JULIA SETS APPEAR QUASICONFORMALLY IN THE MANDELBROT SET, II: A PARABOLIC PROOF

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ABSTRACT. Following the idea of A. Douady, we give an alternative proof of the authors' result as follows: For any boundary point c_0 of the Mandelbrot set M, we can find small quasiconformal copies of M in M that are encaged in nested quasiconformal copies of the totally disconnected Julia set of some parameter which is arbitrarily close to c_0.

1. INTRODUCTION

In the paper by A. Douady, X. Buff, R. Devaney, and P. Sentenac titled "Baby Mandelbrot sets are born in cauliflowers" [**D-BDS**], they showed that we can find small quasiconformal copies of the Mandelbrot set M in M that are encaged in nested quasiconformal copies of an imploded cauliflower (the Julia set of $z \mapsto z^2 + 1/4 + \varepsilon$ for small $\varepsilon > 0$). Indeed, we can always find such a copy near the cusp point of the primitive small copies of M and we can visually observe imploded and nested cauliflowers around them. The proof relies on the parabolic implosion technique developed by Douady, Lavaurs, and Shishikura.

Later in [**KK**], the authors extended this result and showed that fairly large varieties of quadratic Julia sets appear in M, but the proof presented in that paper is based on the shooting technique around Misiurewicz parameters. The aim of this paper is to present an alternative proof *à la Douady*, replacing "Misiurewicz" by "parabolic".

The main result. We will loosely follow Douady's original notation in [**D-BDS**]. We set

$$D(R) := \{z \in \mathbb{C} \mid |z| < R\}, \quad D(\alpha, R) := \{z \in \mathbb{C} \mid |z - \alpha| < R\},$$
$$A(r, R) := \{z \in \mathbb{C} \mid r < |z| < R\} \quad (0 < r < R).$$

For the quadratic map $P_c(z) := z^2 + c$ ($c \in \mathbb{C}$), let $K(P_c)$ and $J(P_c)$ denote the filled Julia set and the Julia set respectively. Now we choose any $\sigma \in \mathbb{C} \setminus M$ such that $J(P_\sigma)$ is a Cantor set. We also choose an $R > 1$ such that

$$J(P_\sigma) \subset A(R^{-1/2}, R^{1/2}),$$

and define the *rescaled Julia set* $\Gamma_0(\sigma)$ by

$$\Gamma_0(\sigma) := J(P_\sigma) \times R^{3/2} = \left\{R^{3/2}\, z \;\middle|\; z \in J(P_\sigma)\right\}$$

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in such a way that $\Gamma_0(\sigma)$ is contained in $A(R, R^2)$.

Let $\Gamma_m(\sigma)$ ($m \in \mathbb{N}$) be the inverse image of $\Gamma_0(\sigma)$ by $z \mapsto z^{2^m}$. Then the sets $\Gamma_0(\sigma), \Gamma_1(\sigma), \Gamma_2(\sigma), \ldots$ are mutually disjoint since $\Gamma_m(\sigma) \subset A(R^{2^{-m}}, R^{2^{-m+1}})$.

We define the *decorated Mandelbrot set* $\mathcal{M}(\sigma)$ by

$$\mathcal{M}(\sigma) := M \cup \Phi_M^{-1}\Big(\bigcup_{m=0}^{\infty} \Gamma_m(\sigma)\Big),$$

where $\Phi_M : \mathbb{C} \smallsetminus M \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ is the conformal isomorphism with $\Phi_M(c)/c \to 1$ as $|c| \to \infty$.

Let X and Y be non-empty compact sets in $\mathbb{C}$. We say Y *contains a quasiconformal copy of* X if there is a quasiconformal map χ on a neighborhood of X such that $\chi(X) \subset Y$ and $\chi(\partial X) \subset \partial Y$. Following A. Douady, we also say X *appears quasiconformally in* Y. Note that the condition $\chi(\partial X) \subset \partial Y$ is to exclude the case $\chi(X) \subset \mathrm{int}(Y)$.

Now we are ready to state the main theorem of this paper:

Theorem 1.1 (Julia sets appear quasiconformally)**.** *For any choices of $c_0 \in \partial M$ and $\varepsilon > 0$, there exists a parameter*

$$\sigma \in \mathbb{D}(c_0, \varepsilon) \smallsetminus M$$

such that M contains a quasiconformal copy of the decorated Mandelbrot set $\mathcal{M}(\sigma)$. Moreover, one can find such a copy in any open disk intersecting with ∂M.

Since $\mathcal{M}(\sigma)$ contains a rescaled Julia set $\Gamma_0(\sigma) = J(P_\sigma) \times R^{3/2}$, we may say that *the Julia set $J(P_\sigma)$ appears quasiconformally in M.*

Note that if $K(P_{c_0})$ has empty interior (i.e., P_{c_0} has no parabolic basins nor Siegel disks), then $J(P_\sigma)$ tends to $J(P_{c_0})$ in the Hausdorff topology as $\varepsilon \to 0$. Even the case when the interior of $K(P_{c_0})$ is non-empty, $J(P_\sigma)$ is contained in a small neighborhood of $K(P_{c_0})$, and a small neighborhood of $J(P_\sigma)$ contains $J(P_{c_0})$. See [**Do**].

Small Mandelbrot sets. The statement of the authors' original theorem (Theorem A or Theorem A' of [**KK**]) has more information about the location of the copy. The precise version can be described in terms of *tuning* and *small Mandelbrot sets*.

Let $s_0 \neq 0$ be a superattracting parameter such that the period of the critical point 0 is more than one. By the Douady-Hubbard tuning theorem [**H**, Théorème 1 du Modulation], there exists a unique compact subset M_{s_0} of M associated with a canonical homeomorphism $\chi_{s_0} : M_{s_0} \to M$ such that $\chi_{s_0}(s_0) = 0$. We also denote M_{s_0} by $s_0 \perp M$ and call it the *small Mandelbrot set with center* s_0. Similarly, for $c_0 \in M$, let $s_0 \perp c_0$ denote the parameter $\chi_{s_0}^{-1}(c_0)$ in M_{s_0}.

Theorem 1.1 can be derived from the following result:

Theorem 1.2 (Theorem A' of [**KK**])**.** *Let $s_0 \neq 0$ be any superattracting parameter and M_{s_0} be the small Mandelbrot set with center s_0. Then for any $c_0 \in \partial M$ and for every $\varepsilon > 0$ and $\varepsilon' > 0$, there exists an $\eta \in \mathbb{C}$ such that $|\eta| < \varepsilon$, $c_0 + \eta \notin M$, and $\mathcal{M}(c_0 + \eta)$ appears quasiconformally in $M \cap \overline{D(c_1, \varepsilon')}$, where $c_1 = s_0 \perp c_0 \in M_{s_0}$.*

(i)

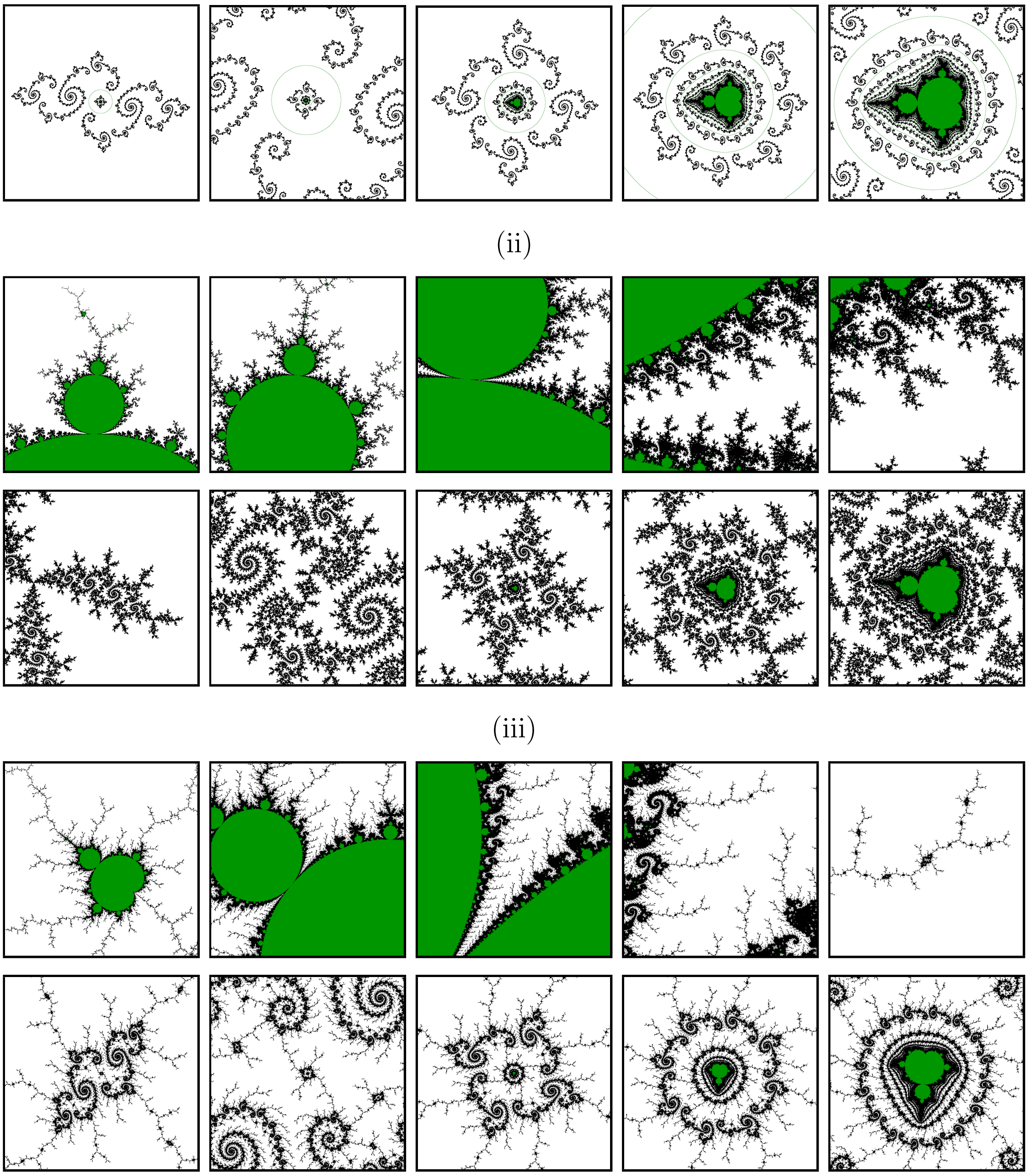

FIGURE 1. (i): The decorated Mandelbrot set $\mathcal{M}(\sigma)$ for $\sigma = -0.77 + 0.18i$ (close to the parabolic parameter $c_0 = -0.75$). (ii) and (iii): Embedded quasiconformal copies of $\mathcal{M}(\sigma)$ above near satellite and primitive small Mandelbrot sets.

Indeed, to obtain Theorem 1.1, we may take any s_0 in Theoerem 1.2 and let $\sigma := c_0 + \eta$. Since any open disk D intersecting with ∂M contains a Misiurewicz parameter c, and there is a sequence of small Mandelbrot sets converging to the c (see [**DH2**, Chapter V]), we may take a small Mandelbrot set M_{s_0} in D and thus we can find such a quasiconformal copy of $\mathcal{M}(\sigma)$ in D.

Organization of the paper. In Section 2, we recall some fundamental facts about quadratic-like maps and tuning of the Mandelbrot set. In Section 3 we summarize the parabolic implosion technique developed by Douady, Lavours, and Shishikura. Sections 4 to 7 are devoted for the main steps (P1)–(P4) of the proof of Theorem 1.2.

Notes. It is well-known that the Mandelbrot set inherits the structure of the quadratic Julia sets. See [**Ber**] [**BH**], [**CRY**], [**GS1**], [**GS2**], [**J**], [**K2**], [**MTU**], [**MNTU**], [**PS**], [**Riv**], [**S**], and [**T**] for example. For more details, readers may consult the introduction of [**KK**].

Similar phenomena to the quadratic family are observed also in the unicritical family $\{z^d + c\}_{c \in \mathbb{C}}$. In particular, similar theorems to our main results should be formalized and proved in the same manner as for the quadratic case.

2. Quadratic-like maps and renormalization

In this section we briefly recall a fundamental theory of quadratic-like maps. See [**DH2**], [**H**, Théorème 1 du Modulation], [**Mil1**], and [**Ly2**] for more details.

Quadratic-like mappings. Let U' and U be topological disks in $\mathbb{C}$ satisfying $U' \Subset U$ (i.e., $\overline{U'} \subset U$). A holomorphic map $h : U' \to U$ is called a *quadratic-like map* if h is a proper branched covering of degree two. We define the *filled Julia set* $K(h)$ and the *Julia set* $J(h)$ of h by

$$K(h) := \bigcap_{n=0}^{\infty} h^{-n}(U'), \quad \text{and} \quad J(h) := \partial K(h).$$

By the Douady-Hubbard straightening theorem [**DH2**, p296, Theorem 1], there exists a quadratic map $P_c(z) = z^2 + c$ and a quasiconformal map $\phi : U \to \phi(U)$ such that $\phi \circ h = P_c \circ \phi$ and $\bar{\partial}\phi = 0$ a.e. on $K(h)$. Such a parameter c is unique when $K(h)$ is connected, and we say *the quadratic-like map h is hybrid equivalent to P_c.*

Primitive vs. satellite. Let $s_0 \neq 0$ be any superattracting parameter (given in the statement of Theorem 1.2) such that the period of the critical point 0 is exactly $p \geq 2$. We say the small Mandelbrot set M_{s_0} is *primitive* if $P_{s_0 \perp (1/4)}$ has a parabolic periodic point with a single petal. Otherwise we say M_{s_0} is *satellite*, in which case $P_{s_0 \perp (1/4)}$ has a parabolic periodic point with more than one petal. One can visually distinguish them by looking at the hyperbolic component X_0 containing s_0: It is primitive if the boundary of X_0 has a cuspidal point at $s_0 \perp (1/4)$; or it is satellite if there is another hyperbolic component X_1 such that $\partial X_1 \cap \partial X_0 = \{s_0 \perp (1/4)\}$. (See Figure 4 of [**KK**], for example.)

By the Douady-Hubbard tuning theorem [**H**, p.42, Théorème 1 du Modulation], there exists a simply connected domain $\Lambda = \Lambda_{s_0}$ in the parameter plane with the following properties:

- For any $c \in \Lambda$, P_c is renormalizable with period p. More precisely, there exist two Jordan domains $\widetilde{U}'_c$ and $\widetilde{U}_c$ with piecewise analytic boundaries such that
$$f_c := P_c^p|_{\widetilde{U}'_c} : \widetilde{U}'_c \to \widetilde{U}_c$$
is a quadratic-like map with a critical point $0 \in \widetilde{U}'_c$. In particular, the boundaries of $\widetilde{U}'_c$ and $\widetilde{U}_c$ move holomorphically with respect to c over Λ.
- There exists a canonical homeomorphism $\chi_{s_0} : \Lambda \to \chi_{s_0}(\Lambda)$ such that $M \smallsetminus \{1/4\} \subset \chi_{s_0}(\Lambda)$ and for each $c \in \Lambda$, $f_c : \widetilde{U}'_c \to \widetilde{U}_c$ is hybrid equivalent to $z \mapsto z^2 + \chi_{s_0}(c)$.
- In both cases, $\chi_{s_0}^{-1}$ restricted to $M \smallsetminus \{1/4\}$ extends to a homeomorphism $\chi_{s_0}^{-1} : M \to \chi_{s_0}^{-1}(M)$. This image $\chi_{s_0}^{-1}(M)$ is the small Mandelbrot set M_{s_0}.
- If M_{s_0} is a primitive small Mandelbrot set, then $M_{s_0} \subset \Lambda$.
- If M_{s_0} is a satellite small Mandelbrot set, then $M_{s_0} \smallsetminus \{s_0 \perp (1/4)\} \subset \Lambda$.

This family $\{f_c\}_{c \in \Lambda}$ of quadratic-like maps above is a key ingredient of the proof of Theorem 1.2.

Outline of the proof of Theorem 1.2. First, since the parabolic parameters (that is, parameter c for which P_c has a parabolic periodic point) are dense in ∂M, we may assume that c_0 in the statement is a parabolic parameter. Then for the parameter $c_1 := s_0 \perp c_0$, we consider perturbation P_c of P_{c_1} such that the parameter c ranges over a sector S attached to c_1.

Next the proof breaks into four steps (P1)–(P4): In Steps (P1) and (P2), we construct two families of quadratic-like maps $\boldsymbol{f} = \{f_c : U'_c \to U_c\}_{c \in S \cap \Lambda}$ and $\boldsymbol{G} = \{G_c : V'_c \to U_c\}_{c \in W}$, that are "nested" in both dynamical and parameter spaces in the sense that $W \Subset S \cap \Lambda$ and $V'_c \Subset U'_c$ for $c \in W$. Then in Steps (P3) and (P4), we check that the first family $\{f_c\}_{c \in S \cap \Lambda}$ restricted on $c \in W$ provides a stable quasiconformal copy of the Julia set $J(P_{c_0+\eta})$ in the statement, and the second family $\{G_c\}_{c \in W}$ provides a quasiconformal copy of the decorated Mandelbrot set $\mathcal{M}(c_0 + \eta)$.

3. Fatou coordinates

In this section we recall some fundamental facts about Fatou coordinates and their perturbations near parabolic parameters that will be mainly used in Step (P2) (Section 5).

Agan let M_{s_0} be the small Mandelbrot set with center $s_0 \neq 0$ such that 0 is a periodic point of period $p \geq 2$, and let $\Lambda = \Lambda_{s_0}$ be the simply connected domain where the family $\{f_c := P_c^p|_{\widetilde{U}'_c} : \widetilde{U}'_c \to \widetilde{U}_c\}_{c \in \Lambda}$ of quadratic-like maps (given in the previous section) is defined. As we have remarked, since the parabolic parameters are dense in ∂M, we may assume that $c_0 \in \partial M$ in Theorem 1.2 is a parabolic parameter. Let $c_1 := s_0 \perp c_0 \in M_{s_0}$. Note that c_1 is also a parabolic parameter.

A pair of petals and the Fatou coordinates. We start with the global dynamics of $P_{c_1} : \mathbb{C} \to \mathbb{C}$ including that of $f_{c_1} = P_{c_1}^p|_{\widetilde{U}'_{c_1}}$. Let Δ be the Fatou component of $K(f_{c_1})$ (i.e., the connected component of the interior of $K(f_{c_1})$) containing 0. The boundary

$\partial\Delta$ contains a unique parabolic periodic point q_{c_1} of f_{c_1} (resp. P_{c_1}) of period k (resp. kp). The multiplier $\mu_{c_1} := (f_{c_1}^k)'(q_{c_1})$ is of the form

$$\mu_{c_1} := (f_{c_1}^k)'(q_{c_1}) = e^{2\pi i\nu'/\nu},$$

where ν' and ν are coprime integers. Since P_{c_1} has only one critical point, q_{c_1} has ν-petals. That is, by choosing an appropriate local coordinate $w = \psi_{c_1}(z)$ near q_{c_1} with $\psi_{c_1}(q_{c_1}) = 0$, we have

$$\psi_{c_1} \circ f_{c_1}^{k\nu} \circ \psi_{c_1}^{-1}(w) = w(1 + w^\nu + O(w^{2\nu})).$$

See [**Bea**, Proof of Theorem 6.5.7] or [**K1**, Appendix A.2].[1] The set of w's with $\arg w^\nu = 0$ (resp. $\arg w^\nu = \pi$) determines *the repelling* (resp. *attracting*) *directions* of this parabolic point. Note that the Fatou component Δ is invariant under $f_{c_1}^{k\nu}$, and it contains a unique attracting direction. In particular, the sequence $f_{c_1}^{k\nu m}(0)$ $(m \in \mathbb{N})$ converges to q_{c_1} within Δ tangentially to the attracting direction.

Set

$$\Omega_{c_1}^+ := \left\{ z = \psi_{c_1}^{-1}(w) \in \mathbb{C} \mid -\frac{2\pi}{3\nu} \le \arg w \le \frac{2\pi}{3\nu},\ 0 < |w| < r \right\},$$
$$\Omega_{c_1}^- := \left\{ z = \psi_{c_1}^{-1}(w) \in \mathbb{C} \mid -\frac{5\pi}{3\nu} \le \arg w \le -\frac{\pi}{3\nu},\ 0 < |w| < r \right\}$$

for some sufficiently small $r > 0$ such that $\Omega_{c_1}^+$ and $\Omega_{c_1}^-$ are a pair of repelling and attracting petals with $\Omega_{c_1}^+ \cap \Omega_{c_1}^- \neq \emptyset$. (See Figure 2.) By multiplying a ν-th root of unity to the local coordinate $w = \psi_{c_1}(z)$ if necessary, we may assume that the attracting petal $\Omega_{c_1}^-$ is contained in Δ.

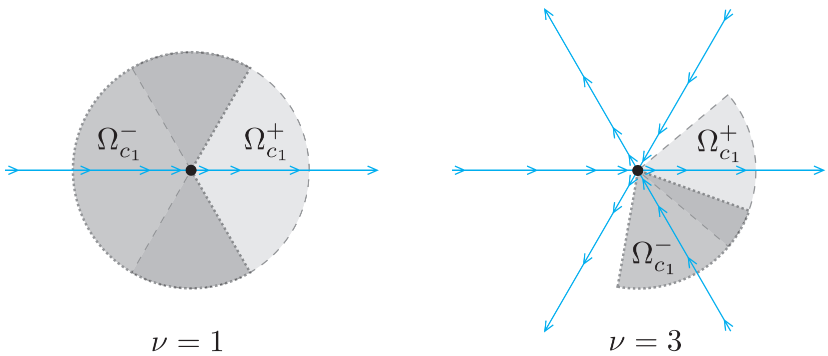

FIGURE 2. We choose a pair of repelling and attracting petals. Their intersection has two components when $\nu = 1$.

For the coordinate $w = \psi_{c_1}(z)$, we consider an additional coordinate change $w \mapsto W = -1/(\nu w^\nu)$. In this W-coordinate, the action of $f_{c_1}^{k\nu}$ on each petal is

$$W \mapsto W + 1 + O(W^{-1}).$$

[1] *A priori* the error term is $O(w^{\nu+1})$, but here it is refined to be $O(w^{2\nu})$.

By taking a smaller r if necessary, there exist conformal mappings $\phi_{c_1}^+ : \Omega_{c_1}^+ \to \mathbb{C}$ and $\phi_{c_1}^- : \Omega_{c_1}^- \to \mathbb{C}$ such that $\phi_{c_1}^\pm(f_{c_1}^{k\nu}(z)) = \phi_{c_1}^\pm(z) + 1$ which are unique up to adding constants. (We will normalize them later.) We call $\phi_{c_1}^\pm$ the *Fatou coordinates*.

Perturbed Fatou coordinates. When $\nu = 1$ (equivalently, $\mu_{c_1} = 1$), the parabolic fixed point q_{c_1} of $f_{c_1}^k$ splits into two distinct fixed points of f_c^k for each $c \neq c_1$. To describe this bifurcation, it is convenient to use a parameter u that satisfies $c = c_1 + u^2$. It is known that there are two holomorphic functions $q_+(u)$ and $q_-(u)$ defined near 0 such that $f_{c_1+u^2}^k(q_\pm(u)) = q_\pm(u)$; $q_{c_1} = q_+(0) = q_-(0)$; and their multipliers satisfy

$$\mu_\pm(u) := (f_{c_1+u^2}^k)'(q_\pm(u)) = 1 \pm A_0 u + O(u^2)$$

for some $A_0 \neq 0$. (See [**DH1**, Exposé XI], [**T2**, Theorem 1.1 (c)], or the primitive case of [**Mil1**, Lemma 4.2].) Note that the maps $u \mapsto \mu_\pm(u)$ are univalent near $u = 0$ and hence locally invertible.

For $r > 0$ we define a sector $S_\mu(r) \subset \mathbb{C}$ attached to 1 by

$$S_\mu(r) := \left\{ \mu \in \mathbb{C} \mid 0 < |\mu - 1| < r \text{ and } \left| \arg(\mu - 1) - \frac{\pi}{2} \right| < \frac{\pi}{8} \right\}.$$

We choose a sufficiently small $r_0 > 0$ such that the set

$$S := \left\{ c = c_1 + u^2 \mid \mu_+(u) \in S_\mu(r_0) \right\}$$

is contained in Λ and that the correspondence between $\mu = \mu_+(u) \in S_\mu(r_0)$ and $c = c_1 + u^2 \in S$ is one-to-one. See Figure 3 (left). We may regard the parameter u that mediates this one-to-one correspondence as a holomorphic branch of $\sqrt{c - c_1}$ over S. We may also regard

$$\begin{aligned} q_c &:= q_+(u) = q_+(\sqrt{c - c_1}) \qquad \text{and} \\ \mu_c &:= \mu_+(u) = 1 + A_0\sqrt{c - c_1} + O(c - c_1) \end{aligned} \tag{3.1}$$

as a fixed point of f_c^k and its multiplier that depend holomorphically on $c \in S$.

When $\nu \geq 2$ (equivalently, $\mu_{c_1} \neq 1$), the parabolic fixed point q_{c_1} of $f_{c_1}^k$ splits into one fixed point q_c and a cycle of period ν of f_c^k for each $c \neq c_1$. By the implicit function theorem, q_c and its multiplier $\mu_c := (f_c^k)'(q_c)$ depend holomorphically on c near c_1, and it is known that

$$\mu_c := (f_c^k)'(q_c) = \mu_{c_1}\left(1 + B_0(c - c_1) + O((c - c_1)^2)\right) \tag{3.2}$$

for some constant $B_0 \neq 0$. (See [**DH1**, Exposé XI], [**T2**, Theorem A.1 (c)], or the satellite case of [**Mil1**, Lemma 4.2].) Note that the map $c \mapsto \mu_c$ is univalent near c_1 and hence locally invertible.

We choose a sufficiently small $r_0 > 0$ such that the set

$$S := \left\{ c \in \mathbb{C} \mid \frac{\mu_c}{\mu_{c_1}} \in S_\mu(r_0) \right\}$$

is contained in Λ and that the correspondence between $\mu = \mu_c \in \mu_{c_1} \times S_\mu(r_0)$ and $c \in S$ is one-to-one. See Figure 3 (right). We call S a *sector* attached to $c_1 \in \partial M_{s_0}$.

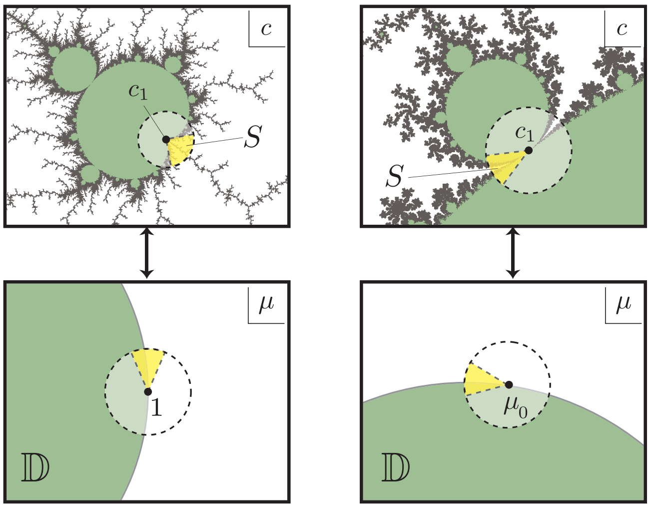

FIGURE 3. The sector S for $\nu = 1$ (left) and $\nu \geq 2$ (right).

Now we consider general ν ($= 1$ or ≥ 2). By taking a sufficiently small r_0, we may assume that for each $c \in S$ there exists a holomorphic local coordinate $w = \psi_c(z)$ near q_c with $\psi_c(q_c) = 0$ such that

$$\psi_c \circ f_c^{k\nu} \circ \psi_c^{-1}(w) = \mu_c^\nu w\,(1 + w^\nu + O(w^{2\nu})),$$

where $\mu_c^\nu \to 1$ and $\psi_c \to \psi$ uniformly as $c \in S$ tends to c_1. See [**K1**, Appendix A.2]. By a further ν-fold coordinate change $W = -\mu_c^{\nu^2}/(\nu w^\nu)$, the action of $f_c^{k\nu}$ is

$$W \mapsto \mu_c^{-\nu^2} W + 1 + O(W^{-1}),$$

where $W = \infty$ is a fixed point with multiplier $\mu_c^{\nu^2}$ that corresponds to the fixed point q_c of $f_c^{k\nu}$. There is another fixed point of the form $W = 1/(1 - \mu_c^{-\nu^2}) + O(1)$ with multiplier close to $\mu_c^{-\nu^2}$ on each branch of the ν-fold coordinate. Note that

$$\mu_c^{\pm\nu^2} = 1 \pm A_0\sqrt{c - c_1} + O(c - c_1) \qquad \text{or}$$
$$\mu_c^{\pm\nu^2} = 1 \pm \nu^2 B_0(c - c_1) + O((c - c_1)^2)$$

according to $\nu = 1$ or $\nu \geq 2$ by (3.1) and (3.2).

It is known that for each c in $S \cup \{c_1\}$ (by taking a smaller r_0 if necessary), there exist (perturbed) *Fatou coordinates* $\phi_c^+ : \Omega_c^+ \to \mathbb{C}$ and $\phi_c^- : \Omega_c^- \to \mathbb{C}$ satisfying the following conditions ([**La**], [**DSZ**] and [**S**, Proposition A.2.1]):

- For any $c \in S$, both $\partial\Omega_c^+$ and $\partial\Omega_c^-$ contain two fixed points q_c and q_c' of $f_c^{k\nu}$ that converge to q_{c_1} as $c \in S$ tends to c_1.
- For any $c \in S \cup \{c_1\}$, $\phi_c^\pm$ is a conformal map from a domain $\Omega_c^\pm$ onto the image in $\mathbb{C}$ that satisfies $\phi_c^\pm(f_c^{k\nu}(z)) = \phi_c^\pm(z) + 1$ if both z and $f_c^{k\nu}(z)$ are contained in $\Omega_c^\pm$.
- (Holomorphic dependence) Every compact set E in $\Omega_{c_1}^\pm$ is contained in $\Omega_c^\pm$ for $c \in S$ sufficiently close to c_1, and $\phi_c^\pm(z)$ depends holomorphically on $c \in S$ near c_1 for each $z \in \Omega_{c_1}^\pm$.

Indeed, Fatou coordinates $\phi_c^{\pm}$ for each $c \in S \cup \{c_1\}$ are uniquely determined up to composition with a translation of $\mathbb{C}$. We will give a specific normalization for them by using the critical orbits in Step (P2) (Section 5). We can also arrange the domains $\Omega_c^{\pm}$ such that $\Omega_c^+ = \Omega_c^- =: \Omega_c^*$ for each $c \in S$. Hence for each $z \in \Omega_c^*$,

$$\tau(c) := \phi_c^+(z) - \phi_c^-(z) \in \mathbb{C}$$

is defined and independent of z. The function $\tau : S \to \mathbb{C}$ is called the *lifted phase*. Note that the value $\tau(c)$ is determined by the unique normalized Fatou coordinates associated with the analytic germ $f_c^{k\nu}$, and it does not depend on the choice of the parametrization.[2] It is known that if $\nu = 1$, then

$$\tau(c) = -\frac{2\pi i}{A_0\sqrt{c-c_1}} + O(1)$$

as $c \in S$ tends to c_1, where A_0 is given in (3.1). Similarly if $\nu \geq 2$, we obtain

$$\tau(c) = -\frac{2\pi i}{\nu^2 B_0(c-c_1)} + O(1)$$

as $c \in S$ tends to c_1, where B_0 is given in (3.2). In both cases it can be also shown that $\tau(c)$ is univalent on S if S is sufficiently small.

4. Step (P1): Definitions of U_c, U'_c and V_c

Now we start the main steps of the proof of Theorem 1.2. Let us briefly recall the notation: The parameter $s_0 \neq 0$ is superattracting for which $P_{s_0}^p(0) = 0$ with $p \geq 2$. The small Mandelbrot set M_{s_0} with center s_0, and the family $\{f_c := P_c^p|_{\widetilde{U}'_c} : \widetilde{U}'_c \to \widetilde{U}_c\}_{c \in \Lambda}$ of quadratic-like maps are associated with it. For a given parabolic parameter $c_0 \in \partial M$, we have considered another parabolic parameter $c_1 = s_0 \perp c_0 \in M_{s_0}$ and its perturbation in a sector S.

For a technical reason, *we first assume that the parabolic parameter $c_1 = s_0 \perp c_0 \in M_{s_0}$ belongs to Λ.* Note that this assumption only excludes the case where M_{s_0} is a satellite small Mandelbrot set and $c_0 = 1/4$. This case will be discussed separately.

Under this assumption, we shall construct a family $\boldsymbol{f} = \{f_c : U'_c \to U_c\}_{c \in S \cap \Lambda}$ of quadratic-like maps and a family $\boldsymbol{g} = \{g_c : V_c \to U_c\}_{c \in S \cap \Lambda}$ of isomorphisms.

We start with the "global" dynamics of P_{c_1}:

Lemma 4.1 (cf. Lemma 4.1 of [**KK**])**.** *There exist Jordan domains U_{c_1}, U'_{c_1} and V_{c_1} with C^1 boundaries and integers N, $j \in \mathbb{N}$ which satisfy the following:*

(1) $0 \in U'_{c_1} \subset \widetilde{U}'_{c_1}$ *and* $f_{c_1} : U'_{c_1} \to U_{c_1}$ *is a quadratic-like map.*
(2) $g_{c_1} := P_{c_1}^N|_{V_{c_1}} : V_{c_1} \to U_{c_1}$ *is an isomorphism and* $\overline{f_{c_1}^j(V_{c_1})} \subset U_{c_1} \smallsetminus \overline{U'_{c_1}}$.
(3) $\overline{V_{c_1}} \subset \Omega_{c_1}^+$. *Also we can take* V_{c_1} *arbitrarily close to* $q_{c_1} \in \partial\Omega_{c_1}^+$.

[2] In [**D-BDS**], the lifted phase for $\nu = 1$ is described in terms of the normalized germ $f_\mu(z) = z + z^2 + \mu + \cdots$ $(\mu \to 0)$. In this case the multiplier for two fixed points of f_μ are $1 \pm 2\sqrt{\mu}i\,(1 + O(\mu))$ and $\tau = -\pi/\sqrt{\mu} + O(1)$ as $\mu \to 0$. In [**DSZ**], they use $\alpha = (\nu/2\pi i)\log(\mu_c/\mu_{c_1} - 1)$ (so that $\mu_c = \exp(2\pi i(\nu' + \alpha)/\nu))$ to parametrize the germs. Any parameterizations are analytically equivalent and they determine the same value τ.

Proof. By shrinking $\widetilde{U}_{c_1}$ and $\widetilde{U}'_{c_1}$ slightly, we can take Jordan domains U_{c_1} and U'_{c_1} with C^1 boundaries which are neighborhoods of $J(f_{c_1})$ and $f_{c_1} : U'_{c_1} \to U_{c_1}$ is a quadratic-like map.

For $j \in \mathbb{N}$ let

$$A_j := f_{c_1}^{-j}(U_{c_1} \smallsetminus \overline{U'_{c_1}}).$$

Since f_{c_1} is conjugate to z^2 on $U'_{c_1} \smallsetminus J(f_{c_1})$, the annulus A_j is uniformly close to $J(f_{c_1})$ and thus $A_j \cap \Omega^+_{c_1} \neq \emptyset$ for every sufficiently large j. Also since the "global" Julia set $J(P_{c_1})$ of P_{c_1} is a connected set containing the "small" Julia set $J(f_{c_1})$, the annulus $U_{c_1} \smallsetminus \overline{U'_{c_1}}$ intersects with $J(P_{c_1})$ and so does A_j. In particular, for every sufficiently large j, $A_j \cap \Omega^+_{c_1}$ contains a point $z_0 \in J(P_{c_1})$ arbitrarily close to the parabolic periodic point $q_{c_1} \in J(f_{c_1})$. Let B be any closed disk in $\Omega^+_{c_1} \cap A_j$ centered at this z_0 with an arbitrarily small radius.

Note that the postcritical set of the map P_{c_1} in $\mathbb{C}$ is contained in $U'_{c_1} \cup P_{c_1}(U'_{c_1}) \cup \cdots \cup P_{c_1}^{p-1}(U'_{c_1})$. There are two disjoint connected components $X := P_{c_1}^{p-1}(U'_{c_1})$ and $-X := \{-x \in \mathbb{C} \,:\, x \in X\}$ of $P_{c_1}^{-1}(U_{c_1})$, where $-X$ does not intersect with neither the postcritical set of P_{c_1} nor the critical point 0. Hence for any $n \in \mathbb{N}$ and any connected component V of $P_{c_1}^{-n}(-X)$, $P_{c_1}^{n+1} : V \to U_{c_1}$ is an isomorphism.

Since the inverse images of $-X$ in the dynamics of P_{c_1} accumulate on any point in the Julia set $J(P_{c_1})$ of P_{c_1} (by Montel's theorem), the shrinking lemma ([**LyMin**, p.86] or [**CT**, Lem.2.9]) implies that we can find a component V_{c_1} of $P_{c_1}^{-N+1}(-X)$ contained in the closed disk B for some $N \in \mathbb{N}$. This gives a desired isomorphism $g_{c_1} := P_{c_1}^N : V_{c_1} \to U_{c_1}$. ■ (Lemma 4.1)

Remark. This proof indicates that there are infinitely many different choices of V_{c_1} and one can choose V_{c_1} with arbitrarily small diameter. Indeed, each choice of V_{c_1} will give a different copy of the decorated Mandelbrot set.

Definition of the families $\boldsymbol{f}$ and $\boldsymbol{g}$. For $\varepsilon' > 0$ given in the statement of Theorem 1.2, we may assume that the sector S attached to c_1 (defined in the previous section) is contained in $D(c_1, \varepsilon') \cap \Lambda$ by taking a smaller $r_0 > 0$ in the definition of S.

Let $U_c := U_{c_1}$ for each $c \in S \subset \Lambda$. By taking a smaller U_{c_1} in the previous lemma and a smaller S if necessary, we obtain a quadratic-like map $f_c : U'_c \to U_c$ with $U'_c := f_c^{-1}(U_c) \subset \widetilde{U}'_c$, a component V_c of $P_c^{-N}(U_c)$ close to V_{c_1} satisfying $f_c^j(V_c) \subset U_c \smallsetminus \overline{U'_c}$, and an isomorphism $g_c : V_c \to U_c \, (\equiv U_{c_1})$ for each $c \in S$. See Figure 4. We also define the *pre-critical point* b_c by $b_c := g_c^{-1}(0) \in V_c$. Hence we have done the construction of the families

$$\boldsymbol{f} = \{f_c : U'_c \to U_c\}_{c \in S} \quad \text{and} \quad \boldsymbol{g} = \{g_c : V_c \to U_c\}_{c \in S}.$$

Satellite roots. Now we deal with the remaining case where M_{s_0} is a satellite small Mandelbrot set with renormalization period p and $c_0 = 1/4$. (We say c_1 for this case is a *satellite root*.) Let Λ be the simply connected domain associated with M_{s_0}. Since M_{s_0}

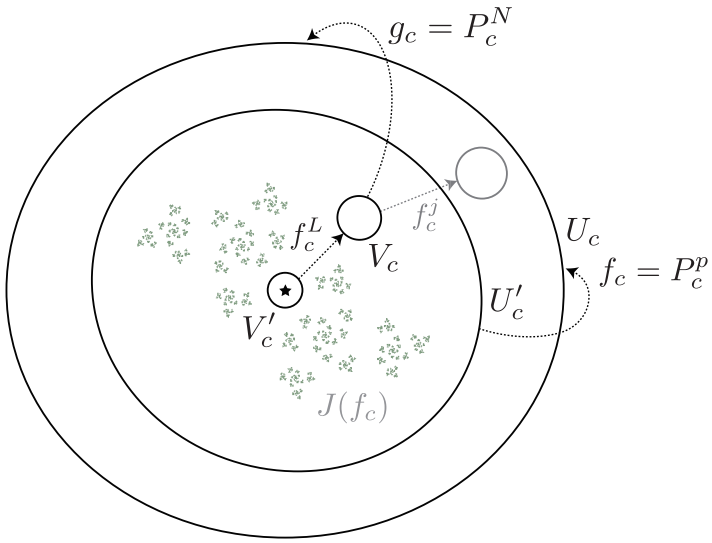

FIGURE 4. The Jordan domains U_c, U_c', V_c, and V_c'.

is satellite, the quadratic-like family $\{f_c = P_c^p|_{\widetilde{U}_c'} : \widetilde{U}_c' \to \widetilde{U}_c\}_{c\in\Lambda}$ excludes the parameter $c_1 = s_0 \perp (1/4) \notin \Lambda$. However, by slightly modifying the notion of quadratic-like map at this parameter, one can establish a version of Lemma 4.1 as follows.

Let q_{c_1} be the fixed point of $P_{c_1}^p$ with multiplier 1, whose petal number is $\nu \geq 2$. (Then the period of q_{c_1} in the dynamics of P_{c_1} is $p' = p/\nu$.) Let $\Omega_{c_1}^+$ be the repelling petal attached to q_{c_1} as in Figure 2 (right). Then we have:

Lemma 4.1 for the Satellite Roots. *There exist Jordan domains U_{c_1}, U_{c_1}' and V_{c_1} with C^1 boundaries and integers N, $j \in \mathbb{N}$ which satisfy the following:*

(1) $0 \in U_{c_1}' \subset U_{c_1}$, $\partial U_{c_1}' \cap \partial U_{c_1} = \{q_{c_1}\}$, *and* $f_{c_1} := P_{c_1}^p|_{U_{c_1}'} : U_{c_1}' \to U_{c_1}$ *is a proper branched covering of degree two.*

(2) $g_{c_1} := P_{c_1}^N|_{V_{c_1}} : V_{c_1} \to U_{c_1}$ *is an isomorphism and* $\overline{f_{c_1}^j(V_{c_1})} \subset U_{c_1} \smallsetminus \overline{U_{c_1}'}$.

(3) $\overline{V_{c_1}} \subset \Omega_{c_1}^+$. *Also we can take V_{c_1} arbitrarily close to $q_{c_1} \in \partial\Omega_{c_1}^+$.*

The proof is analogous to those of Lemmas 4.1. However, to obtain (1) we need an important idea of the proof of the Douady-Hubbard tuning theorem [**H**, §3] which is essential in the construction of Λ. Here we only give a sketch: For each $c \in \Lambda$ there exists a repelling fixed point q_c of P_c^p that depends holomorphically on c and $q_c \to q_{c_1}$ as $c \in \Lambda$ tends to c_1. We define a Jordan domain U_c by adding a small disk centered at q_c to U_{c_1}. (We also modify U_c slightly such that ∂U_c is a C^1 Jordan curve that moves holomorphically with respect to c.) Then for any $c \in \Lambda$ sufficiently close to c_1, we have a quadratic-like map $f_c : U_c' \to U_c$ and an isomorphism $g_c : V_c \to U_c$ where U_c' is a connected component of $P_c^{-p}(U_c)$ with $\overline{U_c'} \subset U_c$ and V_c is a connected component of $P_c^{-N}(U_c)$ that is close to V_{c_1} and $\overline{f_c^j(V_c)} \subset U_c \smallsetminus \overline{U_c'}$. Since q_{c_1} is parabolic with $\nu \geq 2$ petals, we take the sector S attached to c_1 as in Figure 3 (right). By taking S with sufficiently small radius, we obtain the families $\{f_c : U_c' \to U_c\}_{c\in S\cap\Lambda}$ and $\{g_c : V_c \to U_c\}_{c\in S\cap\Lambda}$ over the set $S \cap \Lambda$ together with Fatou coordinates and lifted phase. In conclusion, we have constructed the families

$$\boldsymbol{f} = \{f_c : U_c' \to U_c\}_{c \in S\cap\Lambda} \quad \text{and} \quad \boldsymbol{g} = \{g_c : V_c \to U_c\}_{c \in S\cap\Lambda}.$$

Remark. We have $S \not\subset \Lambda$ for the satellite root c_1, but $S \subset \Lambda$ for the other cases. Hence we regard $\{f_c\}$ and $\{g_c\}$ as families defined over $S \cap \Lambda$ for all cases.

5. Step (P2): Construction of the quadratic-like family $\boldsymbol{G}$

We shall construct the second quadratic-like family

$$\boldsymbol{G} = \boldsymbol{G}_n := \{G_c = G_{c,n} : V_c' = V_{c,n}' \to U_c\}_{c \in W_n}$$

such that $V_c' = V_{c,n}' \subset U_c'$ and $W = W_n \subset S \cap \Lambda$ for every sufficiently large $n \in \mathbb{N}$.

Normalization of the Fatou coordinates. Recall that we have (perturbed) Fatou coordinates $\phi_c : \Omega_c^{\pm} \to \mathbb{C}$ for each $c \in S \cup \{c_1\}$ as given in Section 3. By taking a smaller S if necessary, we may normalize them such that:

- $\overline{V_c} \subset \Omega_c^+$ for any $c \in S \cup \{c_1\}$.
- There exists an $m \in \mathbb{N}$ such that $f_c^{k\nu m}(0) \in \Omega_c^-$ for any $c \in S \cup \{c_1\}$, and ϕ_c^- is normalized such that $\phi_c^-(f_c^{k\nu m}(0)) = m$.
- $\phi_c^+(b_c) = 0$, where $b_c = g_c^{-1}(0) \in V_c$ is the pre-critical point.

Recall also that we may arrange the domains $\Omega_c^{\pm}$ such that $\Omega_c^+ = \Omega_c^- =: \Omega_c^*$ for each $c \in S$.

Definition of W. Now for each $n \geq m$, define

$$W = W_n := \{c \in S \mid f_c^{k\nu i}(0) \in \Omega_c^* \ \text{ for } \ i = m, \ldots, n-1 \ \text{ and } \ f_c^{k\nu n}(0) \in V_c\},$$

that is, we consider the parameter c such that the orbit of 0 by $f_c^{k\nu}$ hits V_c.

Lemma 5.1 (cf. Lemma 4.2 of [**KK**]). *By shrinking $U_c \equiv U_{c_1}$ slightly, the set $W = W_n$ is a non-empty Jordan domain with C^1 boundary for every sufficiently large n. Moreover, there exists an $s_n \in W_n$ such that $f_{s_n}^{k\nu n}(0) = b_{s_n}$, which implies $g_{s_n} \circ f_{s_n}^{k\nu n}(0) = P_{s_n}^{k\nu np+N}(0) = 0$ and hence P_{s_n} has a superattracting periodic point.*

Proof. We observe the dynamics of $f_c^{k\nu}$ near q_c through the perturbed Fatou coordinate $\phi_c^+ : \Omega_c^+ = \Omega_c^* \to \mathbb{C}$ of q_c. Let

$$\widetilde{V}_c := \phi_c^+(V_c),$$

then $c \in W_n$ if and only if $\phi_c^+(f_c^{k\nu n}(0)) \in \widetilde{V}_c$. By the normalization of ϕ_c^+, we have

$$\tau(c) = \phi_c^+(f_c^{k\nu n}(0)) - \phi_c^-(f_c^{k\nu n}(0)) = \phi_c^+(f_c^{k\nu n}(0)) - n.$$

Hence it follows that $c \in W_n$ if and only if

$$\tau(c) + n \in \widetilde{V}_c.$$

Next take a Riemann map

$$u : U_c \equiv U_{c_1} \to \mathbb{D}, \quad u(0) = 0$$

and define

$$v(c,\zeta) := \phi_c^+ \circ (u \circ g_c)^{-1}(\zeta), \quad \zeta \in \mathbb{D} \ \text{ with } \ v(c,0) = 0,$$

which is the inverse of a Riemann map $u \circ g_c \circ (\phi_c^+)^{-1}$ of $\widetilde{V}_c$. Now we solve the equation with respect to the variable c

$$\tau(c) + n = v(c,\zeta) \tag{5.1}$$

for each fixed $\zeta \in \mathbb{D}$.

Case 1 : $\nu = 1$. Since

$$\tau(c) = -\frac{2\pi i}{A_0\sqrt{c-c_1}} + h(c), \quad h(c) = O(1) \quad (c \to c_1),$$

the equation (5.1) can be rewritten as

$$F(c,\zeta) + G(c,\zeta) = 0, \tag{5.2}$$

where

$$F(c,\zeta) := -\frac{2\pi i}{A_0\sqrt{c-c_1}} + n - v(c_1,\zeta), \quad G(c,\zeta) := h(c) - \big(v(c,\zeta) - v(c_1,\zeta)\big).$$

The equation $F(c,\zeta) = 0$ has a unique solution

$$c = c_n(\zeta) := c_1 - \frac{4\pi^2}{A_0^2(n - v(c_1,\zeta))^2}.$$

Let

$$r_n(\zeta) := \left| -\frac{4\pi^2}{A_0^2(n - v(c_1,\zeta))^2} \right| = O(n^{-2})$$

and take any β with $0 < \beta < 1/2$. Consider (5.2) in the disk $D(c_n(\zeta), r_n(\zeta)^{1+\beta})$. Since it is easy to see that

$$|F(c,\zeta)| = O(r_n(\zeta)^{\beta - 1/2}) = O(n^{1-2\beta}), \quad |G(c,\zeta)| = O(1)$$

on the boundary C of this disk, we have $|F(c,\zeta)| > |G(c,\zeta)|$ on C for sufficiently large n. By Rouché's theorem (5.2) has a unique solution $c = \check{c}_n(\zeta)$ in $D(c_n(\zeta), r_n(\zeta)^{1+\beta})$. By using this solution, we can write

$$W_n = \{\check{c}_n(\zeta) \in \mathbb{C} \mid \zeta \in \mathbb{D}\}.$$

Claim. (1) *$\check{c}_n : \mathbb{D} \to W_n$ is holomorphic.*
(2) *For every $r \in (0,1)$, $\check{c}_n$ is univalent on $\mathbb{D}(r)$ for every sufficiently large n.*

Proof. (1) By the argument principle, for each $\zeta \in \mathbb{D}$ we have

$$\check{c}_n(\zeta) = \frac{1}{2\pi i}\int_C H(c,\zeta)c \cdot dc,$$

where

$$H(c,\zeta) := \frac{\frac{\partial}{\partial c}\big(F(c,\zeta) + G(c,\zeta)\big)}{F(c,\zeta) + G(c,\zeta)}, \quad C = \{z \mid |c - c_n(\zeta)| = r_n(\zeta)^{1+\beta}\}$$

Hence if $|\Delta\zeta| \ll 1$ and $c_n(\zeta+\Delta\zeta) \in \mathrm{int}(C)$, we have

$$\check{c}_n(\zeta+\Delta\zeta) = \frac{1}{2\pi i}\int_C H(c, \zeta+\Delta\zeta)c \cdot dc.$$

Then it follows that H is holomorphic with respect to ζ and hence $\check{c}_n(\zeta)$ is holomorphic in a neighborhood of ζ. Thus $\check{c}_n : \mathbb{D} \to W_n$ is holomorphic.
(2) Since

$$\tau(\check{c}_n(\zeta)) + n = v(\check{c}_n(\zeta), \zeta)$$

and $\tau(c)$ is univalent, it is enough to show that $v(\check{c}_n(\zeta), \zeta)$ is univalent on $\overline{\mathbb{D}(r)}$ for sufficiently large n. Note that $v(\check{c}_n(\zeta), \zeta) \to v(c_1, \zeta)$ $(n \to \infty)$ uniformly on $\overline{\mathbb{D}(r)}$.

Set $v_n(\zeta) := v(\check{c}_n(\zeta), \zeta)$ and $v(\zeta) := v(c_1, \zeta)$ for brevity. Now suppose on the contrary that

$$v_n(\zeta_n) = v_n(\zeta_n')$$

for some $\zeta_n,\ \zeta_n' \in \overline{\mathbb{D}(r)},\ \zeta_n \neq \zeta_n'$, where n ranges over a subsequence $\{n_k\}_{k=1}^{\infty}$. By taking a further subsequence, we may assume that

$$\zeta_n \to \hat{\zeta}, \quad \zeta_n' \to \hat{\zeta}' \quad \text{for } \ n = n_k,\ k \to \infty.$$

(a) When $\hat{\zeta} = \hat{\zeta}'$: Let $C_0 := v'(\hat{\zeta}) \neq 0$, then there exists a $\delta > 0$ such that

$$|v'(\zeta) - C_0| \leq \frac{|C_0|}{4} \quad \text{on } \mathbb{D}(\hat{\zeta}, \delta).$$

Since $v_n \to v$ uniformly, we have

$$|v_n'(\zeta) - C_0| \leq \frac{|C_0|}{2} \quad \text{on } \mathbb{D}(\hat{\zeta}, \delta) \text{ for } n \gg 0.$$

Hence for $\zeta,\ \zeta' \in \mathbb{D}(\hat{\zeta}, \delta)$ we have

$$\begin{aligned} \left|\{v_n(\zeta) - v_n(\zeta')\} - C_0(\zeta - \zeta')\right| &= \left|\int_{\zeta}^{\zeta'} (v_n'(\zeta) - C_0) d\zeta\right| \\ &\leq \frac{|C_0|}{2}|\zeta - \zeta'| \end{aligned}$$

It follows that

$$\frac{|C_0|}{2}|\zeta - \zeta'| \leq |v_n(\zeta) - v_n(\zeta')| \leq \frac{3}{2}|C_0||\zeta - \zeta'|.$$

In particular, v_n is injective on $\mathbb{D}(\hat{\zeta}, \delta)$ for $n \gg 0$. However, $\zeta_n,\ \zeta_n' \in \mathbb{D}(\hat{\zeta}, \delta)$ for $n \gg 0$ and this is a contradiction.
(b) When $\hat{\zeta} \neq \hat{\zeta}'$: We have

$$|v(\hat{\zeta}) - v(\hat{\zeta}')| \leq |v(\hat{\zeta}) - v(\zeta_n)| + |v(\zeta_n) - v(\zeta_n')| + |v(\zeta_n') - v(\hat{\zeta}')|.$$

As $n = n_k \to \infty$, the first and the third terms of the right hand side of this inequality tend to 0 by the continuity of v. Also by using $v_n(\zeta_n) = v_n(\zeta_n')$, we have

$$\begin{aligned} |v(\zeta_n) - v(\zeta_n')| &\leq |v(\zeta_n) - v_n(\zeta_n)| + |v_n(\zeta_n') - v(\zeta_n')| \\ &\leq 2 \sup_{\zeta \in \overline{\mathbb{D}(r)}} |v(\zeta) - v_n(\zeta)| \to 0 \quad (n = n_k \to \infty), \end{aligned}$$

since $v_n \to v$ uniformly on $\overline{\mathbb{D}(r)}$. This implies $v(\hat{\zeta}) = v(\hat{\zeta}')$, but this contradicts the univalence of v. ■(Claim)

By shrinking $U_c \equiv U_{c_1}$ slightly and using the Riemann map u of the original $U_c \equiv U_{c_1}$, the boundary of the new $U_c \equiv U_{c_1}$ is parametrized as $u^{-1}(\gamma(t))$, where $\gamma(t) = re^{2\pi it} \subset \mathbb{D}$ $(t \in [0,1])$ and $r \in (0,1)$ is close to 1. Then $\partial\widetilde{V}_c$ is parameterized as $v(c, \gamma(t))$ and hence ∂W_n (for the new W_n) is parameterized as $\check{c}_n(\gamma(t))$ by using the solution $\check{c}_n(\zeta)$ for the equation (5.1). Clearly this is a C^1 Jordan curve and W_n is the image of $\mathbb{D}(r)$ by $\check{c}_n(\zeta)$. This shows that W_n is a non-empty Jordan domain with C^1 boundary.

In particular, let $s_n := \check{c}_n(0)$ then this satisfies $\tau(s_n) + n = 0$. This means that $f_{s_n}^{k\nu n}(0) = b_{s_n}$, which implies $g_{s_n} \circ f_{s_n}^{k\nu n}(0) = P_{s_n}^{k\nu np+N}(0) = 0$. Hence P_{s_n} has a super-attracting periodic point. This completes the proof for Case 1.

Case 2 : $\nu \geq 2$. The argument is completely parallel to the Case 1. In this case the functions $\tau(c)$, $F(c,\zeta)$, $G(c,\zeta)$, $c_n(\zeta)$ and $r_n(\zeta)$ are replaced with

$$\begin{aligned}
&\tau(c) = -\frac{2\pi i}{\nu^2 B_0(c-c_1)} + h(c), \quad h(c) = O(1) \quad (c \to c_1),\\
&F(c,\zeta) = -\frac{2\pi i}{\nu^2 B_0(c-c_1)} + n - v(c_1,\zeta), \quad G(c,\zeta) = h(c) - \big(v(c,\zeta) - v(c_1,\zeta)\big),\\
&c_n(\zeta) = c_1 + \frac{2\pi i}{\nu^2 B_0(n - v(c_1,\zeta))} \quad \text{and} \quad r_n(\zeta) = \left|\frac{2\pi i}{\nu^2 B_0(n - v(c_1,\zeta))}\right| = O(n^{-1}).
\end{aligned}$$

Then we have the estimates

$$|F(c,\zeta)| = O(r_n(\zeta)^{-1+\beta}) = O(n^{1-\beta}), \quad |G(c,\zeta)| = O(1)$$

on $\partial D(c_n(\zeta), r_n(\zeta)^{1+\beta})$ for $0 < \beta < 1/2$. The rest of the argument is the same as in Case 1 and hence the same conclusion follows also in Case 2. This completes the proof. ■ (Lemma 5.1)

Definition of the family $\boldsymbol{G}$. Now we take a sufficiently large n such that the previous lemma holds. We call $s_n \in W_n$ the *center* of W_n. Let $L = L_n := k\nu n$ and $V_c' = V_{c,n}'$ be the component of $f_c^{-L}(V_c)$ containing 0 and define

$$G_c = G_{c,n} := g_c \circ f_c^L : V_c' \to U_c$$

for $c \in W_n = \{c \in S \mid f_c^L(0) \in V_c\}$. The map $f_c^L : V_c' \to V_c$ is a branched covering of degree 2 and $g_c : V_c \to U_c$ is a holomorphic isomorphism. Hence $G_c := g_c \circ f_c^L : V_c' \to U_c$ is a quadratic-like map for each $c \in W_n$ and we define a quadratic-like family $\boldsymbol{G}$ by

$$\boldsymbol{G} = \boldsymbol{G}_n := \{G_c : V_c' \to U_c\}_{c \in W_n}.$$

See Figure 4.

6. Step (P3): Proof for $\boldsymbol{G}$ being a Mandelbrot-like family

In this step we follow Douady and Hubbard's formulation to obtain a (quasiconformal) copy of M in the parameter space of a given quadratic-like family.

Mandelbrot-like families. A family of holomorphic maps $\boldsymbol{h} = \{h_\lambda\}_{\lambda \in W}$ is called a *Mandelbrot-like family* if the following (1)–(8) hold:

(1) $W \subset \mathbb{C}$ is a Jordan domain with C^1 boundary ∂W.

(2) There exists a family of maps $\Theta = \{\Theta_\lambda\}_{\lambda \in W}$ such that for every $\lambda \in W$, $\Theta_\lambda : \overline{A(R, R^2)} \to \mathbb{C}$ is a quasiconformal embedding and that $\Theta_\lambda(Z)$ is holomorphic in λ for every $Z \in \overline{A(R, R^2)}$.

(3) Define $C_\lambda := \Theta_\lambda(\partial D(R^2))$, $C'_\lambda := \Theta_\lambda(\partial D(R))$ and let U_λ (resp. U'_λ) be the Jordan domain bounded by C_λ (resp. C'_λ). Then $h_\lambda : U'_\lambda \to U_\lambda$ is a quadratic-like map with a critical point ω_λ. Also let

$$\mathcal{U} := \{(\lambda, z) \mid \lambda \in W,\ z \in U_\lambda\}, \quad \mathcal{U}' := \{(\lambda, z) \mid \lambda \in W,\ z \in U'_\lambda\}$$

then $\boldsymbol{h} : \mathcal{U} \to \mathcal{U}'$, $(\lambda, z) \mapsto (\lambda, h_\lambda(z))$ is analytic and proper.

(4) Θ is equivariant on the boundary, i.e., $\Theta_\lambda(Z^2) = h_\lambda(\Theta_\lambda(Z))$ for $Z \in \partial D(R)$.

The family of maps $\Theta = \{\Theta_\lambda\}_{\lambda \in W}$ satisfying the above conditions (1)–(4) is called a *tubing.*

(5) $\boldsymbol{h}$ extends continuously to a map $\overline{\mathcal{U}'} \to \overline{\mathcal{U}}$ and $\Theta_\lambda : (\lambda, z) \mapsto (\lambda, \Theta_\lambda(Z))$ extends continuously to a map $\overline{W} \times \overline{A(R, R^2)} \to \overline{\mathcal{U}}$ such that Θ_λ is injective on $A(R, R^2)$ for $\lambda \in \partial W$.

(6) The map $\lambda \mapsto \omega_\lambda$ extends continuously to $\overline{W}$.

(7) $h_\lambda(\omega_\lambda) \in C_\lambda$ for $\lambda \in \partial W$.

(8) *The one turn condition*: When λ ranges over ∂W making one turn, then the vector $h_\lambda(\omega_\lambda) - \omega_\lambda$ makes one turn around 0.

Now let $M_{\boldsymbol{h}}$ be the *connectedness locus* of the family $\boldsymbol{h} = \{h_\lambda\}_{\lambda \in W}$:

$$M_{\boldsymbol{h}} := \{\lambda \in W \mid K(h_\lambda) \text{ is connected}\} = \{\lambda \in W \mid \omega_\lambda \in K(h_\lambda)\}.$$

Douady and Hubbard ([**DH2**, Chapter IV]) showed that there exists a homeomorphism

$$\chi : M_{\boldsymbol{h}} \to M.$$

This is just a correspondence by the Douady-Hubbard straightening theorem, that is, for every $\lambda \in M_{\boldsymbol{h}}$ there exist a unique $c = \chi(\lambda) \in M$ such that h_λ is hybrid equivalent to $P_c(z) = z^2 + c$. Furthermore they showed that this χ can be extended to a homeomorphism $\chi_\Theta : W \to W_M$ by using $\Theta = \{\Theta_\lambda\}_{\lambda \in W}$, where W_M is a neighborhood of M given by

$$W_M := \{c \in \mathbb{C} \mid \mathcal{G}_M(c) < 2 \log R\}, \quad \mathcal{G}_M := \text{the Green function of } M.$$

In particular, $\chi_\Theta(\lambda)$ for $\lambda \in W \smallsetminus M_{\boldsymbol{h}}$ is defined in such a way that

$$\Theta_\lambda^{-1}\big(h_\lambda^k(\omega_\lambda)\big) = \{\Phi_M(\chi_\Theta(\lambda)\}^{2^{k-1}} \tag{6.1}$$

for the unique $k \in \mathbb{N}$ with $h_\lambda^k(\omega_\lambda) \in U_\lambda - U_\lambda'$. Also Lyubich showed that χ_Θ is quasiconformal on any W' with $W' \Subset W$ ([**Ly1**, p.366, THEOREM 5.5 (The QC Theorem)]).

Mandelbrot-like family with a "decorated" tubing. Now recall that there exists a canonical homeomorphism $\chi_{s_0} : \Lambda \to \chi_{s_0}(\Lambda)$ such that for any $c \in \Lambda$, $f_c : \tilde{U}_c' \to \tilde{U}_c$ is hybrid equivalent to some P_α with $\alpha = \chi_{s_0}(c)$ by the Douady-Hubbard tuning theorem. Here we will check:

Lemma 6.1. *Let $W = W_n$, $s = s_n$, and $\sigma = \chi_{s_0}(s)$. Then for sufficiently large $n \in \mathbb{N}$, there exists an $R > 1$ such that the family $\boldsymbol{G} = \{G_c : V_c' \to U_c\}_{c \in W}$ is a Mandelbrot-like family with a tubing*

$$\Theta = \{\Theta_c : \overline{A(R, R^2)} \to \overline{U_c} \smallsetminus V_c'\}_{c \in W}$$

satisfying $\Theta_c(\Gamma(\sigma)) = J(f_c)$ for any $c \in W$.

Proof. Suppose that n is sufficiently large and the quadratic-like family $\boldsymbol{G}_n$ over $W_n \subset S \cap \Lambda$ is defined as in Step (P2). We construct a tubing $\Theta = \Theta_n = \{\Theta_c\}_{c \in W_n}$ for $\boldsymbol{G}_n$ as follows: For $s_n \in W_n$, since $f_{s_n}^L(0) \in V_{s_n}$ and $f_{s_n}^j(V_{s_n}) \subset U_{s_n} \smallsetminus \overline{U_{s_n}'}$, from Lemma 4.1, we have $f_{s_n}^{L+j}(0) \notin U_{s_n}'$. It follows that $J(f_{s_n})$ is a Cantor set, which is quasiconformally homeomorphic to a quadratic Cantor Julia set $J(P_{c_0+\eta_n})$ for some $\eta = \eta_n$ with $\chi_{s_0}(s_n) = c_0 + \eta_n \notin M$ by the Douady-Hubbard straightening theorem. Let Ψ_{s_n} be the quasiconformal straightening map that conjugates f_{s_n} and $P_{c_0+\eta_n}$ defined on a neighborhood of $K(f_{s_n})$. Then the image of $J(f_{s_n})$ by Ψ_{s_n} is $J(P_{c_0+\eta_n})$. Take an $R > 1$ such that $J(P_{c_0+\eta_n}) \subset A(R^{-1/2}, R^{1/2})$. Let Γ be the rescaled Julia set, that is,

$$\Gamma := \Gamma_0(c_0 + \eta_n) = J(P_{c_0+\eta_n}) \times R^{3/2} \subset A(R, R^2).$$

Now we show the following claim:

Claim. *There exists a quasiconformal homeomorphism*

$$\Theta_n^0 : \overline{A(R, R^2)} \to \overline{U}_{s_n} \smallsetminus V_{s_n}'$$

for s_n such that

- Θ_n^0 *is quasiconformal;*
- Θ_n^0 *is equivariant on the boundary, i.e., $\Theta_n^0(Z^2) = G_{s_n}(\Theta_n^0(Z))$ for $|Z| = R$;*
- $\Theta_n^0(Z) = \Psi_{s_n}^{-1}(R^{-3/2}Z)$ *for $Z \in \Gamma_0(c_0 + \eta_n)$; and*
- $\Theta_n^0(\Gamma_0(c_0 + \eta_n)) = J(f_{s_n})$.

Proof of the Claim. Since the boundary components of the closed annuli $\overline{A(R, R^2)}$ and $\overline{U}_{s_n} \smallsetminus V_{s_n}'$ are smooth, we can take a smooth homeomorphism ψ_0 between $\partial D(R^2)$ and ∂U_{s_n}. By letting $\psi_0(Z)$ be an appropriate branch of $G_{s_n}^{-1}(\psi_0(Z^2))$ for $Z \in \partial D(R)$, we have a smooth, equivariant homeomorphism ψ_0 between the boundaries of the closed annuli.

Next we consider $\Gamma = J(P_{c_0+\eta_n}) \times R^{3/2}$. Recall that the quasiconformal (straightening) map Ψ_{s_n} sends a neighborhood of $J(f_{s_n})$ to that of $J(P_{c_0+\eta_n})$. There exists a neighborhood D^* of Γ such that the quasiconformal map $\psi_1 := \Psi_{s_n}^{-1}(R^{-3/2}Z)$ defined on D^* sends Γ to $J(f_{s_n})$. Since Γ is a Cantor set, we may choose D^* such that D^* is a finite union of smooth Jordan domains satisfying $D^* \Subset A(R, R^2)$ and $\psi_1(D^*) \Subset U_{s_n} \setminus \overline{V'_{s_n}}$. Now the sets $A(R, R^2) \setminus \overline{D^*}$ and $U_{s_n} \setminus \overline{V'_{s_n} \cup \psi_1(D^*)}$ are multiply connected domains with the same connectivity. By a standard argument in complex analysis (see [**A**, Chapter 6, Theorem 10] for example), they are conformally equivalent to round annuli with concentric circular slits, and there is a quasiconformal homeomorphism ψ_2 between these domains. Since each component of $\psi_1(D^*)$ is a quasidisk, we can modify ψ_2 such that the boundary correspondence agrees with ψ_0 and ψ_1. Hence we obtain a desired quasiconformal homeomorphism Θ_n^0 by gluing ψ_0, ψ_1, and this modified ψ_2.

■(Claim)

Let us return to the proof of Lemma 6.1. The Julia set $J(f_c) \subset U'_c \setminus \overline{V'_c}$ is a Cantor set for every $c \in W_n$ for the same reason for $J(f_{s_n})$ and this, as well as ∂U_c and $\partial V'_c$ undergo holomorphic motion (see [**S**, p.229]). By Słodkowski's theorem ([**Sł**]) there exists a holomorphic motion ι_c on $\mathbb{C}$ which induces these motions. Finally define $\Theta_c := \iota_c \circ \Theta_n^0$, then $\Theta = \Theta_n := \{\Theta_c\}_{c \in W_n}$ is a desired tubing for $\boldsymbol{G}_n$. To prove this, we have to check that $\boldsymbol{G}_n$ with Θ_n satisfies the conditions (1)–(8) for a Mandelbrot-like family. The condition (1) is already shown in Lemma 5.1. It is easy to check the conditions (2)–(7). Finally the one turn condition (8) is proved as follows: Note that $\check{c}_n(\gamma(t))$ satisfies

$$\tau(\check{c}_n(\gamma(t))) + n = v(\check{c}_n(\gamma(t)), \gamma(t))$$

When c ranges over ∂W_n making one turn, the variable t for both sides varies from $t = 0$ to $t = 1$. Since $v(\check{c}_n(\gamma(t)), \gamma(t))$ is very close to $v(c_1, \gamma(t))$, which is a parameterization of $\partial\widetilde{V}_{c_1}$ for sufficiently large n, $v(\check{c}_n(\gamma(t)), \gamma(t))$ and hence $\tau(\check{c}_n(\gamma(t))) + n$ makes one turn in a very thin tubular neighborhood of $\partial\widetilde{V}_{c_1}$ as t moves from 0 to 1. This implies that $f_c^{k\nu n}(0) = f_c^L(0)$ makes one turn in a very thin tubular neighborhood of ∂V_{c_1}. Hence $G_c(0) - 0 = G_c(0) = g_c \circ f_c^{k\nu n}(0) = g_c \circ f_c^L(0)$ makes one turn in a very thin tubular neighborhood of ∂U_{c_1}. In particular this shows that $G_c(0) - 0$ makes one turn around $0 \in U_{c_1}$. ■(Lemma 6.1)

7. Step (P4): Existence of a copy of $\mathcal{M}(c_0 + \eta)$ in W

Finally we show that the family $\boldsymbol{G}$ provides the desired quasiconformal copy of $\mathcal{M}(c_0 + \eta)$ in W.

End of the proof of Theorem 1.2. Recall that in Step (P1), we took the sector S such that $S \subset D(c_1, \varepsilon')$ for the given $\varepsilon' > 0$ in the statement. Hence $W = W_n \subset S$ is contained in $D(c_1, \varepsilon')$. Its center $s = s_n$ tends to c_1 as $n \to \infty$ by construction. Hence for any $\varepsilon > 0$ given in the statement, we may assume that $|\chi_{s_0}(s) - \chi_{s_0}(c_1)| < \varepsilon$ by taking a sufficiently large n. (Here we used the continuity of $\chi_{s_0} : \Lambda \to \chi_{s_0}(\Lambda)$.) Set $\eta := \chi_{s_0}(s) - \chi_{s_0}(c_1) = \sigma - c_0$ such that $\sigma = c_0 + \eta \in \mathbb{C} \setminus M$. Now we will show that $\mathcal{M}(\sigma)$ appears quasiconformally in $M \cap \overline{W}$.

Let $\Theta = \{\Theta_c\}_{c \in W}$ be the tubing of $\boldsymbol{G}$ given in Lemma 6.1. Let us check that the set

$$\mathcal{N} := M_{\boldsymbol{G}} \cup \{c \mid G_c^k(0) \in \Theta_c(\Gamma_0(\sigma)) \quad \text{for some } k \in \mathbb{N}\}$$

is the image of $\mathcal{M}(\sigma)$ by the quasiconformal map $\chi_\Theta^{-1} : W_R \to W$, where $M_{\boldsymbol{G}} = \chi_\Theta^{-1}(M)$ is the connectedness locus of $\boldsymbol{G}$. Indeed, by (6.1), $G_c^k(0) \in \Theta_c(\Gamma_0(\sigma))$ is equivalent to $\{\Phi_M(\chi_\Theta(c))\}^{2^{k-1}} \in \Gamma_0(\sigma)$ and thus $\Phi_M(\chi_\Theta(c)) \in \Gamma_{k-1}(\sigma)$. This implies that $\chi_\Theta(c) \in \mathcal{M}(\sigma)$.

Finally we check that $\mathcal{N} \subset M$ and $\partial\mathcal{N} \subset \partial M$. If $c \in M_{\boldsymbol{G}}$, then the orbit of the critical point 0 by $G_c = g_c \circ f_c^{k\nu n} = P_c^{k\nu np+N}$ is bounded. Hence we have $c \in M$. If c satisfies

$$G_c^k(0) \in \Theta_c(\Gamma_0(\sigma)) = J(f_c)$$

for some $k \in \mathbb{N}$, it implies $c \in M$ as well since $J(f_c)$ is invariant under $f_c = P_c^p$. So the set $\mathcal{N}$ is a subset of M. To show $\partial\mathcal{N} \subset \partial M$, it is enough to show that the Misiurewicz parameters are dense in $\partial\mathcal{N}$. They are clearly dense in $\partial M_{\boldsymbol{G}}$. Let $\mathcal{L}$ be the set of c such that $G_c^k(0)$ is a repelling periodic point in $J(f_c)$ for some $k \geq 1$. Then $\mathcal{L}$ is a dense subset of $\partial\mathcal{N} \setminus \partial M_{\boldsymbol{G}} = \mathcal{N} \setminus M_{\boldsymbol{G}}$. This completes the proof of Theorem 1.2. ■

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