H_2 := \mathrm{id} : \Omega_2 \to \Omega_2$ on ∂K_c. By Bers' lemma (Lemma 6.3), we have a quasiconformal map $H : \Omega_1 \to \Omega_2$ such that $H(\mathcal{K}_c(c_0 + \eta)) = h(\mathcal{K})$. ■

7. Proof of Theorem C

Almost conformal straightening. Let us start with a formulation that gives a "fine" copy of the Mandelbrot set.

Lemma 7.1 (**Almost conformal straightening**)**.** *Fix two positive constants r and δ. Suppose that for some $R > \max\{r + \delta, 8\}$ we have a function $u = u_c(z) = u(z, c)$ that is holomorphic with $u_c'(0) = 0$ and $|u(z, c)| < \delta$ in both $z \in D(R)$ and $c \in D(r)$. Let*

$$f_c(z) := z^2 + c + u(z, c),$$

$U_c := D(R)$ and $U_c' := f_c^{-1}(U_c)$. Then for any $c \in D(r)$ the map $f_c : U_c' \to U_c$ is a quadratic-like map with a critical point $z = 0$. Moreover, it satisfies the following properties for sufficiently large R:

(1) *There exists a family of smooth* $(1+O(R^{-1}))$*-quasiconformal maps (a tubing)*

$$\Theta = \{\Theta_c : \overline{A(R^{1/2}, R)} \to \overline{U_c} \setminus U_c'\}_{c \in D(r)}$$

such that

- Θ_c *is identity on* $\partial D(R)$ *for each* c*;*
- Θ_c *is equivariant on the boundary, i.e.,* $\Theta_c(z^2) = f_c(\Theta_c(z))$ *on* $\partial D(R^{1/2})$*; and*
- *for each* $z \in \overline{A(R^{1/2}, R)}$ *the map* $c \mapsto \Theta_c(z)$ *is holomorphic in* $c \in D(r)$*.*

(2) *Each* Θ_c *induces a straightening map* $h_c : U_c \to h(U_c)$ *that is uniformly* $(1+O(R^{-1}))$*-quasiconformal for* $c \in D(r)$*.*

Thus we obtain an analytic family of quadratic-like maps $\boldsymbol{f} = \{f_c : U_c' \to U_c\}_{c \in D(r)}$. (Note that $\boldsymbol{f}$ with tubing Θ is not necessarily a Mandelbrot-like family.)

Proof. One can check that $f_c : U_c' \to U_c$ is a quadratic-like map as in Example 1 of [**DH2**, p.329]. Indeed, if $w \in \overline{U_c} = \overline{D(R)}$ and $R > \max\{r+\delta, 8\}$, then $R^2/4 > 2R > r+\delta+R$ and thus the equation $f_c(z) = w$ has two solutions in $D(R/2)$ by Rouché's theorem. By the maximum principle, $f_c : U_c' \to U_c$ is a proper branched covering of degree two. Since $|f_c(0)| \le r+\delta < R$, the critical point 0 is contained in U_c'. Thus the Riemann-Hurwitz formula implies that U_c' is a topological disk contained in $D(R/2) \Subset U_c$.

Next we construct Θ: Let $z = g_c(w)$ be the univalent branch of $f_c^{-1}(w) = \sqrt{w-(c+u)}$ defined on the disk centered at $w(0) := R$ with radius $R/2$ such that $g_c(R)$ is close to $R^{1/2}$. (Note that if $|z| = R > 8$ we have $|f_c(z)| \ge R^2 - (r+\delta) > R(R-1) > 3R/2$. Thus we have such a univalent branch.) We take the analytic continuation of the branch g_c along the curve $w(t) := Re^{it}$ $(0 \le t < 4\pi)$ that is univalent on each $D(w(t), R/2)$. Then

$$\log z = \log g_c(w) = \frac{1}{2}\log w + \frac{1}{2}\log\left(1 - \frac{c+u}{w}\right),$$

where $u = u(g_c(w), c)$ on $\overline{D(w(t), R/2)}$. Let

$$\Upsilon_c(w) := \frac{1}{2}\log\left(1 - \frac{c+u}{w}\right).$$

Then we have $|\Upsilon_c(w)| = O(R^{-1})$ (by taking an appropriate branch) and hence $\dfrac{d\Upsilon_c}{dw}(w(t)) = O(R^{-2})$ by the Schwarz lemma. For $t \in [0, 2\pi)$, set

$$v_c(t) := \Upsilon_c(w(2t))$$

such that $t \mapsto w(t) = \exp(\log R + it)$ and $t \mapsto \exp((\log R)/2 + it + v_c(t))$ parametrize the boundaries of U_c and U_c'. To give a homeomorphism between the closed annuli $\overline{A(R^{1/2}, R)}$ and $\overline{A_c} := \overline{U_c} \setminus U_c'$, we take their logarithms: Set $\ell := (\log R)/2$, and consider the rectangle $E := \{s+it \mid \ell \le s \le 2\ell, 0 \le t \le 2\pi\}$. Fix a smooth decreasing function $\eta_0 : [0,1] \to [0,1]$ such that: $\eta_0(0) = 1$; $\eta_0(1) = 0$; and the j-th derivative of η_0 tends to 0 as $x \to +0$ and as $x \to 1-0$ for any j. Set

$$\eta(s) := \eta_0(s/\ell - 1), \quad s \in [\ell, 2\ell].$$

Then we have $\dfrac{d\eta}{ds}(s) = O(\ell^{-1})$, $v_c(t) = O(R^{-1})$, and $\dfrac{dv_c}{dt}(t) = O(R^{-1})$.

Now consider the smooth map $\theta_c : E \to \mathbb{C}$ defined by

$$\theta_c(s+it) := s + it + \eta(s)v_c(t).$$

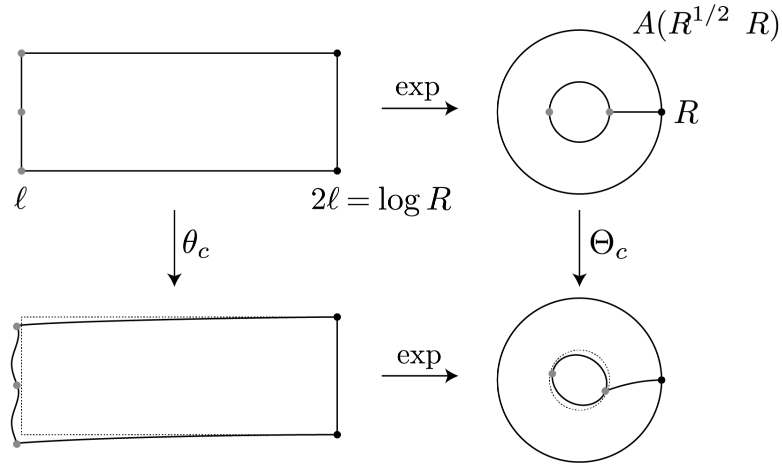

FIGURE 13. Construction of the tubing Θ for $\boldsymbol{f}$.

The map θ_c is injective for sufficiently large R since

$$\begin{aligned} &|\theta_c(s+it) - \theta_c(s'+it')| \\ \geq &|(s-s') + i(t-t')| - |\eta(s)(v_c(t) - v_c(t'))| - |(\eta(s) - \eta(s'))v_c(t')| \\ \geq &|(s-s') + i(t-t')| - O(R^{-1})|t-t'| - O(\ell^{-1}R^{-1})|s-s'|. \end{aligned} \tag{7.1}$$

The Beltrami coefficient of θ_c is given by

$$\mu_{\theta_c} = \frac{(d\eta/ds)\, v_c + \eta\, (dv_c/dt)i}{2 + (d\eta/ds)\, v_c - \eta\, (dv_c/dt)i} = O(R^{-1}).$$

Hence θ_c is an orientation preserving diffeomorphism onto its image for sufficiently large R, and its maximal dilatation is bounded by $1 + O(R^{-1})$. By observing θ_c through the exponential function, we obtain a smooth $(1 + O(R^{-1}))$-quasiconformal homeomorphism $\Theta_c : \overline{A(R^{1/2}, R)} \to \overline{A_c}$ that fixes the outer boundary and satisfies $\Theta_c(z^2) = f_c(\Theta_c(z))$ on the inner boundary. Holomorphic dependence of $c \mapsto \Theta_c(z)$ for each fixed $z \in \overline{A(R^{1/2}, R)}$ is obvious by the construction of θ_c.

Finally we construct the straightening map $h_c : U_c \to \mathbb{C}$ of $f_c : U_c' \to U_c = D(R)$. Let us extend f_c to a smooth quasiregular map $F_c : \mathbb{C} \to \mathbb{C}$ by setting

$$F_c(z) := \begin{cases} f_c(z) & \text{if } z \in U_c', \\ \{\Theta_c^{-1}(z)\}^2 & \text{if } z \in U_c \smallsetminus U_c', \text{ and} \\ z^2 & \text{if } z \in \mathbb{C} \smallsetminus U_c. \end{cases}$$

We define an F_c-invariant Beltrami coefficient μ_c (i.e., $F_c^*\mu_c = \mu_c$) by

$$\mu_c(z) := \begin{cases} 0 & \text{if } z \in K(f_c) \text{ or } z \in \mathbb{C} \smallsetminus U_c, \\ \dfrac{(F_c)_{\overline{z}}(z)}{(F_c)_z(z)} & \text{if } z \in U_c \smallsetminus U_c', \text{ and} \\ (f_c^n)^*\mu_c(z) & \text{if } f_c^n(z) \in U_c \smallsetminus U_c' \text{ for some } n > 0, \end{cases}$$

where

$$(f_c^n)^*\mu_c(z) = \mu_c(f_c^n(z)) \frac{\overline{(f_c^n)'(z)}}{(f_c^n)'(z)}.$$

Then μ_c is supported on $\overline{D(R)}$ and it satisfies $\|\mu_c\|_\infty = O(R^{-1})$. By [**IT**, Theorem 4.24] we have a unique $(1+O(R^{-1}))$-quasiconformal map $h_c : \mathbb{C} \to \mathbb{C}$ that satisfies the Beltrami equation $(h_c)_{\overline{z}} = \mu_c \cdot (h_c)_z$ a.e., $h_c(0) = 0$, and $(h_c)_z - 1 \in L^p(\mathbb{C})$ for some $p > 2$. (The relation between R and p will be more specified in the next lemma.) The condition $(h_c)_z - 1 \in L^p(\mathbb{C})$ implies $w = h_c(z) = z + b_c + O(1/z)$ as $z \to \infty$ for some constant $b_c \in \mathbb{C}$.

Since μ_c is F_c-invariant, the map $P(w) := h_c \circ F_c \circ h_c^{-1}(w)$ is a holomorphic map of degree 2 with a critical point at $h_c(0) = 0$ and a superattracting fixed point at $h_c(\infty) = \infty$. Hence $P(w)$ is a quadratic polynomial. The expansion of the form $w = h_c(z) = z + b_c + O(1/z)$ implies that we actually have $b_c = 0$ and $P(w)$ is of the form $P(w) = w^2 + \chi(c) = P_{\chi(c)}(w)$. Hence the restriction $h_c|_{U_c}$ is our desired straightening map. ■

The next lemma shows that the quasiconformal map $h_c : \mathbb{C} \to \mathbb{C}$ constructed above is uniformly close to the identity on compact sets for sufficiently large R:

Lemma 7.2. *Fix any $p > 2$ and any compact set $E \subset \mathbb{C}$. If R is sufficiently large, then the quasiconformal map h_c in Lemma 7.1 satisfies*

$$|h_c(z) - z| = O(R^{-1+2/p})$$

uniformly for each $c \in D(r)$ and $z \in E$.

Indeed, the estimate is valid for any $R \geq C_0 p^2$, where C_0 is a constant independent of p.

Proof. We have $\|\mu_c\|_\infty \leq C/R =: k$ for some constant C independent of $c \in D(r)$ by the construction of h_c. By [**IT**, Theorem 4.24], we have $(h_c)_z - 1 \in L^p(\mathbb{C})$ for any $p > 2$ satisfying $kC_p < 1$, where C_p is the constant that appears in the Calderon-Zygmund inequality [**IT**, Proposition 4.22]. Gaidashev showed in [**G**, Lemma 6] that $C_p \leq \cot^2(\pi/2p)$. Since $\cot^2(\pi/2p) = (2p/\pi)^2(1 + o(1))$ as $p \to \infty$, the inequality $kC_p \leq (C/R)\cot^2(\pi/2p) \leq 1/2$ is established if we take $R \geq C_0 p^2$ for some constant C_0 independent of $p > 2$. By following the proof of [**IT**, §4, Corollary 2], we have

$$|h_c(z) - z| \leq K_p \cdot \frac{1}{1 - kC_p} \|\mu_c\|_p |z|^{1-2/p}$$

for any $z \in \mathbb{C}$, where $K_p > 0$ is a constant depending only on p and $\|\mu_c\|_p$ is the L^p-norm of μ_c. Since $|\mu_c| \leq C/R$ and μ_c is supported on $\overline{D(R)}$, we have $\|\mu_c\|_p \leq (C/R)(\pi R^2)^{1/p} = C\pi^{1/p}R^{-1+2/p}$. Hence if we take $R \geq C_0 p^2$ such that $kC_p \leq 1/2$, we have

$$|h_c(z) - z| = 2K_p C \pi^{1/p} R^{-1+2/p} |z|^{1-2/p}.$$

This implies that $|h_c(z) - z| = O(R^{-1+2/p})$ on each compact subset E of $\mathbb{C}$. ■

Corollary 7.3. *Fix any $p > 2$. If R is sufficiently large, then for each $c \in D(r)$, f_c is hybrid equivalent to a quadratic polynomial $P_{\chi(c)}(w) = w^2 + \chi(c)$ with*

$$|\chi(c) - c| \leq \delta + O(R^{-1+2/p}).$$

Proof. We have $\chi(c) = h_c(f_c(0))$ since h_c maps the critical value of f_c to that of $P_{\chi(c)}$, For each $c \in D(r)$, $f_c(0) = c + u_c(0)$ is contained in a compact set $\overline{D(r + \delta)}$. By Lemma 7.2, we obtain

$$\chi(c) = h_c(f_c(0)) = c + u_c(0) + O(R^{-1+2/p})$$

for sufficiently large R and this implies the desired estimate. ■

Coordinate changes. Under the same assumption as in Lemma 7.1, we assume in addition that

(i) $\delta < 1$ and $r > 4$; and
(ii) R is large enough such that $\boldsymbol{f} = \{f_c : U_c' \to U_c\}_{c \in D(r)}$ is an analytic family of quadratic-like maps (that may not necessarily be Mandelbrot-like), and that (1) and (2) of Lemma 7.1 hold.

By (2) of Lemma 7.1, each $f_c \in \boldsymbol{f}$ is hybrid equivalent to some quadratic map $P_{\chi(c)}$ by the $(1+O(R^{-1}))$-quasiconformal straightening $h_c : U_c \to h_c(U_c)$. We say the map

$$\chi = \chi_{\boldsymbol{f}} : D(r) \to \mathbb{C}, \quad \chi(c) = h_c(f_c(0))$$

is the *straightening map* of the family $\boldsymbol{f}$ associated with the family $\{h_c\}_{c \in D(r)}$. We say the map $(z, c) \mapsto (h_c(z), \chi_{\boldsymbol{f}}(c))$ defined on $D(R) \times D(r)$ is a *(straightening) coordinate change.*

Now we show that the straightening map is quasiconformal with dilatation arbitrarily close to 1 if we take sufficiently large R and small δ:

Lemma 7.4 (Almost conformal straightening of $\boldsymbol{f}$). *If r and R are sufficiently large and $\delta > 0$ is sufficiently small, then the family $\boldsymbol{f}$ is associated with a $(1+O(\delta)+O(R^{-1}))$-quasiconformal straightening map*

$$\chi = \chi_{\boldsymbol{f}} : D(r) \to \chi(D(r)) \subset \mathbb{C}$$

such that

(1) *$\chi(M_{\boldsymbol{f}}) = M$, where $M_{\boldsymbol{f}}$ is the connectedness locus of $\boldsymbol{f}$;*
(2) *$\chi|_{D(r) \smallsetminus M_{\boldsymbol{f}}}$ is $(1 + O(R^{-1}))$-quasiconformal; and*
(3) *$\chi|_{M_{\boldsymbol{f}}}$ extends to a $(1 + O(\delta))$-quasiconformal map on the plane.*

Proof. By slightly shrinking $r > 4$ if necessary, we may assume that f_c is defined for $c \in \partial D(r)$. When $c = re^{it}$ $(0 \le t \le 2\pi)$, we have $|f_c(0)-0| = |c+u(0, c)| \ge r-\delta > 3$ (since $\delta < 1$). Hence as c makes one turn around the origin so does $f_c(0)$. By [**DH2**, p.328], χ gives a homeomorphism between $M_{\boldsymbol{f}}$ and M. Moreover, $\chi|_{D(r) \smallsetminus M_{\boldsymbol{f}}}$ is $(1 + O(R^{-1}))$-quasiconformal by [**DH2**, Proposition 20, Lemma in p.327], since each Θ_c is $(1+O(R^{-1}))$-quasiconformal.

For the dilatation of $\chi|_{M_{\boldsymbol{f}}}$, we follow the argument of [**Mc2**, Lemma 4.2]: Consider the families $\boldsymbol{f}_t := \{f_{c,t}\}_{c \in D(r)}$ defined for each $t \in \mathbb{D}$, where

$$f_{c,t}(z) := z^2 + c + \frac{t}{\delta}\, u(z, c).$$

By the same argument as above, the connectedness locus $M_{\boldsymbol{f}_t}$ of $\boldsymbol{f}_t$ is homeomorphic to M by the straightening map

$$\chi_t = \chi_{\boldsymbol{f}_t} : M_{\boldsymbol{f}_t} \to M.$$

Then the inverse $\phi_t := \chi_t^{-1} : M \to M_{\boldsymbol{f}_t}$ gives a holomorphic family of injections over $\mathbb{D}$. By Bers and Royden's theorem [**BR**, Theorem 1], each of them extends to a $(1+|t|)/(1-|t|)$-quasiconformal map $\widetilde{\chi}_t$ on $\mathbb{C}$. In particular, $\chi|_{M_{\boldsymbol{f}}} = \chi_\delta$ extends to a $(1 + O(\delta))$-quasiconformal map on $\mathbb{C}$. Now we apply Bers' lemma (Lemma 6.3) to $H_1 := \chi|_{D(r)-M_{\boldsymbol{f}}}$

and $H_2 = \widetilde{\chi}_\delta$. Then H_1 and H_2 are glued along ∂M and the glued map $H : D(r) \to \mathbb{C}$, which coincides with χ, is $(1 + O(\delta) + O(R^{-1}))$-quasiconformal. ∎

Idea of the Proof of Theorem C. The rest of this section is devoted to the proof of Theorem C, which follows the argument of Theorem A. Recall that in the proof of Theorem A, we construct two families of quadratic-like maps $\boldsymbol{f} = \{f_c : U'_c \to U_c\}_{c\in\Lambda}$ ("the first renormalization") and $\boldsymbol{G} = \{G_c : V'_c \to U_c\}_{c\in W}$ ("the second renormalization"), and we conclude that the small Mandelbrot set corresponding to the family $\boldsymbol{G}$ has a desired decoration.

In the following proof of Theorem C, we first take a "thickened" family $\widehat{\boldsymbol{f}} = \{f_c : \widehat{U}'_c \to \widehat{U}_c\}_{c\in\widehat{\Lambda}}$ that contains $\boldsymbol{f} = \{f_c : U'_c \to U_c\}_{c\in\Lambda}$ as a restriction (in both dynamical and parameter spaces), such that $U_c \Subset \widehat{U}_c$ and the modulus of $\widehat{U}_c \smallsetminus \overline{U_c}$ is sufficiently large. Next we construct another "thickened" family $\widehat{\boldsymbol{G}} = \{G_c : \widehat{V}'_c \to \widehat{U}_c\}_{c\in\widehat{W}}$ that contains $\boldsymbol{G} = \{G_c : V'_c \to U_c\}_{c\in W}$ with $V'_c \Subset \widehat{V}'_c$. Then we can apply a slightly modified versions of the lemmas above to the family $\boldsymbol{G}$. Finally we conclude that the small Mandelbrot set corresponding to the family $\boldsymbol{G}$ has a very fine decoration.

Notation. We will use a conventional notation: For complex variables α and β, by $\alpha \asymp \beta$ we mean $C^{-1}|\alpha| \le |\beta| \le C|\alpha|$ for an implicit constant $C > 1$.

Proof of Theorem C: First renormalization. We start with a result by McMullen [**Mc2**, Theorem 3.1] applied to (and modified for) the quadratic family:

Lemma 7.5 (**Misiurewicz cascades).** *For any Misiurewicz parameter m_0 and any arbitrarily large r and R, there exist sequences $\{s_n\}_{n\ge1}$, $\{p_n\}_{n\ge1}$, $\{t_n\}_{n\ge1}$, and $\{\delta_n\}_{n\ge1}$ that satisfy the following conditions for each sufficiently large n:*

(a) *s_n is a superattracting parameter of period p_n with $|s_n - m_0| \asymp \mu_0^{-n}$, where μ_0 is the multiplier of the repelling cycle of P_{m_0} on which the critical orbit lands.*
(b) *$t_n \in \mathbb{C}^*$ and $t_n \asymp \mu_0^{-2n}$.*
(c) *$\delta_n > 0$ and $\delta_n \asymp n\mu_0^{-n}$.*
(d) *Let $X_n : D_n := D(s_n, r|t_n|) \to D(r)$ be the affine map defined by*

$$C = X_n(c) := \frac{c - s_n}{t_n}.$$

Then there exists a non-zero holomorphic function $c \mapsto \alpha(c) = \alpha_c$ defined for $c \in D_n$ such that $\alpha_c \asymp \mu_0^{-n}$ and the map

$$Z = A_c(z) := \frac{z}{\alpha_c}$$

conjugates $P_c^{p_n}$ on $D(R\,|\alpha_c|)$ to the map $F_C := A_c \circ P_c^{p_n} \circ A_c^{-1}$ of the form

$$F_C(Z) = A_c \circ P_c^{p_n} \circ A_c^{-1}(Z) = Z^2 + C + u(Z, C), \tag{7.2}$$

where $u(Z, C) = u_C(Z)$ is holomorphic in both $Z \in D(R)$ and $C \in D(r)$, and satisfies $u'_C(0) = 0$ and $|u(Z, C)| \le \delta_n$.

We may regard the map $(z, c) \mapsto (Z, C) = (A_c(z), X_n(c))$ as an "affine" coordinate change (Figure 14).

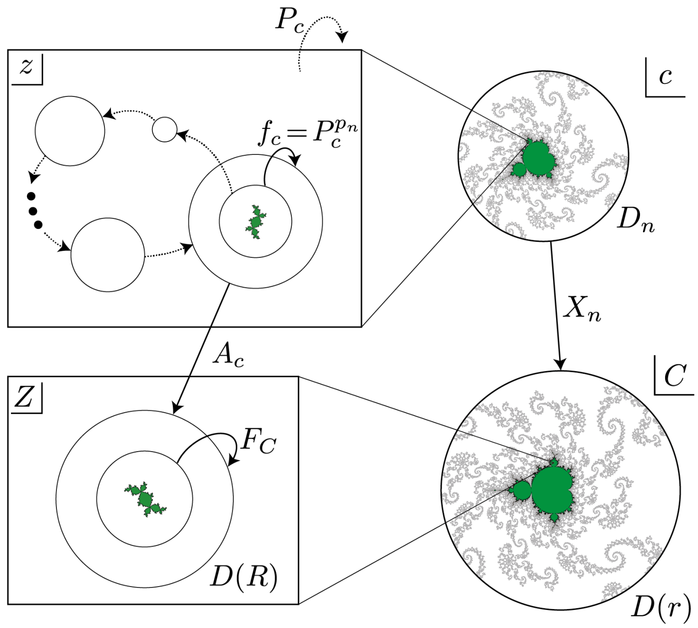

FIGURE 14. An affine coordinate change.

Construction of the family $\widehat{\boldsymbol{f}}$. Let us fix arbitrarily small $\varepsilon > 0$ and $\kappa > 0$ as in the statement. We choose any Misiurewicz parameter m_0 in $\mathrm{int}(B) \cap \partial M$, where B is the closed disk given in the statement. (The Misiurewicz parameters are dense in ∂M.)

For any r and R bigger than 4 (we will replace them with larger ones if necessary, but it will happen finitely many times in what follows), by taking a sufficiently large n in Lemma 7.5 [2], we have an analytic family $\{F_C : D(R) \to \mathbb{C}\}_{C \in D(r)}$ that satisfies the conditions for Lemma 7.1. Moreover, its restriction

$$\boldsymbol{F}_n := \left\{F_C : F_C^{-1}(D(R)) \to D(R)\right\}_{C \in D(r)}$$

is an analytic family of quadratic-like maps that satisfies the conditions for Lemma 7.4. Hence we have an associated straightening coordinate change of the form $(Z, C) \mapsto (H_C(Z), \chi_{\boldsymbol{F}_n}(C))$. More precisely, for each $C \in D(r)$, F_C is hybrid equivalent to $Z \mapsto Z^2 + \chi_{\boldsymbol{F}_n}(C)$ by a $(1+O(R^{-1}))$-quasiconformal straightening H_C by Lemma 7.1, and H_C satisfies the estimate of Lemma 7.2. By Lemma 7.4, the straightening $\chi_{\boldsymbol{F}_n} : D(r) \to \mathbb{C}$ is $(1+O(\delta_n)+O(R^{-1}))$-quasiconformal and satisfies the estimate of Corollary 7.3. Hence we may assume that R and n are large enough such that both $Z \mapsto H_C(Z)$ and $C \mapsto \chi_{\boldsymbol{F}_n}(C)$ are $(1+\kappa)^{1/2}$-quasiconformal for the arbitrarily small $\kappa > 0$ given in the statement.

Let $f_c := P_c^{p_n} : \widehat{U}_c' \to \widehat{U}_c$ be the pull-back of $F_C : F_C^{-1}(D(R)) \to D(R)$ by the "affine" coordinate change $(z, c) \mapsto (Z, C) = (A_c(z), X_n(c))$. (Note that $\widehat{U}_c = D(R\,|\alpha_c|)$ is a round disk.) Set

$$p := p_n, \quad s_0 := s_n, \quad \text{and} \quad \widehat{\Lambda} := D_n = D(s_n,\, r|t_n|).$$

The quadratic-like family

$$\widehat{\boldsymbol{f}} := \left\{f_c : \widehat{U}_c' \to \widehat{U}_c\right\}_{c \in \widehat{\Lambda}}$$

[2]More precisely, we fix r first, and then take a larger R (and an n) if necessary to apply those lemmas.

is our first family of renormalizations whose straightening coordinate change $(z, c) \mapsto (h_c(z), \chi(c))$ is given by

$$(h_c(z), \chi(c)) := (H_C \circ A_c(z), \chi_{\boldsymbol{F}_n} \circ X_n(c)).$$

Note that both $h_c : \widehat{U}_c \to \mathbb{C}$ and $\chi : \widehat{\Lambda} \to \mathbb{C}$ are $(1+\kappa)^{1/2}$-quasiconformal. By Lemma 7.2 and Corollary 7.3, if we fix any $p' > 2$ and any compact subset E of U_c, then for sufficiently large R we have

$$h_c(z) = A_c(z) + O(R^{-1+2/p'})$$

on E and

$$\chi(c) = X_n(c) + O(\delta_n) + O(R^{-1+2/p'})$$

on $\widehat{\Lambda}$. Hence the straightening coordinate change $(z, c) \mapsto (h_c(z), \chi(c))$ is very close to the affine coordinate change $(z, c) \mapsto (A_c(z), X_n(c)) = (z/\alpha_c, (c - s_n)/t_n)$ if we take sufficiently large R and n.

Construction of the family $\boldsymbol{f}$. Let $\rho > 4$ be an arbitrarily large number. By taking sufficiently large r, R and n if necessary, we may assume the following:

- $F_C : F_C^{-1}(D(\rho)) \to D(\rho)$ is quadratic-like for each $C \in D(r)$.
- The set $\Omega(\rho) := \{C \in D(r) \mid F_C(0) \in D(\rho)\}$ gives a Mandelbrot-like family

$$\boldsymbol{F}_n(\rho) := \{F_C : F_C^{-1}(D(\rho)) \to D(\rho)\}_{C \in \Omega(\rho)}.$$

Now we define the Mandelbrot-like family

$$\boldsymbol{f} = \boldsymbol{f}(\rho) = \{f_c : U_c' \to U_c\}_{c \in \Lambda}$$

as the pull-back of $\boldsymbol{F}_n(\rho)$ by the "affine" coordinate change $(z, c) \mapsto (A_c(z), X_n(c))$ above. More precisely, we let $U_c := A_c^{-1}(D(\rho)) = D(\rho|\alpha_c|)$ for each $c \in \widehat{\Lambda}$, and consider the restriction $f_c : U_c' \to U_c$ of $f_c : \widehat{U}_c' \to \widehat{U}_c$. Then we define the subset Λ of $\widehat{\Lambda}$ by $\Lambda := X_n^{-1}(\Omega(\rho))$ such that the family $\boldsymbol{f}$ above becomes a Mandelbrot-like family.

Let $M_{\boldsymbol{f}} = s_0 \perp M$ be the connectedness locus of the family $\boldsymbol{f}$, which coincides with that of $\widehat{\boldsymbol{f}}$. Note that $\boldsymbol{f}$ has the same straightening coordinate change $(z, c) \mapsto (h_c(z), \chi(c))$ as $\widehat{\boldsymbol{f}}$ such that $\chi(M_{\boldsymbol{f}}) = M$.

Second renormalization. For a given Misiurewicz or parabolic parameter c_0 in the statement of Theorem C, we define a Misiurewicz or parabolic parameter $c_1 \in M_{\boldsymbol{f}}$ by

$$c_1 := \chi^{-1}(c_0) = s_0 \perp c_0.$$

Let q_{c_1} be a repelling or parabolic periodic point of P_{c_1} that belongs to the postcritical set. More precisely, when c_1 is Misiurewicz, the orbit of 0 eventually lands on q_{c_1} that is a repelling periodic point. When c_1 is parabolic, the orbit of 0 accumulates on q_{c_1} that is a parabolic periodic point.

The rest of the proof of Theorem C is divided into Claims 1 to 7 below and their proofs. For the first two claims, we may simply apply the argument of Steps (M1)–(M2) or Steps (P1)–(P2):

Claim 1. *There exists a Jordan domain* $\widehat{V}_{c_1}$ *with* C^1 *boundary such that*

(1) $\widehat{V}_{c_1}$ *is a connected component of* $P_{c_1}^{-N}(\widehat{U}_{c_1})$ *for some* N *such that* $P_{c_1}^N : \widehat{V}_{c_1} \to \widehat{U}_{c_1}$ *is an isomorphism.*

(2) *There exists a* $j \in \mathbb{N}$ *such that* $f_{c_1}^j(\widehat{V}_{c_1}) = P_{c_1}^{pj}(\widehat{V}_{c_1}) \Subset \widehat{U}_{c_1} \setminus \overline{\widehat{U}'_{c_1}}$.

In particular, $\widehat{V}_{c_1}$ *can be arbitrarily close to* q_{c_1} *by taking sufficiently large* N *and* j.

Claim 2. *There exists a Jordan domain* $\widehat{W} \Subset \Lambda \setminus M_{\boldsymbol{f}}$ $(\subset \widehat{\Lambda} \setminus M_{\boldsymbol{f}})$ *arbitrarily close to* c_1 *that satisfies the following:*

(1) *There is a holomorphic motion of* $\widehat{V}_{c_1}$ *over* $\widehat{W}$ *that generates a family of Jordan domains* $\{\widehat{V}_c\}_{c \in \widehat{W}}$ *with* C^1 *boundaries such that for each* $c \in \widehat{W}$, $f_c^j(\widehat{V}_c) \Subset \widehat{U}_c \setminus \overline{\widehat{U}'_c}$ *and* $P_c^N : \widehat{V}_c \to \widehat{U}_c$ *is an isomorphism.*

(2) *There exists an* L *such that* $f_c^L(0) = P_c^{pL}(0) \in \widehat{V}_c$ *for any* $c \in \widehat{W}$.

(3) *For* $c \in \partial\widehat{W}$, *we heve* $P_c^{pL+N}(0) \in \partial\widehat{U}_c$. *Moreover, when* c *makes one turn along* $\partial\widehat{W}$, *then* $P_c^{pL+N}(0)$ *makes one turn around the origin.*

Indeed, such a domain $\widehat{W}$ is given by

$$\widehat{W} = \left\{ c \in \Lambda \setminus M \mid f_c^L(0) \in \widehat{V}_c \right\}.$$

See Figure 15. Note that if we take sufficiently large N and j, we may always assume that $\widehat{V}_c \Subset U'_c$ as depicted in Figure 15. Moreover, the proof of Lemma 4.1 indicates that we can choose $\widehat{V}_c$ with arbitrarily small diameter.

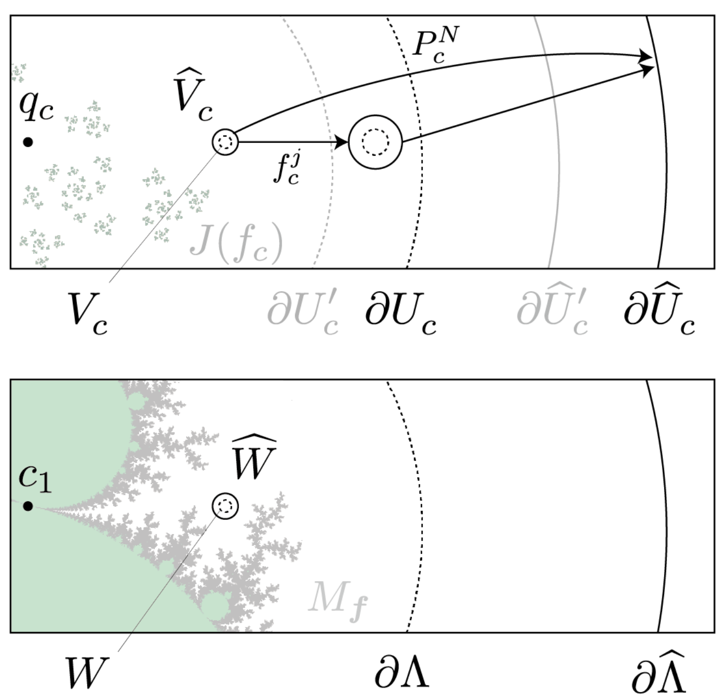

FIGURE 15. For any $c \in \widehat{W}$, the Julia set $J(f_c)$ moves only a little from $J(f_s)$ of the center $s \in W \Subset \widehat{W}$.

Definition of the center of $\widehat{W}$ and W. By the same argument as in Step (M2) and Step (P2), there exists a unique superattracting parameter $s \in \widehat{W}$ such that $P_s^{pL+N}(0) = 0$. We call s the *center* of $\widehat{W}$.

By Claim 2, we can find a family of Jordan domains $\{V_c\}_{c \in \widehat{W}}$ with C^1 boundaries such that $V_c \Subset \widehat{V}_c$ and that $P_c^N : V_c \to U_c$ is an isomorphism for each $c \in \widehat{W}$. Let $W \Subset \widehat{W}$ be the set of c such that $f_c^L(0) \in V_c$. By the same argument as in Step (M3) and Step (P3), one can check that both $\widehat{W}$ and W are Jordan domains with C^1 boundaries. Note that s is the center of W as well. (See Figure 15 again.)

Moreover, we have:

Claim 3 (Straightening the center). *By choosing $\widehat{V}_c$ in Claim 1 close enough to q_{c_1}, we can find an $\eta \in D(\varepsilon)$ with $c_0 + \eta \notin M$ such that $f_s : \widehat{U}'_s \to \widehat{U}_s$ is hybrid equivalent to a quadratic-like restriction of $P_{c_0+\eta}$ with $(1+\kappa)^{1/2}$-quasiconformal straightening map. In particular, $J(f_s)$ is a $(1+\kappa)^{1/2}$-quasiconformal image of $J_{c_0+\eta}$.*

Proof. By the construction of $\widehat{W}$ in Claim 2 (following Step (M2) or Step (P2)), we can take $\widehat{W}$ arbitrarily close to c_1. Since the straightening map $\chi = \chi_{\boldsymbol{f}} : \widehat{\Lambda} \to \mathbb{C}$ is continuous, we have $|\chi(s) - \chi(c_1)| = |\chi(s) - c_0| < \varepsilon$ by taking $s \in \widehat{W}$ close enough to c_1. Set $\eta := \chi(s) - c_0$. Then we have $\chi(s) = c_0 + \eta \in \mathbb{C} \setminus M$ since $\widehat{W} \subset \widehat{\Lambda} \setminus M_{\boldsymbol{f}}$. By the construction of the first renormalization, f_s is conjugate to $P_{\chi(s)}$ by the $(1+\kappa)^{1/2}$-quasiconformal straightening map h_s such that $h_s(J(f_s)) = J_{c_0+\eta}$. ■

Remark. Since $h_s(z) = H_{X_n(s)} \circ A_s(z) = z/\alpha_s + o(1)$, $J(f_s)$ is actually an "almost affine" (even better than "almost conformal"!) copy of $J_{c_0+\eta}$.

Holomorphic motion of the Cantor Julia sets. Since $\widehat{W} \subset \widehat{\Lambda} \setminus M_{\boldsymbol{f}}$, the Julia set $J(f_c)$ for each $c \in \widehat{W}$ is a Cantor set that is a $(1+\kappa)^{1/2}$-quasiconformal image of $J_{\chi(c)}$. Moreover, the Julia set $J(f_c)$ moves holomorphically for $c \in \widehat{W}$:

Claim 4 (Cantor Julia moves a little). *There exists a holomorphic motion $\iota : J(f_s) \times \widehat{W} \to \mathbb{C}$ such that $\iota_c(z) := \iota(z, c)$ maps $J(f_s)$ bijectively to $J(f_c)$ for each $c \in \widehat{W}$. Moreover, if R is sufficiently large, then ι_c extends to a $(1+\kappa)^{1/2}$-quasiconformal homeomorphism on the plane for each $c \in W \Subset \widehat{W}$.*

A direct corollary of Claims 3 and 4 is:

Corollary 7.6 (Julia appears in Julia). *The Julia set J_c contains a $(1+\kappa)$-quasiconformal copy of $J_{c_0+\eta}$ for any $c \in W \Subset \widehat{W}$.*

Proof of Claim 4. Since $J(f_c)$ is a hyperbolic set for each $c \in \widehat{W}$, it has a local holomorphic motion near c. (See [**S**, p.229].) The holomorphic motion extends to that of $J(f_s)$ over $\widehat{W}$ as in the statement since $\widehat{W}$ is simply connected (and isomorphic to $\mathbb{D}$).

By [**DH2**, Proposition 20, Lemma in p.327], the annulus $\widehat{W} \setminus \overline{W}$ is a $(1 + O(R^{-1}))$-quasiconformal image of $\widehat{U}_s \setminus \overline{U_s}$. Hence we have

$$\begin{aligned} \operatorname{mod}(\widehat{W} \setminus \overline{W}) &= (1 + O(R^{-1})) \operatorname{mod}(\widehat{U}_s \setminus \overline{U_s}) \\ &= (1 + O(R^{-1})) \operatorname{mod}(A(\rho, R)). \end{aligned}$$

By taking R relatively larger than ρ, this modulus is arbitrarily large. Now let us choose a uniformization $\psi : \widehat{W} \to \mathbb{D}$ such that $\psi(s) = 0$. For an arbitrarily small $\nu > 0$, we may assume that $\psi(W) \subset D(\nu)$ when $\mathrm{mod}(\widehat{W} \smallsetminus \overline{W})$ is sufficiently large. (See [**Mc1**, Theorems 2.1 and 2.4]. Indeed, it is enough to take R such that $\nu \asymp \rho/R$.) By the Bers-Royden theorem ([**BR**, Theorem 1]), each $\iota_c : J(f_s) \to J(f_c)$ extends to a $(1+\nu)/(1-\nu)$-quasiconformal map on $\mathbb{C}$. Thus the dilatation is uniformly smaller than $(1+\kappa)^{1/2}$ if we choose a sufficiently large R. ∎

Definition of the families $\widehat{G}$ and G. For each $c \in \widehat{W}$, let $\widehat{V}_c'$ be the connected component of $f_c^{-L}(\widehat{V}_c)$ (or, that of $P_c^{-pL-N}(\widehat{U}_c)$) containing the critical point 0. We define $G_c : \widehat{V}_c' \to \widehat{U}_c$ by the restriction of $P_c^N \circ f_c^L = P_c^{pL+N}$ on $\widehat{V}_c'$. Then we have a family of quadratic-like maps

$$\widehat{\boldsymbol{G}} := \{G_c : \widehat{V}_c' \to \widehat{U}_c\}_{c \in \widehat{W}}.$$

Similarly, for each $c \in W$, let V_c' be the connected component of $f_c^{-L}(V_c)$ (or, that of $P_c^{-pL-N}(U_c)$) containing 0. Then we have a quadratic-like family

$$\boldsymbol{G} = \boldsymbol{G}(\rho) := \{G_c : V_c' \to U_c\}_{c \in W},$$

where ρ comes from the definition of $U_c = A_c^{-1}(D(\rho))$.

Note that both the annuli $\widehat{U}_c \smallsetminus \overline{\widehat{V}_c'}$ and $U_c \smallsetminus \overline{V_c'}$ compactly contain the Cantor Julia set $J(f_c)$ for each $c \in \widehat{W}$.

Claim 5 (Extending the holomorphic motion). *If R is sufficiently large and relatively larger than ρ, we have the following extensions of the holomorphic motion ι of the Julia set $J(f_s)$ given in Claim 4:*

(1) *An extension to the holomorphic motion of $J(f_s) \cup \partial \widehat{V}_s' \cup \partial U_s \cup \partial \widehat{U}_s$ over $\widehat{W}$ that is equivariant to the action of $G_c : \partial \widehat{V}_c' \to \partial \widehat{U}_c$.*
(2) *A further extension of (1) to the motion of the closed annulus $\overline{\widehat{U}_s} \smallsetminus \widehat{V}_s'$ over $\widehat{W}$.*
(3) *An extension to the holomorphic motion of $J(f_s) \cup \partial V_s' \cup \partial U_s$ over W that is equivariant to the action of $G_c : \partial V_c' \to \partial U_c$.*
(4) *A further extension of (3) to the motion of the closed annulus $\overline{U_s} \smallsetminus V_s'$ over W.*

In particular, the quasiconformal map $\iota_c : \overline{U_s} \smallsetminus V_s' \to \overline{U_c} \smallsetminus V_c'$ induced by (4) extends to a $(1+\kappa)^{1/2}$-quasiconformal map on the plane for each $c \in W$.

See Figure 16.

Proof. (1) The sets ∂U_c, $\partial \widehat{V}_c'$, and $\partial \widehat{U}_c$ are all images of round circles by analytic families of locally conformal injections over $\widehat{W}$. In particular, they never intersect with the Julia set $J(f_c)$ for each $c \in \widehat{W}$. Hence the extension of ι to $J(f_s) \cup \partial \widehat{V}_s' \cup \partial U_s \cup \partial \widehat{U}_s$ over $\widehat{W}$ is straightforward.
(2) By Słodkowski's theorem, it extends to the motion of $\mathbb{C}$, and its restriction to the closed annulus $\overline{\widehat{U}_s} \smallsetminus \widehat{V}_s'$ is our desired motion. Note that $\iota_c : \overline{\widehat{U}_s} \smallsetminus \widehat{V}_s' \to \overline{\widehat{U}_c} \smallsetminus \widehat{V}_c'$ is uniformly $(1+\kappa)^{1/2}$-quasiconformal for $c \in W \Subset \widehat{W}$ by taking a sufficiently large R that is relatively larger than ρ. (See the proof of Claim 4.)
(3) Similarly, $\partial V_c'$ is an image of a round circle ∂U_c by an analytic family of injections (that is locally a univalent branch of G_c^{-1}) with $c \in W$. Since $V_c' \Subset \widehat{V}_c'$, V_c' never intersects with $J(f_c)$ for $c \in W$ and we obtain an extension of the motion of $J(f_s)$ to that of

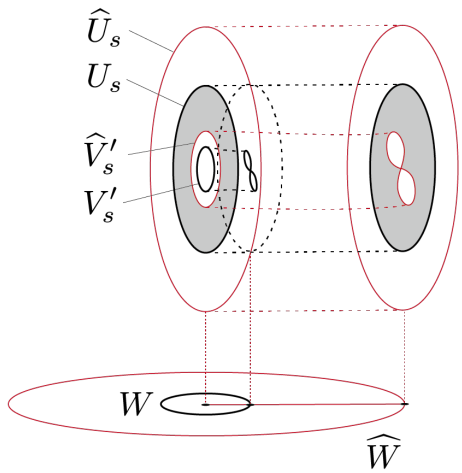

FIGURE 16. Extending the holomorphic motion to the closed annuli. The motion of $J(f_s)$ is contained in the motion of shadowed annuli. (Note that each nested annuli are extremely thick in our setting.)

$J(f_s) \cup \partial V'_s \cup \partial U_s$ over W which satisfies $G_c \circ \iota_c = \iota_c \circ G_s$ on $\partial V'_s$.
(4) To extend (3) to the closed annulus $\overline{U_s} \smallsetminus V'_s$, we divide the annulus into two annuli $\overline{U_s} \smallsetminus \widehat{V}'_s$ and $\overline{\widehat{V}'_s} \smallsetminus V'_s$. The desired motion of $\overline{U_s} \smallsetminus \widehat{V}'_s$ over W is contained in the motion given in (2). For the annulus $\overline{\widehat{V}'_s} \smallsetminus V'_s$, we note that the map $G_c : \overline{\widehat{V}'_c} \smallsetminus V'_c \to \overline{\widehat{U}_c} \smallsetminus U_c$ is a holomorphic covering of degree two. Hence we can pull-back the motion of $\overline{\widehat{U}_s} \smallsetminus U_s$ over W that is contained in the motion given in (2) by these covering maps. More precisely, we can construct an analytic family $\iota_c : \overline{\widehat{V}'_s} \smallsetminus \widehat{V}'_s \to \overline{\widehat{V}'_c} \smallsetminus \widehat{V}'_c$ of $(1+\kappa)^{1/2}$-quasiconformal maps that agrees with the motion of $J(f_s) \cup \partial V'_s \cup \partial U_s$, by taking a branch of $G_c^{-1} \circ \iota_c \circ G_s$ with ι_c given in (2). ■

Claim 6 (Decorated tubing). *By taking larger R, r, and ρ if necessary, there exist a $\rho' > 0$ and a tubing*

$$\check{\Theta} := \left\{ \check{\Theta}_c : \overline{A(\check{R}, \check{R}^2)} \to \overline{U_c} \smallsetminus V'_c \right\}_{c \in W}$$

of the family $\boldsymbol{G}$ with the following properties:

(1) *$\check{R} = \rho/\rho'$ and $\Gamma_0(c_0 + \eta) = \Gamma_0(c_0 + \eta)_{\rho',\rho}$ is contained in $A(\check{R}, \check{R}^2)$.*
(2) *$\check{\Theta}_s(\check{z}) = h_s^{-1}(((\rho')^2/\rho) \cdot \check{z})$ for $\check{z} \in \Gamma_0(c_0+\eta)$ such that $\check{\Theta}_s$ maps $\Gamma_0(c_0+\eta)$ onto $J(f_s)$.*
(3) *Each $\check{\Theta}_c : \overline{A(\check{R}, \check{R}^2)} \to \overline{U_c} \smallsetminus V'_c$ is a $(1+\kappa)$-quasiconformal embedding that is compatible with the holomorphic motion of $\overline{U_s} \smallsetminus V'_s$ over W given in (2) of Claim 5. More precisely, we have $\check{\Theta}_c = \iota_c \circ \check{\Theta}_s$ for each $c \in W$, where $\iota_c : \overline{U_s} \smallsetminus V'_s \to \overline{U_c} \smallsetminus V'_c$ is the quasiconformal map induced by the motion.*

We call this tubing $\check{\Theta}$ a *decorated tubing* of $\boldsymbol{G}$.

Proof. For each $c \in \widehat{W}$, the map $G_c = P_c^{pL+N}|_{\widehat{V}'_c}$ can be decomposed as $G_c = Q_c \circ P_0$, where $P_0(z) = z^2$ and $Q_c : P_0(\widehat{V}'_c) \to \widehat{U}_c$ is an isomorphism. Let

$$\beta_c := Q'_c(0) \quad \text{and} \quad \gamma_c := Q_c(0).$$

Note that $\gamma_c = G_c(0) \in U_c$ if $c \in W$.

Since $\widehat{U}_c$ and U_c are round disks of radii $R|\alpha_c|$ and $\rho|\alpha_c|$ respectively, we apply the Koebe distortion theorem to $Q_c^{-1}:\widehat{U}_c\to P_0(\widehat{V}_c')$ and obtain

$$z=Q_c^{-1}(w)=\beta_c^{-1}(w-\gamma_c)(1+O(\rho/R)) \tag{7.3}$$

for $w\in U_c$. Indeed, by the Koebe distortion theorem ([**D**, §2.3]), we have

$$\left|\frac{(Q_c^{-1})'(w)}{(Q_c^{-1})'(\gamma_c)}\right|=1+O(\rho/R)\quad\text{and}\quad\arg\frac{(Q_c^{-1})'(w)}{(Q_c^{-1})'(\gamma_c)}=O(\rho/R)$$

for $w\in U_c$. Hence we have $(Q_c^{-1})'(w)=\beta_c^{-1}(1+O(\rho/R))$ on U_c. By integrating the function $(Q_c^{-1})'(w)-\beta_c^{-1}$ along the segment joining γ_c to w in U_c, we obtain

$$|Q_c^{-1}(w)-\beta_c^{-1}(w-\gamma_c)|=|w-\gamma_c||\beta_c|^{-1}O(\rho/R)$$

that is equivalent to (7.3).

This implies that

$$G_c(z)=Q_c(z^2)=\gamma_c+\beta_c z^2\,(1+O(\rho/R))$$

on V_c'. By an affine coordinate change

$$\check{z}=\check{A}_c(z):=\beta_c\,z,$$

we obtain a quadratic-like map $\check{G}_c:\check{V}_c'\to\check{U}_c$ of the form

$$\check{w}=\check{G}_c(\check{z}):=\check{A}_c\circ G_c\circ\check{A}_c^{-1}(\check{z})=\beta_c\gamma_c+\check{z}^2(1+O(\rho/R)), \tag{7.4}$$

where $\check{V}_c':=\check{A}_c(V_c')$ and $\check{U}_c:=\check{A}_c(U_c)=D(\rho|\alpha_c||\beta_c|)$.

Now suppose that $c=s$. Then the condition $G_s(0)=0$ implies $\gamma_s=0$. Hence we have

$$\check{G}_s(\check{z})=\check{z}^2(1+O(\rho/R))\quad\text{and}\quad\check{G}_s^{-1}(\check{w})=\sqrt{\check{w}(1+O(\rho/R))}. \tag{7.5}$$

Let $\check{R}:=(\rho|\alpha_s||\beta_s|)^{1/2}$. Then $\check{U}_s=D(\check{R}^2)$ and $\partial\check{V}_s'$ is C^1-close to a circle $\partial D(\check{R})$. In other words, the annulus $\mathcal{A}:=\check{U}_s\smallsetminus\overline{\check{V}_s'}$ is close to a round annulus $A(\check{R},\check{R}^2)$. Moreover, the annulus $\mathcal{A}$ contains the compact set $\mathcal{J}:=\check{A}_s(J(f_s))=J(f_s)\times\beta_s$.

Let us define $\rho'>0$ such that $\rho/\rho'=\check{R}$, i.e.,

$$\rho':=\frac{\rho}{\check{R}}=\left(\frac{\rho}{|\alpha_s||\beta_s|}\right)^{1/2}.$$

By taking a sufficiently large N in Claim 1, we may assume that the diameter of V_s' is sufficiently small (equivalently, $|\beta_s|$ is sufficiently large) such that

$$J_{c_0+\eta}\subset A(\rho',\rho).$$

Hence the rescaled Julia set

$$\mathcal{J}_0:=\Gamma_0(c_0+\eta)_{\rho',\rho}=J_{c_0+\eta}\times\frac{\rho}{(\rho')^2}$$

is contained in the annulus $A(\check{R},\check{R}^2)$.

Lemma 7.7. *There exists a $(1+\kappa)^{1/2}$-quasiconformal map $\Psi:\overline{A(\check{R},\check{R}^2)}\to\overline{\mathcal{A}}=\overline{\check{U}_s}\smallsetminus\check{V}_s'$ such that $\Psi(\mathcal{J}_0)=\mathcal{J}$ and $\Psi(\check{z}^2)=\check{G}_s(\Psi(\check{z}))$ for any $\check{z}\in\partial D(\check{R})$.*

Proof of Lemma 7.7. We will construct such a Ψ for $\partial A(\check{R},\check{R}^2)$ and for $\mathcal{J}_0$ separately, then use the Bers-Royden theorem to interpolate them.

Let us start with the boundary of the annulus: By (7.5), we have

$$\log \check{G}_s^{-1}(\check{w}) = \frac{1}{2}\log\check{w} + \frac{1}{2}\log\left(1+O(\rho/R)\right) = \frac{1}{2}\log\check{w} + O(\rho/R)$$

near $\partial\check{U}_s = \partial D(\check{R})$. Hence we may apply the same argument as the proof of Lemma 7.1 for sufficiently large $\check{R}$ and small ρ/R. Indeed, we let

$$\check{\Upsilon}(\check{w}) := \log \check{G}_s^{-1}(\check{w}) - \frac{1}{2}\log\check{w} = O(\rho/R)$$

and $\check{v}(t) := \check{\Upsilon}(\check{w}(2t))$, where $\check{w}(2t) = \check{R}^2 e^{2ti}$ makes two turns along $\partial D(\check{R}^2)$ as t varies from 0 to 2π. Set $\check{\ell} := \log\check{R}$ and $\check{\eta}(s) := \eta_0(s/\check{\ell}-1)$, where η_0 is defined in the proof of Lemma 7.1. Then the map

$$\check{\theta}_\xi(s+it) := s+it+\xi\check{\eta}(s)\check{v}(t)$$

with a parameter $\xi\in\mathbb{C}$ is defined for $(s,t)\in[\check{\ell},2\check{\ell}]\times[0,2\pi]$C and $\check{\theta}_\xi$ is a $(1+O(|\xi|\rho/R))$-quasiconformal map for sufficiently large $\check{R}$ and small ρ/R. We rather fix such ρ, R and $\check{R}$, and obtain a holomorphic family of injections $\left\{\check{\theta}_\xi\right\}_\xi$ with parameter ξ in a disk $D(d_0)$ of radius $d_0 \asymp R/\rho$. (This bound comes from the estimate like (7.1).) By observing the motion through the exponential map, we obtain a holomorphic motion $\psi : \partial A(\check{R},\check{R}^2)\times D(d_0)\to\mathbb{C}$ of $\partial A(\check{R},\check{R}^2)$ over $D(d_0)$ with $\psi(\check{z},\xi) := \psi_\xi(\check{z})$. In particular, by letting $\xi := 1$, the map $\Psi := \psi_1$ satisfies $\Psi(\check{z}^2) = \check{G}_s(\Psi(\check{z}))$ by construction.

Next we consider the Julia set: Let $\Psi : \mathcal{J}_0\to\mathcal{J}$ be the quasiconformal map given by composing the four maps

$$\mathcal{J}_0 = \Gamma(c_0+\eta)_{\rho',\rho} \xrightarrow{(1)} J_{c_0+\eta} \xrightarrow{(2)} J(F_{X_n(s)}) \xrightarrow{(3)} J(f_s) \xrightarrow{(4)} \mathcal{J},$$

where (1) is the affine map $\check{z}\mapsto(\rho/(\rho')^2)^{-1}\check{z}$; (2) is the inverse of the $(1+O(R^{-1}))$-quasiconformal straightening $H := H_{X_n(s)}$ of $F_{X_n(s)}$ to $P_{c_0+\eta}$; (3) is the inverse of the affine map $A_s : z\mapsto z/\alpha_s$; and (4) is the affine map $\check{A}_s : \check{z}\mapsto\beta_s\check{z}$. The straightening map H in (2) extends to a $(1+O(R^{-1}))$-quasiconformal map on the plane as in Lemma 7.1. Let $\check{\mu} := \mu_{H^{-1}}$ be the Beltrami coefficient of the *inverse* of such an extended H with $\|\check{\mu}\|_\infty = O(R^{-1})$. Then the Beltrami equation for $\check{\mu}_\xi := \xi\cdot\check{\mu}$ with a complex parameter ξ has a solution if $\xi\in D(d_1)$ with $d_1\asymp R$. Let ϕ_ξ be the unique normalized solution such that $\phi_\xi(0)=0$ and $(\phi_\xi)_z - 1\in L^p(\mathbb{C})$ for some $p>2$. Then the map $\psi_\xi(\check{z}) := \check{A}_s\circ A_s^{-1}\circ\phi_\xi((\rho/(\rho')^2)^{-1}\check{z})$ gives a holomorphic motion $\psi : \mathcal{J}_0\times D(d_1)\to\mathbb{C}$ of $\mathcal{J}_0$ over $D(d_1)$ with $\psi(\check{z},\xi) = \psi_\xi(\check{z})$. In particular, by letting $\xi := 1$, the map

$$\Psi(\check{z}) := \psi_1(\check{z}) = \check{A}_s\circ A_s^{-1}\circ H^{-1}_{X_n(s)}((\rho/(\rho')^2)^{-1}\check{z}) = \check{A}_s\circ h_s^{-1}((\rho/(\rho')^2)^{-1}\check{z}) \tag{7.6}$$

satisfies $\Psi(\mathcal{J}_0) = \mathcal{J}$.

Now by taking R relatively larger than ρ, we may assume that $d_0 < d_1$. Let us check that the unified map $\psi_\xi : \partial A(\check{R},\check{R}^2)\cup\mathcal{J}_0\to\mathbb{C}$ gives a holomorphic family of injections for $\xi\in D(d_0)$ if $\check{R}$ is sufficiently large. Indeed, it is enough to check that the distance between $\psi_\xi(\partial A(\check{R},\check{R}^2))$ and $\psi_\xi(\mathcal{J}_0)$ is bounded from below for $\xi\in D(d_0)$.

Let us fix a constant $0<\sigma<1$ such that $J_{c_0+\eta}\subset A(\sigma\rho,\rho)$. Hence $\mathrm{dist}(0,\mathcal{J}_0) > \sigma\check{R}^2$. Note that we can replace $\check{R}$ by an arbitrarily larger one with only a slight change of σ, because in Claim 1 we can replace $\widehat{V}_{c_1}$ by an arbitrarily smaller one such that the location of the center s of $\widehat{W}$ changes only a little (relatively to the size of $\widehat{\Lambda}$). Hence we may assume that $\check{R}$ is large enough such that $\mathrm{dist}(\partial D(\check{R}), \mathcal{J}_0) \geq \mathrm{dist}(0, \mathcal{J}_0) - \check{R} > \check{R}^2(\sigma - 1/\check{R}) \asymp \check{R}^2$. By taking a sufficiently large ρ, we have $J_{c_0+\eta} \subset D(\rho/2)$ and thus $\mathrm{dist}(\partial D(\check{R}^2), \mathcal{J}_0) \geq \check{R}^2/2$. Hence we conclude that $\mathrm{dist}(\partial A(\check{R}, \check{R}^2), \mathcal{J}_0) \asymp \check{R}^2$.

Now suppose that $\xi \in D(d_0)$. Since $d_0 \asymp R/\rho$, an explicit calculation shows that $\psi_\xi(\check{z}) = \check{z}$ on $\partial D(\check{R}^2)$ and $\psi_\xi(\check{z}) = \check{z}(1+\xi O(\rho/R))$ on $\partial D(\check{R})$. Hence $\mathrm{dist}(0, \psi_\xi(\partial D(\check{R}))) \asymp \check{R}$.

On the other hand, since $\check{\mu}_\xi = O(|\xi| R^{-1}) = O(\rho^{-1})$ for $\xi \in D(d_0)$, we have $\phi_\xi(z) = z + O(\rho^{-1+2/p})$ for some $p > 2$ on $J_{c_0+\eta}$ (cf. Lemma 7.2). Hence $\mathrm{dist}(\mathcal{J}_0, \psi_\xi(\mathcal{J}_0)) \asymp O(\rho^{-1+2/p})\check{R}^2$ for sufficiently large ρ. It follows that if ρ, R, and $\check{R}$ are sufficiently large and ρ/R are sufficiently small, then we have

$$
\begin{aligned}
&\mathrm{dist}(\psi_\xi(\partial A(\check{R}, \check{R}^2)), \psi_\xi(\mathcal{J}_0)) \\
&\geq \mathrm{dist}(\partial A(\check{R}, \check{R}^2), \mathcal{J}_0) - \mathrm{dist}(\partial A(\check{R}, \check{R}^2), \psi_\xi(\partial A(\check{R}, \check{R}^2))) - \mathrm{dist}(\mathcal{J}_0, \psi_\xi(\mathcal{J}_0)) \\
&\asymp \check{R}^2(1 - O(1/\check{R}) - o(1)) \asymp \check{R}^2
\end{aligned}
$$

for $\xi \in D(d_0)$.

By the Bers-Royden theorem, the holomorphic motion ψ of $\partial A(\check{R}, \check{R}^2) \cup \mathcal{J}_0$ over $D(d_0)$ extends to that of $\mathbb{C}$ and the dilatation of ψ_1 is bounded by $1 + O(1/d_0) = 1 + O(\rho/R)$. It is $(1+\kappa)^{1/2}$-quasiconformal by taking R relatively larger than ρ. Thus the restriction Ψ of ψ_1 on $\overline{A(\check{R}, \check{R}^2)}$ is our desired map. ■(Lemma 7.7)

Proof of Claim 6, continued. Let $\check{\Theta}_s := \check{A}_s^{-1} \circ \Psi$ (hence $\check{\Theta}_s(\check{z}) = h_s^{-1}((\rho/(\rho')^2)^{-1}\check{z})$ for $\check{z} \in \Gamma(c_0+\eta)_{\rho',\rho}$ by (7.6)) and $\check{\Theta}_c := \iota_c \circ \check{\Theta}_s$ for $c \in W$, where $\iota_c : \overline{U_s} \setminus V_s' \to \overline{U_c} \setminus V_c'$ is a $(1+\kappa)^{1/2}$-quasiconformal map given in Claim 5. Then $\check{\Theta}_c$ is $(1+\kappa)$-quasiconformal for each $c \in W$ with desired properties. ■

Almost conformal embedding of the model. We finish the proof of Theorem C by the next claim:

Claim 7 (Almost conformal straightening). *The family*

$$\boldsymbol{G} = \{G_c : V_c' \to U_c\}_{c \in W}$$

is a Mandelbrot-like family whose straightening map $\chi_{\boldsymbol{G}} : W \to \mathbb{C}$ induced by the decorated tubing $\check{\Theta}$ is $(1+\kappa)$-quasiconformal. Moreover, the inverse of $\chi_{\boldsymbol{G}}$ realizes a $(1+\kappa)$-quasiconformal embedding of the model $\mathcal{M}(c_0+\eta)_{\rho',\rho}$.

Proof. The argument is analogous to that of Lemmas 7.1 – 7.4: For each $c \in W$, the decorated tubing $\check{\Theta}_c$ constructed in Claim 6 induces a $(1+\kappa)$-quasiconformal straightening map $\check{h}_c : U_c \to \mathbb{C}$ if we take sufficiently large r, R, ρ, and $\check{R}$ with sufficiently small ρ/R. Hence the straightening map $\chi_{\boldsymbol{G}} : W \to \mathbb{C}$ is given by $\chi_{\boldsymbol{G}}(c) := \check{h}_c(G_c(0))$.

Let us show that the map $\chi_{\boldsymbol{G}}$ is $(1+\kappa)$-quasiconformal by following the argument of Lemma 7.4: It is $(1+\kappa)$-quasiconformal on $W \setminus M_{\boldsymbol{G}}$ because $\check{\Theta}_c$ is $(1+\kappa)$-quasiconformal for any $c \in W$. By Bers' lemma (Lemma 6.3), it is enough to show that $\check{\chi} : M_{\boldsymbol{G}} \to M$ extends to a $(1+\kappa)$-quasiconformal map on $\mathbb{C}$.

Recall that the quadratic-like map $\check{G}_c : \check{V}_c' \to \check{U}_c$ given in (7.4) is of the form

$$\check{w} = \check{G}_c(\check{z}) = \check{A}_c \circ G_c \circ \check{A}_c^{-1}(\check{z}) = \check{z}^2 + \check{c} + \check{u}(\check{z}, \check{c}),$$

where $\check{c} = \check{X}(c) := \beta_c \gamma_c$ and $\check{u}(\check{z}, \check{c}) = \check{z}^2\, O(\rho/R)$. Note that the map $c \mapsto \check{c} = \check{X}(c)$ is univalent near $M_{\boldsymbol{G}}$ and $\check{X}(s) = 0$. Moreover, $\check{u}(\check{z}, \check{c}) = \check{u}_{\check{c}}(\check{z})$ satisfies $\check{u}'_{\check{c}}(0) = 0$, and the value

$$\check{\delta} := \sup\{|\check{u}(\check{z}, \check{c})| \mid (\check{z}, \check{c}) \in \overline{D(4)} \times \overline{D(4)}\}$$

is $O(\rho/R)$. We may assume that ρ/R is small enough such that $\check{\delta} < 1$. As in the proof of Lemma 7.4, we consider the analytic family

$$\check{\boldsymbol{G}}(t) = \left\{\check{z} \mapsto \check{z}^2 + \check{c} + t\check{u}(\check{z}, \check{c})/\check{\delta}\right\}_{\check{c} \in D(4)}$$

with parameter $t \in \mathbb{D}$ whose connectedness locus $\check{M}(t)$ is homeomorphic to M. Then we have $\check{\boldsymbol{G}}(\check{\delta}) = \{\check{G}_c\}_{c \in W}$ and the straightening map $\check{\chi} : \check{M}(\check{\delta}) \to M$ extends to a quasiconformal map on $\mathbb{C}$ with dilatation $1 + O(\check{\delta}) = 1 + O(\rho/R)$ by the Bers-Royden theorem. Hence if ρ/R is sufficiently small, $\check{\chi}$ is $(1+\kappa)$-quasiconformal and so is the map $\chi_{\boldsymbol{G}}(c) = \check{\chi}(\check{X}(c))$.

As in the proof of Theorem A, the inverse of $\chi_{\boldsymbol{G}}(c)$ realizes a $(1+\kappa)$-quasiconformal embedding of $\mathcal{M}(c_0 + \eta)_{\rho', \rho}$ into W. ∎

8. Proof of Corollary D

Proof. We recall the setting of Theorem A. Take any small Mandelbrot set M_{s_0}, where $s_0 \neq 0$ is a superattracting parameter and take any Misiurewicz or parabolic parameter $c_0 \in \partial M$. Then Theorem A shows that $\mathcal{M}(c_0 + \eta)$ appears quasiconformally in M in a small neighborhood of $c_1 := s_0 \perp c_0$. Now let c be a parameter which belongs to the quasiconformal image of the decoration of $\mathcal{M}(c_0 + \eta)$. This means that

$$G_c^k(0) \in Y := \Theta_c(\Gamma_0(c_0 + \eta)) = J(f_c) \quad \text{for some } k \in \mathbb{N}.$$

Since $G_c = P_c^{p'}$ for some $p' \in \mathbb{N}$, we have $P_c^{p'k}(0) \in Y$. On the other hand, Y is f_c $(= P_c^p)$-invariant, that is, $P_c^{pn}(Y) \subseteq Y$ for every n. Then for a fixed $0 \leq r < p$, we have $P_c^{pn+r}(Y) \subseteq P_c^r(Y)$ for every n and each $P_c^r(Y)$ is apart from 0. Therefore for every $i \geq p'k_0$ we have $P_c^i(Y) \in \bigcup_{r=0}^{p-1} P_c^r(Y)$, which implies that the orbit of 0 under the iterate of P_c does not accumulate on 0 itself. Moreover, P_c has no parabolic periodic point since $Y = J(f_c)$ is a Cantor Julia set of a hyperbolic quadratic-like map $f_c : U'_c \to U_c$. This shows that P_c is semihyperbolic. Since Misiurewicz or parabolic parameters are dense in ∂M, we can find decorations in every small neighborhood of every point in ∂M. Also there are only countably many Misiurewicz parameters. Hence it follows that the parameters which are not Misiurewicz and non-hyperbolic but semihyperbolic are dense in ∂M. ∎

9. Concluding Remarks

We have shown that we can see quasiconformal images of some Julia sets in the Mandelbrot set M. But this is not satisfactory, because these images are all Cantor sets and disconnected. On the other hand, M is connected and so what we have detected is only a small part of the whole structure of M. From computer pictures, it is observed that the points in these Cantor sets are connected by some complicated filament structures. This looks like a picture which is obtained from the picture of K_c for $c \in \operatorname{int}(M_{s_1})$ by replacing all small filled Julia sets with small Mandelbrot sets. It would be interesting to explain this mathematically. A similar phenomena as in the quadratic family are observed also in the unicritical family $\{z^d + c\}_{c\in\mathbb{C}}$ by computer pictures. These phenomena should be proved in the same manner as for the quadratic case.

Appendix A. Proof of Lemma 4.3

(Misiurewicz case): In what follows the symbol $'$ always means $\frac{d}{dc}$, that is, the derivative with respect to the variable c. By the argument principle we have

$$\check{c}_n(t) = \frac{1}{2\pi i}\int_{|c-c_n(t)|=r_n(t)} \frac{F'(c)+G'(c)}{F(c)+G(c)}\cdot c\,dc, \quad \text{where} \quad r_n(t) := |d_n(t)|^{1+\beta}.$$

By the change of variable $\zeta := (c - c_n(t))/r_n(t)$ we have

$$\check{c}_n(t) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{F'(c_n(t)+r_n(t)\zeta)+G'(c_n(t)+r_n(t)\zeta)}{F(c_n(t)+r_n(t)\zeta)+G(c_n(t)+r_n(t)\zeta)}\cdot (c_n(t)+r_n(t)\zeta)r_n(t)d\zeta.$$

Then it is easy to see that the integrand is a C^1 function of t and hence $\check{c}_n(t)$ is of class C^1.

Since we already showed in Step (M3) of section 4 that $z_{\check{c}_n(t)}(t) \to z_{c_1}(t) \quad (n\to\infty)$ uniformly on t, here we show that

$$\frac{d}{dt}\Big(z_{\check{c}_n(t)}(t)\Big) \to \frac{d}{dt}z_{c_1}(t) \quad (n\to\infty).$$

Now by differentiating both sides of $z_{\check{c}_n(t)}(t) = \mu(\check{c}_n(t))^n\tau(\check{c}_n(t))$ we have

$$\frac{d}{dt}\check{c}_n(t) = \frac{\frac{\partial}{\partial t}z_c(t)\big|_{c=\check{c}_n(t)}}{n\mu(\check{c}_n(t))^{n-1}\mu'(\check{c}_n(t))\tau(\check{c}_n(t)) + \mu(\check{c}_n(t))^n\tau'(\check{c}_n(t)) - \frac{\partial}{\partial c}z_c(t)\big|_{c=\check{c}_n(t)}}.$$

Each term of the denominator has the following estimate:

$$\begin{aligned}
\text{(I)} &:= n\mu(\check{c}_n(t))^{n-1}\mu'(\check{c}_n(t))\tau(\check{c}_n(t)) \\
&= n\{\mu_{c_1} + M_0(\check{c}_n(t)-c_1)^\alpha + o(|\check{c}_n(t)-c_1|^\alpha)\}^{n-1} \\
&\quad \cdot\{M_0\alpha(\check{c}_n(t)-c_1)^{\alpha-1} + o(|\check{c}_n(t)-c_1|^{\alpha-1})\}\cdot\{K_0(\check{c}_n(t)-c_1) + o(|\check{c}_n(t)-c_1|)\}.
\end{aligned}$$

Since

$$\check{c}_n(t) - c_1 \sim d_n(t) = \frac{z_{c_1}(t)}{\mu_{c_1}^n K_0},$$

we have

$$\text{(I)} \sim n\mu_{c_1}^{n-1}\cdot M_0\alpha d_n(t)^{\alpha-1}\cdot K_0 d_n(t) = n\mu_{c_1}^{n-1}\cdot M_0K_0\alpha d_n(t)^\alpha.$$

So if $\alpha = 1$, then (I) $\sim \frac{M_0\alpha z_{c_1}(t)}{\mu_{c_1}}\cdot n \to \infty \quad (n\to\infty)$ and thus (I) $= O(n)$. If $\alpha \geq 2$, then (I) $\to 0 \quad (n\to\infty)$. Next we obtain

$$\begin{aligned}
\text{(II)} &:= \mu(\check{c}_n(t))^n\tau'(\check{c}_n(t)) \\
&= \{\mu_{c_1} + M_0(\check{c}_n(t)-c_1)^\alpha + o(|\check{c}_n(t)-c_1|^\alpha)\}^n\cdot\{K_0 + o(|\check{c}_n(t)-c_1|)\} \\
&\sim K_0\mu_{c_1}^n \to \infty \quad (n\to\infty)
\end{aligned}$$

and thus (II) $= O(\mu_{c_1}^n)$. For the same numerator we have

$$\text{(III)} := \frac{\partial}{\partial c}z_c(t)\big|_{c=\check{c}_n(t)} \to \frac{\partial}{\partial c}z_c(t)\big|_{c=c_1} \quad (n\to\infty).$$

It follows that the denominator goes to ∞ as $n \to \infty$ and so

$$\frac{d}{dt}\check{c}_n(t) \to 0 \quad (n \to \infty).$$

Hence we have

$$\begin{aligned}\frac{d}{dt}\Big(z_{\check{c}_n(t)}(t)\Big) &= \left(\frac{\partial}{\partial c}z_c(t)\Big|_{c=\check{c}_n(t)}\right) \times \frac{d}{dt}\check{c}_n(t) + \frac{\partial}{\partial t}z_c(t)\big|_{c=\check{c}_n(t)} \\ &\to \left(\frac{\partial}{\partial c}z_c(t)\Big|_{c=c_1}\right) \times 0 + \frac{d}{dt}z_{c_1}(t) = \frac{d}{dt}z_{c_1}(t) \quad (n \to \infty)\end{aligned}$$

and this convergence is uniform on t. This completes the proof of Lemma 4.3 for Misiurewicz case.

(Parabolic case): The proof is parallel to the Misiurewicz case. First we can show that $\check{c}_n(t)$ is of class C^1 in the same way. Then we have only to show that

$$\frac{d}{dt}\Big(z_{\check{c}_n(t)}(t)\Big) \to \frac{d}{dt}z_{c_1}(t) \ \ (n \to \infty).$$

By differentiating both sides of $z_{\check{c}_n(t)}(t) = \tau(\check{c}_n(t)) + n$ we have

$$\frac{d}{dt}\check{c}_n(t) = \frac{\frac{\partial}{\partial t}z_c(t)\big|_{c=\check{c}_n(t)}}{\tau'(\check{c}_n(t)) - \frac{\partial}{\partial c}z_c(t)\big|_{c=\check{c}_n(t)}}.$$

When $n \to \infty$, we have

$$\frac{\partial}{\partial t}z_c(t)\big|_{c=\check{c}_n(t)} \to z'_{c_1}(t), \quad \frac{\partial}{\partial c}z_c(t)\big|_{c=\check{c}_n(t)} \to \frac{\partial}{\partial c}z_c(t)\big|_{c=c_1}.$$

On the other hand we can show $\tau'(\check{c}_n(t)) \to \infty \quad (n \to \infty)$ as follows:

(Case $\nu = 1$): $\tau'(\check{c}_n(t)) = \dfrac{\pi i}{A_0(\check{c}_n(t) - c_1)^{3/2}} + O(1) \to \infty.$

(Case $\nu \geq 2$): $\tau'(\check{c}_n(t)) = \dfrac{2\pi i}{\nu^2 B_0(\check{c}_n(t) - c_1)^2} + O(1) \to \infty.$

It follows that $\frac{d}{dt}\check{c}_n(t) \to 0 \quad (n \to \infty)$ and hence we have

$$\begin{aligned}\frac{d}{dt}\Big(z_{\check{c}_n(t)}(t)\Big) &= \left(\frac{\partial}{\partial c}z_c(t)\Big|_{c=\check{c}_n(t)}\right) \times \frac{d}{dt}\check{c}_n(t) + \frac{\partial}{\partial t}z_{\check{c}_n(t)}(t) \\ &\to \left(\frac{\partial}{\partial c}z_c(t)\Big|_{c=c_1}\right) \times 0 + \frac{d}{dt}z_{c_1}(t) = \frac{d}{dt}z_{c_1}(t) \quad (n \to \infty)\end{aligned}$$

and this convergence is uniform on t. This completes the proof of Lemma 4.3 for Parabolic case. ■

Appendix B. Proof of Lemma 4.4

Suppose on the contrary that there exists a sequence of curves $z_{n_k}(t)$ $(n_k \nearrow \infty,\ k \in \mathbb{N})$ which are not Jordan curves. Then we have

$$z_{n_k}(t_{1k}) = z_{n_k}(t_{2k}), \quad \text{for some} \quad t_{1k} \neq t_{2k} \in S^1 = \mathbb{R}/\mathbb{Z}.$$

Suppose there exists an $\varepsilon_0 > 0$ such that $d_{S^1}(t_{1k}, t_{2k}) \geq \varepsilon_0$ for infinitely many k, where d_{S^1} denotes a natural distance of S^1. By taking a subsequence if necessary, we can assume that

$$t_{1k} \to t_{1\infty}, \quad t_{2k} \to t_{2\infty} \quad \text{and} \quad d_{S^1}(t_{1\infty}, t_{2\infty}) \geq \varepsilon_0. \tag{B.1}$$

Then since the convergence $z_n(t) \to z(t)$ is uniform, it follows that

$$z_{n_k}(t_{1k}) \to z(t_{1\infty}), \quad \text{and} \quad z_{n_k}(t_{2k}) \to z(t_{2\infty})$$

and hence from (B.1), we have

$$z(t_{1\infty}) = z(t_{2\infty}), \quad t_{1\infty} \neq t_{2\infty}.$$

This contradicts the fact that $z(t)$ is a Jordan curve. Therefore it follows that for every $\varepsilon_0 > 0$, $d_{S^1}(t_{1k}, t_{2k}) < \varepsilon_0$ holds for all but finite number of k, that is, we have $d_{S^1}(t_{1k}, t_{2k}) \to 0$ $(k \to \infty)$. Then again by taking a subsequence if necessary, we can assume that

$$t_{1k},\ t_{2k} \to t_\infty \quad (k \to \infty)$$

for some t_∞. Now let $x(t)$ and $x_n(t)$ be the real part of $z(t)$ and $z_n(t)$, respectively. By rotating the curve if necessary, we can assume that there exists a small interval $I_\infty \subset S^1$ centered at t_∞ such that $x'(t) \neq 0$ on I_∞. Then since the convergence $z_n(t) \to z(t)$ is C^1, it follows that $x'_{n_k}(t) \neq 0$ on I_∞ for sufficiently large k. On the other hand from (B.1) we have $x_{n_k}(t_{1k}) = x_{n_k}(t_{2k})$ and $t_{1k},\ t_{2k} \in I_\infty$ for sufficiently large k. By Rolle's theorem there exists $t_{3k} \in I_\infty$ between t_{1k} and t_{2k} such that $x'_{n_k}(t_{3k}) = 0$, which contradicts the fact that $x'_{n_k}(t) \neq 0$ on I_∞. This completes the proof of Lemma 4.4. ■

References

[**B**] A.F. Beardon, *Iteration of Rational Functions*, Springer Verlag, New York, Berlin and Heidelberg, 1991.

[**BH**] X. Buff and C. Henriksen, Julia sets in parameter spaces, *Comm. Math. Phys.* **220** (2001), no. 2, 333–375.

[**BR**] L. Bers and H.L. Royden, Holomorphic families of injections. *Acta Math.* **157** (1986), 259–286.

[**CJY**] L. Carleson, P.W. Jones and J.-C. Yoccoz, *Julia and John*. Bol. Soc. Brasil. Mat. (N.S.) **25** (1994), no. 1, 1–30.

[**D**] P.L. Duren, *Univalent Functions*, Springer-Verlag, 1983.

[**D-BDS**] A. Douady, X. Buff, R. Devaney and P. Sentenac, Baby Mandelbrot sets are born in cauliflowers, In *The Mandelbrot set, theme and variations*, London Math. Soc. Lecture Note Ser. **274**, Cambridge Univ. Press, Cambridge (2000), 19–36.

[**DH1**] A. Douady and J. H. Hubbard, Etude dynamique des polynômes complexes I & II. *Publ. Math. Orsay, Université de Paris-Sud*, Département de Mathématiques, Orsay, 1984/85, 84-2, 85-4.

[**DH2**] A. Douady and J. Hubbard, On the dynamics of polynomial-like mappings, *Ann. Sci. École Norm. Sup.***18**, (1985), 287–343.

[**DSZ**] A. Douady, P. Sentenac and M. Zinsmeister, Implosion parabolique et dimension de Hausdorff, *Compte Rendu Acad. Sci., Sér I*, **325** (1997), 765–772.

[**G**] D. Gaidashev, Cylinder renormalization for Siegel discs and a constructive measurable Riemann mapping theorem. *Nonlinearity* **20** (2007) 713–741.

[**H**] P. Haïssinsky, Modulation dans l'ensemble de Mandelbrot, In *The Mandelbrot set, theme and variations*, London Math. Soc. Lecture Note Ser. **274**, Cambridge Univ. Press, Cambridge (2000), 37–66.

[**IT**] Y. Imayoshi and M. Taniguchi *An Introduction to Teichmüller Spaces*. Springer, 1992.

[**J**] W. Jung, personal communication, 2015. See `http://www.mndynamics.com/indexp.html` (his web page) and also `http://www.mrob.com/pub/muency/embeddedjuliaset.html`.

[**K1**] T. Kawahira, Tessellation and Lyubich-Minsky laminations associated with quadratic maps, I: pinching semiconjugacies *Ergod. Th. & Dynam. Sys.* **29** (2009), 579–612.

[K2] T. Kawahira, Quatre applications du lemme de Zalcman à la dynamique complexe. *J. Anal. Math.* **124** (2014), 309–336.

[La] P. Lavaurs, Systèmes dynamiques holomorphes, explosion de points périodiques paraboliques, *Thèse de doctorat*, Université Paris-Sud in Orsay, 1989.

[Ly1] M. Lyubich, On the Lebesgue measure of the Julia set of a quadratic polynomial, *Preprint* IMS at Stony Brook, #1991/10.

[Ly2] M. Lyubich, Feigenbaum-Coullet-Tresser universality and Milnor's hairiness conjecture, *Ann. of Math.* (2) **149**(1999), no. 2, 319–420.

[Ly3] M. Lyubich, *Conformal Geometry and Dynamics of Quadratic Polynomials, vol I–II.* Available in his web page (`http://www.math.stonybrook.edu/~mlyubich/book.pdf`).

[LyMin] M. Lyubich and Y. Minsky, Laminations in holomorphic dynamics, *J. Differential Geom.* **47** (1997), no. 1, 17–94.

[Ma] R. Mañé, *On a theorem of Fatou*, Bol. Soc. Brasil. Mat. (N.S.) **24** No.1 (1993), 1–11.

[Mc1] C.T. McMullen, Renormalization and complex dynamics. Ann. of Math. Studies **135**, Princeton Univ. Press, 1994.

[Mc2] C.T. McMullen, The Mandelbrot set is universal, In *The Mandelbrot set, theme and variations*, London Math. Soc. Lecture Note Ser. **274**, Cambridge Univ. Press, Cambridge (2000), 1–17.

[Mil1] J. Milnor, Periodic orbits, externals rays and the Mandelbrot set: an expository account, *Géométrie complexe et systèmes dynamiques* (Orsay, 1995). Astérisque No.261 (2000), xiii, 277–333.

[Mil2] J. Milnor, *Dynamics in one complex variable, third ed.*, Annals of Mathematics Studies, vol. **160**, Princeton University Press, Princeton, NJ, 2006.

[MNTU] S. Morosawa, Y. Nishimura, M. Taniguchi and T. Ueda, *Holomorphic dynamics,* Translated from the 1995 Japanese original and revised by the authors. Cambridge Studies in Advanced Mathematics, **66**. Cambridge University Press, Cambridge, 2000.

[MTU] S. Morosawa, M. Taniguchi and T. Ueda, *Holomorphic dynamics* (in Japanese), Baifuukan (publisher), 1995.

[PS] H.-O. Peitgen and D. Saupe (eds.), *The science of fractal images*, Springer-Verlag, New York, 1988.

[Ric] S. Rickmann, Removability theorems for quasiconformal mappings. *Ann. Acad. Sci. Fenn.* **449** (1969), 1–8.

[Riv] J. Rivera-Letelier, On the continuity of Hausdorff dimension of Julia sets and similarity between the Mandelbrot set and Julia sets,*Fund. Math.* **170** (2001), no. 3, 287–317.

[S] M. Shishikura, The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets,*Ann. of Math.* (2), **147** (2), (1998), 225–267.

[T1] Tan L., Similarity between the Mandelbrot set and Julia sets, *Comm. Math. Phys.* **134**(1990), no. 3, 587–617.

[T2] Tan L., Local properties of the Mandelbrot set at parabolic points, In *The Mandelbrot set, theme and variations*, London Math. Soc. Lecture Note Ser. **274**, Cambridge Univ. Press, Cambridge (2000), 133–160.

Department of Mathematics, Graduate School of Science, Tokyo Institute of Technology, Tokyo 152-8551, Japan

Mathematical Science Team, RIKEN Center for Advanced Intelligence Project (AIP), 1-4-1 Nihonbashi, Chuo-ku, Tokyo 103-0027, Japan

E-mail address: `kawahira@math.titech.ac.jp`

Graduate School of Human and Environmental Studies, Kyoto University, Kyoto 606-8501, Japan

E-mail address: `kisaka@math.h.kyoto-u.ac.jp`