Semiconjugacies between the Julia sets of geometrically finite rational maps II

Tomoki Kawahira
Graduate School of Mathematical Science
Nagoya University

Abstract

For a geometrically finite rational map f, there exists a perturbation into another geometrically finite rational map which preserves the dynamics on the Julia set nicely. Indeed, we can perturb parabolic cycles of f into attracting cycles and repelling cycles in any combination.

1 Introduction

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. We call such a map *geometrically finite* if all critical points contained in the Julia set $J(f)$ are eventually periodic.

Perturbations of f. Let us fix a small $\epsilon_0 > 0$ and consider a family of rational maps of degree d, $\{f_\epsilon : \epsilon \in [0, \epsilon_0]\}$ with $f_0 = f$ and $f_\epsilon \to f$ uniformly and continuously with respect to the spherical metric. We represent this family in the convergence form, $f_\epsilon \to f$, and call it a *perturbation* of f.

From the viewpoint of J-stability, a good perturbation $f_\epsilon \to f$ is accompanied by a conjugacy $h_\epsilon : J(f_\epsilon) \to J(f)$ for each $\epsilon \in [0, \epsilon_0]$ with $h_\epsilon^{-1} \to \mathrm{id}$ as $\epsilon \to 0$. (That is, the dynamics on the Julia set is continuously perturbed.) In [3], the author gave some conditions for this as described later in §3. In particular, it is shown that even if a perturbation of f is not accompanied by such a conjugacy, it may be accompanied by a semiconjugacy(that is, continuous and surjective map $h_\epsilon : J(f_\epsilon) \to J(f)$ with $h_\epsilon \circ f_\epsilon = f \circ h_\epsilon$) which still preserves the dynamics of the Julia set except on a countable subset.

In this paper, following the result of [3] and as an application of Shishikura's perturbation in [5], we establish:

Theorem 1.1 *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a geometrically finite rational map of degree d with $J(f) \neq \hat{\mathbb{C}}$. Then there exists a perturbation $f_\epsilon \to f$ such that*

(1) f_ϵ *is also geometrically finite;*

(2) One can choose the direction of the perturbation such that the parabolic cycles of f *are perturbed into repelling, parabolic and attracting cycles of* f_ϵ *in any combination;*

(3) For each ϵ *which is sufficiently small, there exists a unique semiconjugacy* $h_\epsilon : J(f_\epsilon) \to J(f)$ *with* $\sup\{d_{\hat{\mathbb{C}}}(h_\epsilon(x), x) : x \in J(f_\epsilon)\} \to 0$ $(\epsilon \to 0)$*; and*

(4) If $\mathrm{card}(h_\epsilon^{-1}(y)) \geq 2$ *for some* $y \in J(f)$*, then there exists an* n *such that* $f^n(y)$ *is a parabolic periodic point of* f *and* $\mathrm{card}(h_\epsilon^{-1}(y)) = \deg(f^n, y) \cdot p(f^n(y))$*, where* $\deg(f^n, y)$ *is the local degree of* f^n *at* y*, and* $p(f^n(y))$ *is the number of attracting petals at* $f^n(y)$*.*

(5) h_ϵ *is a homeomorphism (thus conjugacy) between the Julia sets if and only if none of parabolic cycles of* f *is perturbed into an attracting cycle.*

The proof is given in the following sections.

Notes.

1. The case when $J(f) = \hat{\mathbb{C}}$ is somehow easier, since f is postcritically finite. By Thurston's theorem, if the orbifold of f is not of type $(2, 2, 2, 2)$, there are only trivial perturbations by Möbius conjugations.

2. Property (3) of the theorem guarantees continuity of the Julia set along the perturbation with respect to the Hausdorff topology.

3. Property (4) of the theorem implies that the injectivity of h_ϵ may break only at countably many points. (We may say "h_ϵ is almost bijective".)

4. Our theorem generalize the results of [2]. One can find a similar result with a sophisticated investigation in [1].

Notation. Here we list some notation.

- $A(f)$: the set of all parabolic periodic points of f.
- $C(f)$: the set of all critical points of f.
- $P(f) := \overline{\{f^n(c) : c \in C(f), n = 1, 2, \ldots\}}$; the postcritical set of f.
- $CP(f) := C(f) \cup P(f)$.
- $n \gg 0$ means that $n > 0$ is sufficiently large.
- $\epsilon \ll 1$ means that $\epsilon > 0$ is sufficiently small.

2 Shishikura's perturbation

For a rational map of given degree, M.Shishikura developed the method of perturbation which makes indifferent cycles into attracting cycles, and gave a sharp estimate of the number of non-repelling cycles[5]. This method is well prepared for application, like this:

Proposition 2.1 *For geometrically finite rational map f with $J(f) \neq \hat{\mathbb{C}}$, there exists a perturbation $f_\epsilon \to f$ with properties (1) and (2) of the Theorem 1.1.*

Proof. The rest of this section is devoted for the proof of this proposition.

Assumptions. By the assumption of geometric finiteness of f and $J(f) \neq \hat{\mathbb{C}}$, there is at least one attracting or parabolic cycle. By taking a Möbius conjugacy, we may also assume that ∞ is in the Fatou set and non-periodic. (We will add more precise condition later.)

Polynomial η. Let M be the maximal number of attracting petals over all $z \in A(f)$. (That is, the maximal multiplicity of parabolic cycles is $M+1$.) We take a monic polynomial $\eta(z)$ of degree $k \gg 0$ such that

- $\eta(z) = 0$ when z is in $CP(f) \cap J(f) - A(f)$ or in an attracting cycle; and
- for a parabolic cycle $\alpha = \{z_1, \ldots, z_p\} \subset A(f)$ with m (attracting) petals,
  - $\eta(z_i) = 0$;
  - $\eta'(z_i) = 1,\ 0$, or -1 according to whether we want to perturb α into a repelling, parabolic, or attracting cycle respectively; and
  - $\eta^{(j)}(z_i) = 0$ for $2 \leq j \leq M+1$.

  Thus η has an expansion of the form

  $$\eta(z) = \sigma_\alpha(z - z_i) + a_{i,M+2}(z - z_i)^{M+2} + \cdots + (z - z_i)^k$$

  about each $z_i \in \alpha$, where $\sigma_\alpha = \eta'(z_i)$.

Quasiregular map g_ϵ. Let $\rho : [0, \infty) \to [0, 1]$ be a smooth non-increasing function such that $\rho(t) = 1$ when $t \in [0, 1]$ and $\rho(t) = 0$ when $t \in [2, \infty)$. In particular, we can take such a ρ with bounded derivative. Set $H_\epsilon(z) = z + \epsilon\eta(z)\rho(\epsilon^{1/k}|z|)$. Then $H_\epsilon : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal map if $\epsilon \ll 1$, and satisfies $H_\epsilon \to \mathrm{id}$ and its maximum dilatation tends to 0 as $\epsilon \to 0$.

Next we set $g_\epsilon := f \circ H_\epsilon$. Then

- $g_\epsilon : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiregular map of degree d with $g_\epsilon \to f$;

- $g_\epsilon(z) = f(z)$ at z in an attracting or parabolic cycle, or in $CP(f) \cap J(f)$; and

- $\deg(g_\epsilon, c) = \deg(f, c)$ at $c \in C(f) \cap J(f)$.

Perturbation of non-repelling cycles. For each attracting or parabolic cycle $\alpha = \{z_1, \ldots z_p\}$, we define two open sets $E(\alpha)$ and $E_\epsilon(\alpha)$ with $f(E(\alpha)) \subset E(\alpha)$ and $g_\epsilon(E_\epsilon(\alpha)) \subset E_\epsilon(\alpha)$ as following.

Attracting case. When α is attracting, we define $E(\alpha)$ by a disjoint union of p small topological disks near α with $f(\overline{E(\alpha)}) \subset E(\alpha)$. Set $E_\epsilon(\alpha) := E(\alpha)$. Then $g_\epsilon(\overline{E_\epsilon(\alpha)}) \subset E_\epsilon(\alpha)$ for $\epsilon \ll 1$ since g_ϵ converges uniformly to f.

Parabolic case. When α is parabolic with m (attracting) petals, we first consider the local dynamics near z_1. Set $\lambda := (f^p)'(z_1)$. (Then $\lambda^m = 1$.) It is known that we can take a conformal map $z = \psi(\zeta)$ defined near $\zeta = 0$ with $z_1 = \psi(0)$ and

$$F(\zeta) := \psi^{-1} \circ f^p \circ \psi(\zeta) = \lambda\zeta\big\{1 - \zeta^m + O(\zeta^{m+1})\big\}.$$

Fix an $r > 0$ and set $E' := \{\zeta : 0 < |\zeta| < r, |\arg \zeta^m| < \pi/3\}$. Then one can show that $F(\overline{E'}) \subset E' \cup \{0\}$ if $r \ll 1$.

On the other hand, a basic calculation shows that

$$G_\epsilon(\zeta) := \psi^{-1} \circ g_\epsilon^p \circ \psi(\zeta) = \lambda\zeta\big\{(1 + \sigma\epsilon)^p - \zeta^m + O(\epsilon\sigma\zeta) + O(\zeta^{m+1})\big\},$$

where $\sigma = \sigma_\alpha$. For $\epsilon \ll 1$, we define an open set E'_ϵ as following:

- If $\sigma = 1$, $E'_\epsilon := E' \cap \big\{|\zeta| > \epsilon^{2/(2m-1)}\big\}$;

- If $\sigma = 0$, $E'_\epsilon := E'$; and

- If $\sigma = -1$, $E'_\epsilon := E' \cup \big\{|\zeta| < \epsilon^{2/(2m-1)}\big\}$.

Then calculations as in [5, §4] show that if $\epsilon \ll 1$, $G_\epsilon(\overline{E'_\epsilon}) \subset E'_\epsilon$ when $\sigma = \pm 1$ and $G_\epsilon(\overline{E'_\epsilon}) \subset E'_\epsilon \cup \{0\}$ when $\sigma = 0$ (Figure 1).

Finally we set $E(\alpha) := \bigcup_{j=0}^{p-1} f^j(\psi(E'))$ and $E_\epsilon(\alpha) := \bigcup_{j=0}^{p-1} g_\epsilon^j(\psi(E'_\epsilon))$. Clearly they have the properties we desired.

Assumptions on infinity (Again). Set $E := \bigcup_\alpha E(\alpha)$ and $E_\epsilon := \bigcup_\alpha E_\epsilon(\alpha)$, where α ranges over all non-repelling cycles of f. Since there is at least one attracting or parabolic cycle in the Fatou set, E and E_ϵ are non-empty. Now we may assume that $\infty \in f^{-1}(E) - \overline{E}$ by taking a Möbius conjugacy. Then E and E_ϵ are uniformly bounded if $\epsilon \ll 1$. We have that $f(\infty) \in E$ and $g_\epsilon(\infty) \in E_\epsilon$ for all $\epsilon \ll 1$.

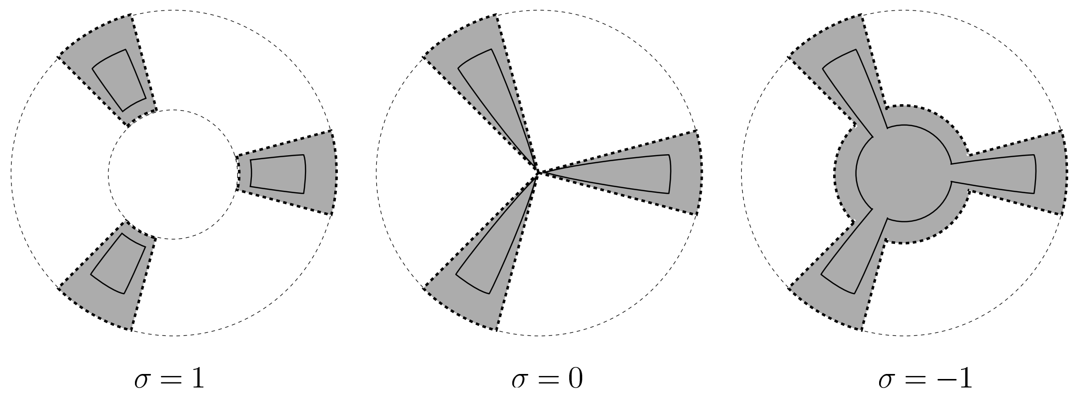

Figure 1: The case of $m = 3$. The shadowed region is E'_ϵ, with the boundary of $G_\epsilon(E'_\epsilon)$ approximately drawn in.

Getting rational perturbation. (See Lemma 3 of [5].) Now the quasiregular map g_ϵ is holomorphic except $V_\epsilon := \{z : |z| > \epsilon^{-1/k}\}$. Note that $f(V_\epsilon) \subset E$ and $g_\epsilon(V_\epsilon) \subset E_\epsilon$ if $\epsilon \ll 1$ by assumptions. Let σ_0 denote the standard complex structure on $\hat{\mathbb{C}}$. For $\epsilon \ll 1$, we put an almost complex structure σ_ϵ defined by $(g_\epsilon^n)^*(\sigma_0)$ on $g_\epsilon^{-n}(E_\epsilon)$ and by σ_0 otherwise. Then σ_ϵ is g_ϵ-invariant and we can find a quasiconformal map Φ_ϵ such that $\Phi_\epsilon^* \sigma_0 = \sigma_\epsilon$ a.e., and thus $f_\epsilon := \Phi_\epsilon \circ g_\epsilon \circ \Phi_\epsilon^{-1}$ is a rational map of degree d. Since $\Phi_\epsilon \to \mathrm{id}$ by the definition of g_ϵ and the continuous dependence of Φ_ϵ on its Beltrami differential, we obtain a perturbation $f_\epsilon \to f$ with (1) and (2) of Theorem 1.1 ■

Remarks.

- If f is a polynomial, $f_\epsilon \to f$ preserves the superattracting fixed point of degree d. Thus we actually have a polynomial perturbation.
- Perturbation of indifferent cycles as in [5, §4] does not guarantee continuity of the global dynamics in general. However, if f is geometrically finite, we have continuity of the dynamics at least on the Julia sets. (We need extra care of critical orbits, though.)

3 Existence of the semiconjugacy

Here we check that the perturbation $f_\epsilon \to f$ given in the previous section is accompanied by semiconjugacy as in Theorem 1.1. This is an application of a theorem from [3]. To state the theorem we introduce the notions of horocyclic perturbation and J-critical relations.

Horocyclic perturbation. We say a perturbation $f_\epsilon \to f$ is *horocyclic* if each parabolic periodic point a of f with m petals satisfies the following:

(a) If $f^p(a) = a$ and $(f^p)'(a) = \lambda$ (thus $\lambda^m = 1$), there exists a_ϵ with $f_\epsilon^p(a_\epsilon) = a_\epsilon \to a$, $(f_\epsilon^p)'(a_\epsilon) = \lambda_\epsilon \to \lambda$ as $\epsilon \to 0$;

(b) There is a neighborhood D of a with local coordinates $\phi_\epsilon,\ \phi : D \to \mathbb{C}$ such that:

1. $a_\epsilon \in D$ and $\phi_\epsilon(a_\epsilon) = \phi(a) = 0$;
2. $\phi_\epsilon \to \phi$ uniformly on D; and
3. If we represent the actions of f_ϵ^{pm} and f^{pm} on D by ϕ_ϵ and ϕ respectively, we obtain the local representation of the perturbation as:

$$\begin{aligned} & \phi_\epsilon \circ f_\epsilon^{pm} \circ \phi_\epsilon^{-1}(z) = \lambda_\epsilon^m z + z^{m+1} + O(z^{m+2}) \\ \longrightarrow \quad & \phi \circ f^{pm} \circ \phi^{-1}(z) = z + z^{m+1} + O(z^{m+2}). \end{aligned} \tag{3.1}$$

In particular, $\phi,\ \phi_\epsilon$ are not necessarily conformal; they can be just homeomorphisms from D to their images.

(c) If we set $\exp(L_\epsilon + i\theta_\epsilon) := \lambda_\epsilon^m$, which tends to 1 as $\epsilon \to 0$, then $\theta_\epsilon^2 = o(|L_\epsilon|)$ as $L_\epsilon,\ \theta_\epsilon \to 0$.

Horocyclic perturbation was originally defined as *horocyclic convergence* of rational maps by C. McMullen[4, §7-9].

J-critical relations. Let $c_1, \ldots, c_N$ be all critical points of f contained in $J(f)$, where N is counted *without* multiplicity. A *J-critical relation of f* is a set of non-negative integers (i, j, m, n) such that $f^m(c_i) = f^n(c_j)$.

We say a perturbation $f_\epsilon \to f$ *preserves the J-critical relations of f* if:

- For all $i = 1, \ldots, N$, the maps f_ϵ have critical points $c_i(\epsilon)$ (may be in the Fatou set) satisfying $c_i(\epsilon) \to c_i$ and $\deg(f_\epsilon, c_i(\epsilon)) = \deg(f, c_i)$ as $\epsilon \to 0$; and
- For each J-critical relation (i, j, m, n) of f, f_ϵ satisfies $f_\epsilon^m(c_i(\epsilon)) = f_\epsilon^n(c_j(\epsilon))$.

For such a perturbation $f_\epsilon \to f$, if f is geometrically finite, then the maps f_ϵ are also geometrically finite. If f is hyperbolic or parabolic, then $C(f) \cap J(f) = \emptyset$ and any small perturbation of f automatically preserves its J-critical relations.

Now we can state the following theorem:

Theorem 3.1 ([3]) *Let f be a geometrically finite rational map of degree ≥ 2, and $f_\epsilon \to f$ a horocyclic perturbation which preserves the J-critical relations of f. Then for each $\epsilon \ll 1$, we have a unique semiconjugacy $h_\epsilon : J(f_\epsilon) \to J(f)$ with properties (3), (4) and (5) of Theorem 1.1.*

Hence it is enough to show that $f_\epsilon \to f$ in the previous section is horocyclic and preserving the J-critical relations of f. In particular, since the J-critical relations are clearly preserved by the construction of f_ϵ, we only need to check each condition of horocyclic perturbation.

Condition (a) is clear by definition. Condition (c) is also easy since $\lambda_\epsilon^m = (1 \pm \sigma\epsilon)^{pm} \in \mathbb{R}$ and $\theta_\epsilon = 0$. For condition (b), since $\Phi_\epsilon \circ \psi$ converges to ψ uniformly near parabolic points of f, it is enough to consider $G_\epsilon \to F$ in the previous section. Now we complete the proof of Theorem 1.1 by claiming this:

Proposition 3.1 *By taking suitable local coordinates near* 0*, the convergence* $G_\epsilon^m \to F^m$ *can be viewed as in (3.1).*

Proof. (See also Propositions 7.1 and 7.2 of [4].) First let us check that the original $G_\epsilon^m \to F^m$ has the form

$$\begin{aligned} & G_\epsilon^m(\zeta) = \lambda_\epsilon^m \zeta + O(\sigma\epsilon\zeta^2) + C\zeta^{m+1} + O(\zeta^{m+2}) \\ \longrightarrow \quad & F^m(\zeta) = \zeta + C\zeta^{m+1} + O(\zeta^{m+2}) \end{aligned}$$

with $C = -m$. For $j \geq 1$, we may set

$$\begin{aligned} & G_\epsilon^j(\zeta) = \lambda_\epsilon^j \zeta + O(\sigma\epsilon\zeta^2) + C_j'\zeta^{m+1} + O(\zeta^{m+2}) \quad \text{and} \\ & F^j(\zeta) = \lambda^j \zeta + C_j\zeta^{m+1} + O(\zeta^{m+2}) \end{aligned}$$

with $C_1' = C_1 = -\lambda$. By comparing $G_\epsilon \circ G_\epsilon^j$ and $F \circ F^j$ with $G_\epsilon^j \circ G_\epsilon$ and F^{j+1} respectively, we have

$$C_j' = \lambda\lambda_\epsilon^{j-1} \cdot \frac{\lambda_\epsilon^{jm} - 1}{\lambda_\epsilon^m - 1} = -j\lambda^j(1 + O(\sigma\epsilon))$$

and $C_j = -j\lambda^j$. By putting m into j, we have the form of convergence as above. If $\sigma = 0$ then C can be normalized to be 1 by making a linear coordinate change $\zeta \mapsto Z = C^{1/m}\zeta$. Thus we consider the case of $\sigma = \pm 1$.

Now G_ϵ^m has the form

$$G_\epsilon^m(\zeta) = \lambda_\epsilon^m \zeta + A_\epsilon \zeta^N + O(\epsilon\zeta^{N+1}) + C\zeta^{m+1} + O(\zeta^{m+2})$$

where $2 \leq N \leq m$ and $A_\epsilon = O(\epsilon)$. Set $\Lambda_\epsilon := \lambda_\epsilon^m = 1 \pm mp\epsilon + O(\epsilon^2)$. Note that Λ_ϵ and A_ϵ are analytic functions of ϵ by the definitions of g_ϵ and G_ϵ. Consider a local coordinate

$$Z = \Psi_\epsilon(\zeta) = \zeta - B_\epsilon\zeta^N \quad \text{with} \quad B_\epsilon = \frac{A_\epsilon}{\Lambda_\epsilon(\Lambda_\epsilon^{N-1} - 1)}.$$

Then Ψ_ϵ converges uniformly to another coordinate change $\Psi(\zeta) = \zeta - B\zeta^N$ near 0 as $\epsilon \to 0$. Let us check that $G_\epsilon^m \to F^m$ is locally represented as

$$\begin{aligned} & \Psi_\epsilon \circ G_\epsilon^m \circ \Psi_\epsilon^{-1}(Z) = \Lambda_\epsilon Z + O(\epsilon Z^{N+1}) + CZ^{m+1} + O(Z^{m+2}) \\ \longrightarrow \quad & \Psi \circ F^m \circ \Psi^{-1}(Z) = Z + CZ^{m+1} + O(Z^{m+2}). \end{aligned}$$

One can easily check the form of $\Psi\circ F^m\circ\Psi^{-1}$. For $\Psi_\epsilon\circ G_\epsilon^m\circ\Psi_\epsilon^{-1}(Z)$, set $\tilde{G}:=G_\epsilon^m$. Note that $\tilde{G}(\zeta)=\Lambda_\epsilon\zeta+O(\epsilon\zeta^N)+O(\zeta^{m+1})$ and thus

$$\begin{aligned}\tilde{G}'(\zeta) &= \Lambda_\epsilon+O(\epsilon\zeta^{N-1})+O(\zeta^m);\\ \tilde{G}^{(j)}(\zeta) &= O(\epsilon)+O(\zeta^{m-j+1}) \quad \text{for } 2\le j\le m; \text{ and}\\ \tilde{G}^{(j)}(\zeta) &= O(1) \quad \text{for } j\ge m+1.\end{aligned}$$

By a careful calculation, we have

$$\begin{aligned}\tilde{G}(\zeta) &= \tilde{G}'(Z+B_\epsilon\zeta^N)\\ &= \tilde{G}(Z)+\tilde{G}'(Z)B_\epsilon\zeta^N+\cdots+\frac{\tilde{G}^{(j)}(Z)}{j!}(B_\epsilon\zeta^N)^j+\cdots\\ &= \Lambda_\epsilon Z+A_\epsilon Z^N+\Lambda_\epsilon B_\epsilon\zeta^N+O(\epsilon Z^{N+1})+CZ^{m+1}+O(Z^{m+2}) \qquad (3.2)\end{aligned}$$

and

$$\begin{aligned}\tilde{G}(\zeta)^N &= (\Lambda_\epsilon\zeta)^N\big(1+O(\epsilon\zeta^{N-1})+O(\zeta^m)\big)^N\\ &= \Lambda_\epsilon^N\zeta^N+O(\epsilon\zeta^{2N-1})+O(\zeta^{m+N})\\ &= \Lambda_\epsilon^N\zeta^N+O(\epsilon Z^{N+1})+O(Z^{m+2}). \qquad (3.3)\end{aligned}$$

By (3.2) and (3.3), we have

$$\begin{aligned}\Psi\circ\tilde{G}(\zeta) &= \tilde{G}(\zeta)-B_\epsilon\tilde{G}(\zeta)^N\\ &= \Lambda_\epsilon Z+A_\epsilon(Z^N-\zeta^N)+O(\epsilon Z^{N+1})+CZ^{m+1}+O(Z^{m+2}).\end{aligned}$$

Since $Z^N-\zeta^N=O(\zeta^{2N-1})=O(Z^{N+1})$ and $A_\epsilon=O(\epsilon)$, we have the form of $\Psi_\epsilon\circ G_\epsilon^m\circ\Psi_\epsilon^{-1}(Z)=\Psi\circ\tilde{G}(\zeta)$ as desired.

We can iterate this coordinate change until the coefficient of Z^m vanishes, and final linear coordinate changes normalize the coefficients of Z^{m+1} to be 1. Now we obtain the convergence of the form (3.1). ■

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