Semiconjugacies between the Julia sets of geometrically finite rational maps

Tomoki Kawahira *
University of Tokyo

October 22, 2003

Abstract

A rational map f is called *geometrically finite* if every critical point contained in its Julia set is eventually periodic. If a perturbation of f into another geometrically finite rational map is horocyclic and preserves the critical orbit relations with respect to the Julia set of f, then we can construct a semiconjugacy or a topological conjugacy between their dynamics on the Julia sets.

1 Introduction

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. We call such a map *geometrically finite* if all critical points contained in the Julia set $J(f)$ are eventually periodic. A geometrically finite rational map can have (super)attracting and parabolic basins, but no Siegel disks or Herman rings. In particular, if a rational map is (sub)hyperbolic or parabolic, then it is geometrically finite.

In this paper, we discuss perturbations of a geometrically finite rational map f within Rat_d, the space of all rational maps of degree d. The topology of this space is defined by uniform convergence on the sphere with respect to the spherical distance $d_\sigma(\cdot,\cdot)$. Our aim is to study the dynamical stability of f on its Julia set; that is, structural stability of f restricted on the Julia set.

Perturbations of f. Let us consider a family of rational maps of degree $d \geq 2$, $\{f_\epsilon \in \mathrm{Rat}_d : \ \epsilon \in [0,1]\}$ with the following conditions:

- $f_0 = f$; and

*Research partially supported by JSPS Research Fellowships for Young Scientists. Appeared in *Erg. Th. & Dyn. Sys.* **23**(2003), 1125–1152.

- $\sup_{x\in\hat{\mathbb{C}}} d_\sigma(f_\epsilon(x), f(x)) \to 0$ as $\epsilon \searrow 0$.

We represent this family in the convergence form, $f_\epsilon \to f$, and call it a *perturbation* of f.

For this perturbation $f_\epsilon \to f$, let us consider whether the dynamics on $J(f)$ is perturbed continuously to that on $J(f_\epsilon)$. More precisely, we consider the existence of a map $h_\epsilon : J(f_\epsilon) \to J(f)$ for each $\epsilon \in [0, 1]$ such that

- h_ϵ is a homeomorphism with $h_\epsilon \circ f_\epsilon = f \circ h_\epsilon$ on $J(f_\epsilon)$; and
- $h_\epsilon^{-1} : J(f) \to J(f_\epsilon)$ tends to $\mathrm{id} : J(f) \to J(f)$ as $\epsilon \to 0$.

Such an h_ϵ with the first condition is called a *(topological) conjugacy* between f_ϵ and f on their respective Julia sets. In addition, for the first condition, if h_ϵ is not a homeomorphism but merely continuous and surjective, then such an h_ϵ is called a *semiconjugacy* between f_ϵ and f on their respective Julia sets.

By the Mañé-Sad-Sullivan theory[15], if f has a connected neighborhood $U \subset \mathrm{Rat}_d$ where each $f_\epsilon \in U$ has the same number of attracting cycles as f, then for each $f_\epsilon \in U$ there exists a unique quasiconformal conjugacy $h_\epsilon : J(f_\epsilon) \to J(f)$ as above. This means any small perturbations of f have desired conjugacies. For example, hyperbolic rational maps have this property.

On the other hand, when f is geometrically finite f can have parabolic cycles: As we will describe, those parabolic cycles may change into attracting cycles under some perturbations. Thus the number of attracting cycles may change and we cannot apply the Mañé-Sad-Sullivan theory. Moreover, by a perturbation of parabolic cycles into attracting cycles, the topology of $J(f)$ may change and we cannot even hope that $J(f)$ and $J(f_\epsilon)$ are homeomorphic in general.

However, in our main theorem (Theorem 1.1), we will give a sufficient condition for perturbations $f_\epsilon \to f$ to be accompanied by such conjugacies as above or best possible semiconjugacies between the dynamics on their Julia sets.

Parabolic points. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$, and let a be a periodic point of f with period l and multiplier $(f^l)'(a) =: \lambda$. We say a is a *parabolic (periodic) point* if λ is a root of unity.

Now let us suppose that a is a parabolic point and λ is a primitive q-th root of unity. Taking a local coordinate near a which maps a to 0, we obtain

$$f^{lq}(z) = z + A_{p+1}z^{p+1} + O(z^{p+2}) \tag{1.0}$$

with $A_{p+1} \neq 0$ and $p \geq 1$. (Moreover, we can normalize A_{p+1} to be 1 by using a linear transformation.) It is known that p is a multiple of q which does not depend on the choice of local coordinates. We call $p = p(a)$ the *petal number* of a. We also say that *a has p petals*.

Note that a is a fixed point of f^{lq} of multiplicity $p+1$. By a perturbation of f into f_ϵ, a splits into $p+1$ fixed points of f_ϵ^{lq} counting with multiplicity. This may cause drastic change of the dynamics, so we have to control the perturbation in order to change the original dynamics tamely.

Horocyclic perturbations. After C. McMullen, we say a perturbation $f_\epsilon \to f$ is *horocyclic* if each parabolic point a of f as above satisfies the following:

(a) There are fixed points a_ϵ of f_ϵ^l with multipliers $(f_\epsilon^l)'(a_\epsilon) = \lambda_\epsilon$ satisfying $a_\epsilon \to a$ and $\lambda_\epsilon \to \lambda$;

(b) There is a neighborhood D of a with local coordinates $\phi_\epsilon,\ \phi : D \to \mathbb{C}$ such that:

1. $a_\epsilon \in D$ and $\phi_\epsilon(a_\epsilon) = \phi(a) = 0$;
2. $\phi_\epsilon \to \phi$ uniformly on D; and
3. If we represent the actions of f_ϵ^{lq} and f^{lq} on D by ϕ_ϵ and ϕ respectively, we obtain the local representation of the perturbation as:

$$f_\epsilon^{lq}(z) = \lambda_\epsilon^q z + z^{p+1} + O(z^{p+2}) \to f^{lq}(z) = z + z^{p+1} + O(z^{p+2}). \quad (1.1)$$

(c) If we set $\exp(L_\epsilon + i\theta_\epsilon) := \lambda_\epsilon^q$, which tends to 1 as $\epsilon \to 0$, then $\theta_\epsilon^2 = o(|L_\epsilon|)$ as $L_\epsilon,\ \theta_\epsilon \to 0$.

Form (1.1) implies that the symmetry of the local dynamics near a is preserved by the perturbation. In particular, $\phi,\ \phi_\epsilon$ are not necessarily conformal, can be just homeomorphisms from D to their images. By condition (c), a avoids being perturbed into an irrationally indifferent periodic point. See §2 for more details.

Horocyclic perturbation was originally defined as *horocyclic convergence* of rational maps, to study the continuity of the Hausdorff dimensions of the Julia sets of geometrically finite rational maps[12, §7-9].

J-critical relations. A geometrically finite rational map may have critical points in its Julia set. Here we introduce a condition which controls the perturbations of the orbits of such critical points.

Let $c_1, \dots, c_N$ be all critical points of f contained in $J(f)$, where N is counted *without* multiplicity. A *J-critical relation of f* is a set of non-negative integers (i, j, m, n) such that $f^m(c_i) = f^n(c_j)$.

Let $\deg(f, x)$ denote the local degree of f at x. We say a perturbation $f_\epsilon \to f$ *preserves the J-critical relations of f* if:

- For all $i = 1, \dots, N$, the maps f_ϵ have critical points $c_i(\epsilon)$ (may be in the Fatou set) satisfying $c_i(\epsilon) \to c_i$ and $\deg(f_\epsilon, c_i(\epsilon)) = \deg(f, c_i)$ as $\epsilon \to 0$; and
- For each J-critical relation (i, j, m, n) of f, f_ϵ satisfies $f_\epsilon^m(c_i(\epsilon)) = f_\epsilon^n(c_j(\epsilon))$.

If f is geometrically finite, then the maps f_ϵ are also geometrically finite. If f is hyperbolic or parabolic, then $C(f) \cap J(f) = \emptyset$ and any small perturbation of f automatically preserves its J-critical relations.

Our main result is:

Theorem 1.1 *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a geometrically finite rational map of degree d, and $f_\epsilon \to f$ a horocyclic perturbation which preserves the J-critical relations of f.*

For each ϵ which is sufficiently small, there exists a unique semiconjugacy $h_\epsilon : J(f_\epsilon) \to J(f)$ with the following properties:

1. *If $\mathrm{card}(h_\epsilon^{-1}(y)) \geq 2$ for some $y \in J(f)$, then there exists an n such that $f^n(y)$ is a parabolic point of f and $\mathrm{card}(h_\epsilon^{-1}(y)) = \deg(f^n, y) \cdot p(f^n(y))$.*

2. *h_ϵ can be arbitrarily close to the identity on $J(f_\epsilon)$. That is, if we fix an arbitrarily small $r > 0$, then for all sufficiently small ϵ, h_ϵ satisfies*

$$\sup \{d_\sigma(h_\epsilon(x), x) : x \in J(f_\epsilon)\} < r.$$

Property 1 implies that the injectivity of h_ϵ may break on the backward orbits of parabolic points of f. Since such points are countable, we say that h_ϵ is *almost bijective*. However, even though f has parabolic points, h_ϵ can give a topological conjugacy. The precise condition for this is described in Corollary 7.1. In addition, Property 2 implies:

Corollary 1.1 *For $f_\epsilon \to f$ as above, $J(f_\epsilon)$ converges to $J(f)$ in the Hausdorff topology.*

For a given geometrically finite rational map, the existence of such perturbations is guaranteed by [10].

Example 1. Let us consider perturbations of a geometrically finite map $f(z) = z(1+z)^m$ with $m \geq 2$. Now -1 is a preparabolic critical point and 0 is a parabolic fixed point with one petal. Here are two typical perturbations:

- $f_\epsilon(z) = \lambda_\epsilon z(1+z)^m$ with real $\lambda_\epsilon \searrow 1$
- $f_\epsilon(z) = \lambda_\epsilon z(1+z)^m$ with real $\lambda_\epsilon \nearrow 1$

For both cases, 0 is split into a pair of attracting and repelling fixed points, 0 and $-1 + 1/\sqrt[m]{\lambda_\epsilon}$. For the first case, 0 is the repelling one, and for the second case, the attracting one. In Figure 1, curves roughly show the shape of the Julia sets for $m = 3$. These split fixed points and their first preimages are shown by heavy dots.

Both two perturbations are horocyclic and preserving the J-critical relations of f. For the first case, we obtain h_ϵ as a topological conjugacy. For the second case, h_ϵ is a semiconjugacy which pinches the backward images of $-1 + 1/\sqrt[m]{\lambda_\epsilon}$ onto those of 0. The injectivity is broken only at these points.

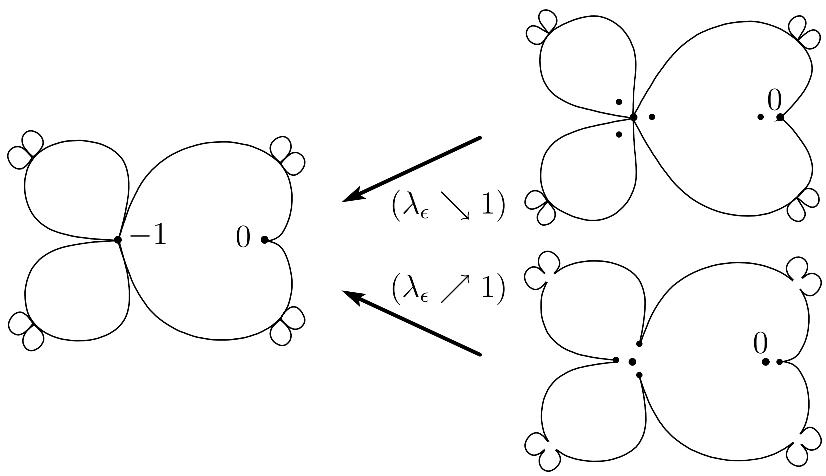

Figure 1: The perturbations $f_\epsilon(z) = \lambda_\epsilon z(1+z)^3$ with real $\lambda_\epsilon \to 1$

Remark on the Goldberg-Milnor conjecture. Theorem 1.1 gives a partial and affirmative answer to the following Goldberg-Milnor conjecture[6]: *For a polynomial f which has a parabolic cycle, there exists a small perturbation of f such that*

- *the immediate basin of the parabolic cycle is converted to basins of some attracting cycles; and*
- *the perturbed polynomial on its Julia set is topologically conjugate to the original polynomial f on $J(f)$.*

Some horocyclic perturbations of a geometrically finite polynomial explicitly give such perturbations. For example, the first perturbation in Example 1 gives an affirmative answer to this conjecture for $f(z) = z(1+z)^m$.

In general, any geometrically finite rational map has such a perturbation. See [10]. For other partial solutions of this conjecture, see [3] and [7].

Example 2. Let us consider a Blaschke product $f(z) = (3z^2+1)/(3+z^2)$ with a parabolic fixed point at $z = 1$, which has 2 petals. The critical points of f are 0 and ∞. The Julia set is the unit circle and the Fatou set is the parabolic basin of $z = 1$.

Let us consider perturbations of f of the form

$$f_\epsilon(z) = \frac{(2+\lambda_\epsilon)z^2 + 2 - \lambda_\epsilon}{2 + \lambda_\epsilon + (2-\lambda_\epsilon)z^2} \quad \text{with real} \quad \lambda_\epsilon \to 1.$$

For $\epsilon \ll 1$, f_ϵ are also Blaschke products and the Julia sets are contained in the unit circle. By this perturbation, the parabolic point $z = 1$ of f splits into the following three fixed points (counting with multiplicity): $z_0 = 1$ with multiplier

λ_ϵ, $z_1 = (-\lambda_\epsilon + 2\sqrt{-1+\lambda_\epsilon})/(-2+\lambda_\epsilon)$ and $z_2 = (-\lambda_\epsilon - 2\sqrt{-1+\lambda_\epsilon})/(-2+\lambda_\epsilon)$ with the same multipliers $-1 + 2/\lambda_\epsilon$.

Now consider the case of real λ_ϵ with (a)$\lambda_\epsilon \searrow 1$ or (b)$\lambda_\epsilon \nearrow 1$. For each cases, one can check that $f_\epsilon \to f$ is a horocyclic perturbation.

When (a), $z_0 = 1$ is repelling and z_1, z_2 are attracting. The Julia set of f_ϵ is also the unit circle. By Theorem 1.1, there is a conjugacy between f_ϵ and f on the unit circle.

When (b), $z_0 = 1$ is attracting and z_1, z_2 are repelling. The Julia set of f_ϵ is a Cantor set contained in the unit circle. By Theorem 1.1, there is a semiconjugacy between f_ϵ and f on their respective Julia sets. Note that the semiconjugacy maps a Cantor set *onto* the unit circle.

Sketch of the proof of the main theorem. Let us roughly sketch the proof of Theorem 1.1; the construction of the semiconjugacy between f_ϵ and f on their respective Julia sets.

Let f be a geometrically finite rational map and let $f_\epsilon \to f$ be a horocyclic perturbation which preserves the J-critical relations of f. We investigate the properties of such a perturbation in §2.

In §3, we prepare the ingredients for the semiconjugacy. For f, we construct a compact set Ω such that $J(f) \subset \Omega \subset f(\Omega)$. Correspondingly, for each fixed f_ϵ, we construct a compact set Ω_ϵ such that $J(f_\epsilon) \subset \Omega_\epsilon \subset f_\epsilon(\Omega_\epsilon)$. We also construct a certain surjective map $h_0 (= h_{0,\epsilon}) : \Omega_\epsilon \to \Omega$ as the "0-th" step to the semiconjugacy.

Then in §4, we inductively construct a sequence of "lifts"

$$\left\{h_n (= h_{n,\epsilon}) : f_\epsilon^{-n}(\Omega_\epsilon) \to f^{-n}(\Omega)\right\}_{n=1}^{\infty}$$

satisfying $f \circ h_{n+1} = h_n \circ f_\epsilon$. In §5, we investigate the expanding property of f; in other words, the contracting property of f^{-1}. By using this property, in §6, we show that $\{h_n\}$ converges uniformly to a surjective map h_ϵ on $J(f_\epsilon)$ if $\epsilon \ll 1$.

$$\begin{array}{ccc}
\vdots & & \vdots \\
{\scriptstyle f_\epsilon}\downarrow & & \downarrow{\scriptstyle f} \\
f_\epsilon^{-2}(\Omega_\epsilon) & \xrightarrow{h_2} & f^{-2}(\Omega) \\
{\scriptstyle f_\epsilon}\downarrow & & \downarrow{\scriptstyle f} \\
f_\epsilon^{-1}(\Omega_\epsilon) & \xrightarrow{h_1} & f^{-1}(\Omega) \\
{\scriptstyle f_\epsilon}\downarrow & & \downarrow{\scriptstyle f} \\
\Omega_\epsilon & \xrightarrow{h_0} & \Omega
\end{array}
\qquad\qquad
\begin{array}{ccc}
J(f_\epsilon) & \xrightarrow{h_\epsilon} & J(f) \\
{\scriptstyle f_\epsilon}\downarrow & & \downarrow{\scriptstyle f} \\
J(f_\epsilon) & \xrightarrow{h_\epsilon} & J(f)
\end{array}$$

In §7, we check that h_ϵ satisfies the properties in Theorem 1.1. To simplify the argument, from §3 to §7, we suppose that $J(f) \neq \hat{\mathbb{C}}$. The case of $J(f) = \hat{\mathbb{C}}$ is treated in §8.

Notes.

1. For the basic properties of the Julia sets and parabolic points, refer to [1], [2] and [14], etc.

2. If f is hyperbolic, we obtain h_ϵ as a topological conjugacy. In particular, by uniqueness, h_ϵ coincides with the quasiconformal conjugacy obtained by using λ-Lemma in [15]. In general, for a perturbation $f_\epsilon \to f$ as Theorem 1.1, if each f_ϵ for $\epsilon \in (0,1]$ is hyperbolic, then each h_ϵ is characterized as a uniform limit of quasiconformal conjugacies.

3. If a rational map f has no Siegel disks or Herman rings and $f_\epsilon \to f$ horocyclically, it is known that $J(f_\epsilon) \to J(f)$ in the Hausdorff topology[8],[12, Theorem 9.1]. Corollary 1.1 gives another proof of this fact in a special case by using the existence of the semiconjugacy.

4. Theorem 1.1 is an improvement of an author's result on horocyclic perturbation of parabolic rational maps in [9] or [8].

Notation. Here we list some notation used throughout this paper.

- $\sigma := 2|dz|/(1+|z|^2)$ is the spherical metric on the Riemann sphere $\hat{\mathbb{C}}$.
- $d_\sigma(\cdot,\cdot)$: the spherical distance measured in σ.
- $B_\sigma(x,r) := \{y \in \hat{\mathbb{C}} : d_\sigma(x,y) < r\}$
- $F(f)$: the Fatou set of f
- $C(f)$: the set of all critical points of f.
- $P(f) := \overline{\{f^n(c) : c \in C(f), n = 1,2,\ldots\}}$; the postcritical set of f.
- For any map f, f^0 denotes the identity map on the domain of f.
- $n \gg 0$ means that $n > 0$ is sufficiently large.
- $\epsilon \ll 1$ means that $\epsilon > 0$ is sufficiently small.

2 Horocyclic perturbations

Bifurcations of parabolic periodic points have a strong effect on the local dynamics as well as the global dynamics. In this section, we describe a horocyclic perturbation $f_\epsilon \to f$ of a geometrically finite rational map f in further detail. In particular, we introduce the notion of planet and satellite for periodic points generated by perturbation of parabolic points. Roughly speaking, a planet is the central periodic point which determines the properties of the perturbed local dynamics. Satellites accompany a planet. Moreover, we will show a key lemma on horocyclic perturbation (Lemma 2.1), and see the local dynamics near parabolic points change tamely under such perturbations.

2.1 Planets and satellites.

First we consider condition (b)-3 of horocyclic perturbation. Let a be a parabolic point of f as in the preceding section, which has a local representation as (1.0).

As we will see afterward, condition (b)-3 is important to keep the original symmetry of the local dynamics for the petals of a. However, if we suppose only conditions (a), (b)-1 and (b)-2 for $f_\epsilon \to f$, we just obtain a local representation of the convergence near a as the following:

$$\begin{aligned} f_\epsilon^{lq}(z) &= \lambda_\epsilon^q z + A_{\epsilon,r} z^r + \cdots + A_{\epsilon,p+1} z^{p+1} + O(z^{p+2}) \\ \to f^{lq}(z) &= z + A_{p+1} z^{p+1} + O(z^{p+2}) \quad (\epsilon \to 0), \end{aligned} \tag{2.1}$$

where $2 \leq r \leq p$. In [12, §7], C. McMullen gave some conditions which insure form (2.1) becomes form (1.1) by taking suitable local coordinates. One of such conditions is:

Proposition 2.1 *For the form (2.1) above, if $A_{\epsilon,i}/(\lambda_\epsilon^q - 1)$ converges as $\epsilon \to 0$ for each $r \leq i \leq p$, then through a continuous change of coordinates near a, we obtain the normalized form of the convergence as (1.1).*

For example, if $p = q$ then $A_{\epsilon,i}$ satisfies this condition. See [12, Proposition 7.2 and 7.3] for more details.

Planets and satellites. Next, we consider the effect of condition (c) of horocyclic perturbation. Let $f_\epsilon \to f$ be a horocyclic perturbation. Now $\lambda_\epsilon^q = \exp(L_\epsilon + i\theta_\epsilon)$, with the assumption that $\theta_\epsilon^2 = o(|L_\epsilon|)$ as $L_\epsilon,\ \theta_\epsilon \to 0$. By this relation, $L_\epsilon = 0$ implies $\theta_\epsilon = 0$. In other words, if $|\lambda_\epsilon^q| = 1$ then a_ϵ is persistently a parabolic point of f_ϵ with the same multiplier λ as a. This means, perturbations of a into another kind of indifferent periodic point are prohibited.

Let us look the relation $\theta_\epsilon^2 = o(|L_\epsilon|)$ in the complex plane. If we fix a pair of arbitrarily small closed disks on the both sides of the imaginary axis, so that they are tangent to the axis at the origin, then they contain $L_\epsilon + i\theta_\epsilon$ for all $\epsilon \ll 1$. Thus $L_\epsilon + i\theta_\epsilon$ cannot converge to 0 along the imaginary axis, but can converge along a curve tangent to the imaginary axis with order < 2.

From (1.1), the solutions of the equation $f_\epsilon^{lq}(z) = z$ near the origin are $z = 0$ and $z \approx (1 - \lambda_\epsilon^q)^{1/p}$ and they correspond to the symmetrically arrayed fixed points of f_ϵ^{lq} generated by the perturbation of a (See Figure 2). We classify them into two types: *planet* and *satellite*.

First, we consider the case of multiple petals: That is, $p \geq 2$. Then we have the following three cases corresponding to $L_\epsilon = 0, < 0$, or > 0:

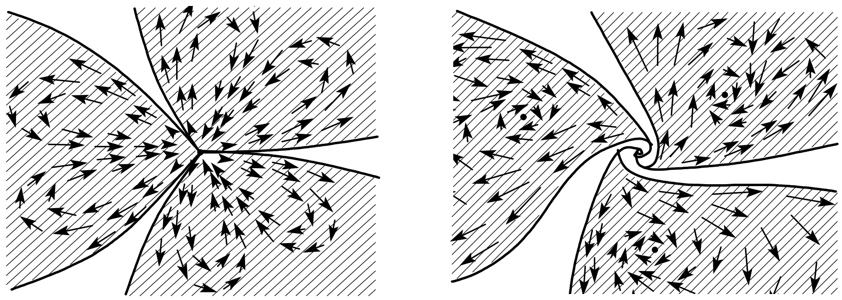

Figure 2: A horocyclic perturbation of a parabolic fixed point of f^{lq} of 3 petals (left) into a repelling fixed point of f_ϵ^{lq} (right).

(1) a_ϵ is persistently a parabolic point with p petals and the multiplier $\lambda_\epsilon = \lambda$;

(2) a_ϵ is an attracting periodic point, and there are p symmetrically arrayed repelling periodic points near a_ϵ; or

(3) a_ϵ is a repelling periodic point, and there are p symmetrically arrayed attracting periodic points near a_ϵ.

For cases (2) and (3), these symmetrically arrayed periodic points have the same period lq and the multipliers $\approx \lambda_\epsilon^{-pq}$. Moreover, they are contained in an open ball centered at a_ϵ with radius $O(|1-\lambda_\epsilon^q|^{1/p})$. We call them the *satellites* of a_ϵ and a_ϵ itself the *planet*. In particular, for case (2), we say that *the parabolic point a is perturbed into an attracting planet a_ϵ*. As we will see in the following sections, attracting planets are the cause of non-injectivity of the semiconjugacies. For case (1), we also call a_ϵ the planet, although it has no satellite.

Next, we consider the case of one petal. Now $p = 1$, then automatically $q = 1$ and $\lambda = 1$. If $\lambda_\epsilon = \lambda(= 1)$, a_ϵ is persistently a parabolic point with one petal. In this case, we also call a_ϵ the planet. If $\lambda_\epsilon \neq \lambda$, a splits into a pair of repelling and attracting periodic points. Which one is suitable for the planet? To define the planet in this case, we need to consider the J-critical relations.

Preparabolic critical orbits in $J(f)$. Let b be a preimage of a such that $a = f^i(b) = f^{i+l}(b)$. If $\deg(f^i, b) = m$, we can take a local coordinate near b such that $\zeta(b) = 0$ and

$$f^{-i} \circ f^{lq} \circ f^i(\zeta) = \zeta + \zeta^{mp+1} + O(\zeta^{mp+2}),$$

with a suitable branch of f^{-i}. This implies that there are mp petals attached to b as preimages of the petals of a.

Let us suppose that a horocyclic perturbation $f_\epsilon \to f$ preserves the J-critical relations of f. Then there exists b_ϵ such that $a_\epsilon = f_\epsilon^i(b_\epsilon) = f_\epsilon^{i+l}(b_\epsilon)$ and $\deg(f_\epsilon^i, b_\epsilon) =$

m. Taking a suitable local coordinate near b_ϵ such that $\zeta(b_\epsilon) = 0$, we obtain the corresponding normalized form of f_ϵ;

$$f_\epsilon^{-i} \circ f_\epsilon^{lq} \circ f_\epsilon^{i}(\zeta) = \lambda_\epsilon^q \zeta + \zeta^{mp+1} + O(\zeta^{mp+2}).$$

If $\lambda_\epsilon^q \neq 1$ (that is, $L_\epsilon \neq 0$) and $p \geq 2$, there are symmetrically arrayed mp "satellites" near b_ϵ as the preimages of the satellites of a_ϵ. Recall that a_ϵ may be attracting: this implies, b_ϵ may be in the Fatou set.

Now let us return to the definition of the planet when a has one petal. In the case of $\lambda_\epsilon = \lambda (= 1)$, it has been defined by a_ϵ. In the case of $\lambda_\epsilon \neq \lambda$, a splits into a pair of repelling and attracting fixed points of f_ϵ^l, say a_ϵ^+ and a_ϵ^- respectively. If a has a critical point in its preimages, then either a_ϵ^+ or a_ϵ^- has a critical point in its preimages because the J-critical relations are preserved. In this case, we define the planet as one containing a critical point in its preimages, and the satellite bas the other one. In particular, if a_ϵ^- is the planet, we also say that *a is perturbed into an attracting planet a_ϵ^-*. If a has no critical point in its preimages, then we formally define the planet as a_ϵ^+ and the satellite as a_ϵ^-.

Example. Let us consider perturbations of $f(z) = z(1+z)^m$ with $m > 1$ again. Recall that 0 is a parabolic fixed point with one petal.

For both perturbations in Example 1, 0 is the planet and $-1 + 1/\sqrt[m]{\lambda_\epsilon}$ is the satellite (See Figure 1). For the second perturbation, 0 is perturbed into an attracting planet.

On the other hand, for a trivial perturbation $f_\epsilon(z) = z(1+\lambda_\epsilon z)^m$ with $\lambda_\epsilon \to 1$, where f_ϵ are conjugate to f by linear transformations, 0 is the planet with no satellite.

Prerepelling critical orbits in $J(f)$. By geometric finiteness of f, some critical orbits in $J(f)$ land on repelling cycles. Since the J-critical relations are preserved, such repelling cycles are perturbed into repelling cycles of f_ϵ for $\epsilon \ll 1$. Let us consider local representations of the perturbations near such cycles.

Let b be a repelling periodic point of f in $P(f) \cap J(f)$, with multiplier λ and period l. Then there exists a repelling periodic point b_ϵ of f_ϵ in $P(f_\epsilon) \cap J(f_\epsilon)$, with multiplier λ_ϵ and period l, such that $b_\epsilon \to b$ and $\lambda_\epsilon \to \lambda$. By using a fundamental fact about linearization near repelling fixed points, we can take suitable local coordinates ψ_ϵ, ψ on a neighborhood of b such that $\psi_\epsilon(b_\epsilon) = \psi(b) = 0$ and

$$\psi_\epsilon \circ f_\epsilon^l \circ \psi_\epsilon^{-1}(z) = \lambda_\epsilon z \to \psi \circ f^l \circ \psi^{-1}(z) = \lambda z, \tag{2.2}$$

where ψ_ϵ converges to ψ uniformly near b. See [14, 8.3 Remark].

2.2 Key lemma on horocyclic perturbation.

Here we show a key lemma on horocyclic perturbation, which describes the perturbation of an orbit which accumulates on parabolic periodic points. We will see how horocyclic perturbations control the parabolic bifurcations.

Let a_0 be a periodic point of f with period l. The *cycle* α of a_0 is defined by

$$\alpha := \{a_0, f(a_0), \ldots, f^{l-1}(a_0)\}.$$

When a_0 is parabolic (resp. attracting, etc.), we call α a *parabolic (resp. attracting, etc.) cycle*.

Let us fix an $x \in \hat{\mathbb{C}}$ whose orbit accumulates on a parabolic cycle α. For an *arbitrarily* small $\delta > 0$, set $\Delta = \Delta(\delta) := \bigcup_{a\in\alpha} B_\sigma(a,\delta)$, and take $N_0 = N_0(x,\delta) \gg 0$ such that $f^n(x)$ are contained in Δ for all $n \geq N_0$. Now the key lemma is described as:

Lemma 2.1 *If the perturbation $f_\epsilon \to f$ is horocyclic, then there exists an $N \geq N_0$ such that $f_\epsilon^n(x)$ are contained in Δ for all $n \geq N$ and all $\epsilon \ll 1$.*

To simplify the proof of this lemma, we use "linearization" of parabolic bifurcations due to C. McMullen[12].

Proof. We begin the proof with constructing a simpler representation of the perturbation.

Linearizing parabolics. Let us take an integer k so that $f^k(a) = a$ and $(f^k)'(a) = 1$ for any $a \in \alpha$, and replace f by f^k. Then we may assume that $\alpha = \{a\}$ is a fixed point with multiplier 1 and that $\Delta = B_\sigma(a,\delta)$. It is sufficient to prove the statement in this case.

From the conditions of horocyclic perturbation, there exists a fixed point a_ϵ of f_ϵ converging to a. We may assume $\epsilon \ll 1$ such that a_ϵ is contained in Δ and sufficiently close to a. Now we set

$$\lambda_\epsilon = \exp(L_\epsilon + i\theta_\epsilon) := 1/f_\epsilon'(a_\epsilon),$$

which tends to 1 with $\theta_\epsilon^2 = o(|L_\epsilon|)$.

By replacing $\Delta = \Delta(\delta)$ with smaller δ and the definition of horocyclic perturbation, we can take a normalized convergent form on Δ as (1.1);

$$f_\epsilon(z) = \lambda_\epsilon^{-1} z + z^{p+1} + O(z^{p+2}) \to f(z) = z + z^{p+1} + O(z^{p+2})$$

where $z(a_\epsilon) = z(a) = 0$ and p is the petal number of a. Moreover, we take a simpler form of the convergence as follows.

First, by using local coordinates such that $z(a_\epsilon) = z(a) = \infty$, we obtain

$$f_\epsilon(z) = \lambda_\epsilon z + z^{1-p} + O(z^{-p}) \to f(z) = z + z^{1-p} + O(z^{-p}) \tag{2.3}$$

as a normal form of the convergence. Next, by using [12, Theorem 8.3] and additional linear conjugacies, we can show that there exist quasiconformal maps

$\phi_{\epsilon,0}$, ϕ_0 with $\phi_{\epsilon,0} \to \phi_0$ near infinity and $\phi_{\epsilon,0}(\infty) = \phi_0(\infty) = \infty$ such that

$$\begin{aligned} T_\epsilon(z) &:= \phi_{\epsilon,0} \circ f_\epsilon \circ \phi_{\epsilon,0}^{-1}(z) = (\lambda_\epsilon^p z^p + 1)^{1/p} \\ \to T(z) &:= \phi_0 \circ f \circ \phi_0^{-1}(z) = (z^p + 1)^{1/p}. \end{aligned} \tag{2.4}$$

Where p-th roots are taken so that $(\lambda_\epsilon^p z^p + 1)^{1/p} = \lambda_\epsilon z + O(1)$ and $(z^p+1)^{1/p} = z + O(1)$. Note that T_ϵ and T are p-fold branched coverings of linear transformations $\tilde{T}_\epsilon(w) = \lambda_\epsilon^p w + 1$ and $\tilde{T}(w) = w + 1$ respectively (where $w = z^p$). We call this form (2.4) a *linearized model* of the perturbation $f_\epsilon \to f$ near a.

Let ϕ_ϵ (resp. ϕ) be the composition of local coordinates of a_ϵ (resp. a) as (2.3) with $\phi_{\epsilon,0}$ (resp. ϕ_0) as (2.4). Then we obtain $\phi_\epsilon \to \phi$, a uniformly convergent family of local coordinates near a, which satisfies $\phi_\epsilon(a_\epsilon) = \phi(a) = \infty$ and conjugates $f_\epsilon \to f$ to $T_\epsilon \to T$. Finally, by replacing $\Delta = \Delta(\delta)$ with much smaller δ, we may assume that Δ is the domains of ϕ_ϵ and ϕ.

Now let us show the lemma by using the linearized model as (2.4). Take a constant $R \gg 0$ and a closed disk $D := \{|z| \geq R\}$, such that D is contained in both $\phi_\epsilon(\Delta)$ and $\phi(\Delta)$. Then there exists an $N_1 \geq N_0$ such that $\phi(f^n(x)) \in D$ for all $n \geq N_1$. Moreover, by uniform convergence of $f_\epsilon \to f$ and $\phi_\epsilon \to \phi$, we may assume that $\phi_\epsilon(f_\epsilon^{N_1}(x)) \in D$. To prove the lemma, it is enough to show that there exists an $N \geq N_1$ such that $\phi_\epsilon(f_\epsilon^n(x)) \in D$ for all $n \geq N$.

The proof breaks into the cases of $p = 1$ and $p \geq 2$.

Case 1: $p = 1$**.** Now $\phi_\epsilon \to \phi$ conjugates $f_\epsilon \to f$ to

$$T_\epsilon(z) = \lambda_\epsilon z + 1 \to T(z) = z + 1 \tag{2.5}$$

on D, with $\phi_\epsilon(a_\epsilon) = \phi(a) = \infty$. (See Figure 3. The four regions are centered at infinity.)

When $\lambda_\epsilon = 1$, T_ϵ is still parabolic and

$$T_\epsilon^k(\phi_\epsilon(f_\epsilon^{N_1}(x))) = \phi_\epsilon(f_\epsilon^{N_1}(x)) + k \in D$$

for all $k \geq 0$. This implies that $f_\epsilon^{N_1+k}(x)$ never escapes from $\phi_\epsilon^{-1}(D) \subset \Delta$ for all $k \geq 0$. Hence we take N_1 as N in this case.

We henceforth assume that $|\lambda_\epsilon| \neq 1$. By the perturbation, a splits into a pair of attracting and repelling fixed points. We may suppose that a_ϵ is the repelling one, and let b_ϵ denote the attracting one. (Here we do not consider which the planet is.) Then $|1/\lambda_\epsilon| = |f_\epsilon'(a_\epsilon)| > 1$, that is, $L_\epsilon \nearrow 0$. Moreover, in the linearized model (2.5), $\phi_\epsilon(b_\epsilon)$ must be the attracting fixed point of T_ϵ; thus $\phi_\epsilon(b_\epsilon) = (1 - \lambda_\epsilon)^{-1} =: b_\epsilon'$, and the multiplier of b_ϵ' is λ_ϵ.

Since the real part of $T^n(z)$ tends to infinity, there exists an integer $N \geq N_1$ such that $\phi(f^N(x))$ is in $D \cap \{|\arg z| < \pi/4\}$. By uniform convergence of $f_\epsilon \to f$

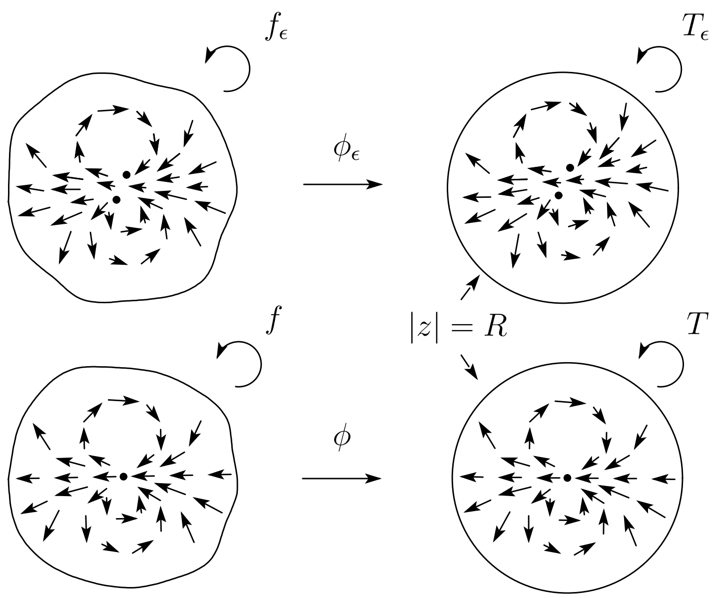

Figure 3: The dynamics on a neighborhood of infinity.

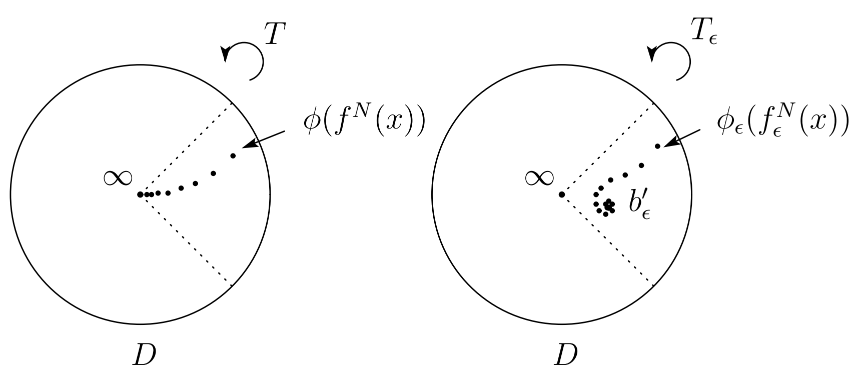

Figure 4: The orbits of $f^N(x)$ and $f_\epsilon^N(x)$ in the model.

and $\phi_\epsilon \to \phi$, we may also assume that $y := \phi_\epsilon(f_\epsilon^N(x))$ is in $D \cap \{|\arg z| < \pi/4\}$ for all $\epsilon \ll 1$(Figure 4).

To see the dynamics of T_ϵ in detail, we take a Möbius conjugacy of T_ϵ by

$$w = \psi_\epsilon(z) = \frac{z - b'_\epsilon}{y - b'_\epsilon},$$

which maps $\infty \mapsto \infty$, $b'_\epsilon \mapsto 0$ and $y \mapsto 1$. This conjugates the action of T_ϵ to $w \mapsto \lambda_\epsilon w$ with $|\lambda_\epsilon| < 1$. Hence $1 = \psi_\epsilon(y)$ is attracted to $0 = \psi_\epsilon(b'_\epsilon)$ by the iteration of $w \mapsto \lambda_\epsilon w$.

Now we claim: *For any fixed $\epsilon \ll 1$, $f_\epsilon^n(x)$ is contained in Δ for all $n \geq N$, and converges to b_ϵ as $n \to \infty$.* In other words, the whole orbit of $1 = \psi_\epsilon(y)$ is contained in $\psi_\epsilon(D)$ where the conjugation between T_ϵ and $w \mapsto \lambda_\epsilon w$ holds.

Set $B := \hat{\mathbb{C}} - D$ and $B' := \psi_\epsilon(B)$. Then B' is defined by this inequality:

$$\left| w - \frac{b'_\epsilon}{b'_\epsilon - y} \right| < \frac{R}{|b'_\epsilon - y|}. \tag{2.6}$$

We will show that the orbit of 1, that is, $\{1 = \psi_\epsilon(y),\ \lambda_\epsilon,\ \lambda_\epsilon^2, \ldots\}$, never enters B'.

For all $\epsilon \ll 1$, the center $b'_\epsilon/(b'_\epsilon - y)$ of B' is approximately $1 - y(L_\epsilon + i\theta_\epsilon)$. On the other hand, for any k such that $|k(L_\epsilon + i\theta_\epsilon)| \ll 1$, λ_ϵ^k is approximately $1 + k(L_\epsilon + i\theta_\epsilon)$. Since $|\arg y| < \pi/4$, the direction of first several points of the orbit $\{1,\ \lambda_\epsilon,\ \lambda_\epsilon^2, \ldots\}$ is opposite to the center of B' with respect to 1. This means, at least, the orbit does not go to B' immediately (Figure 5).

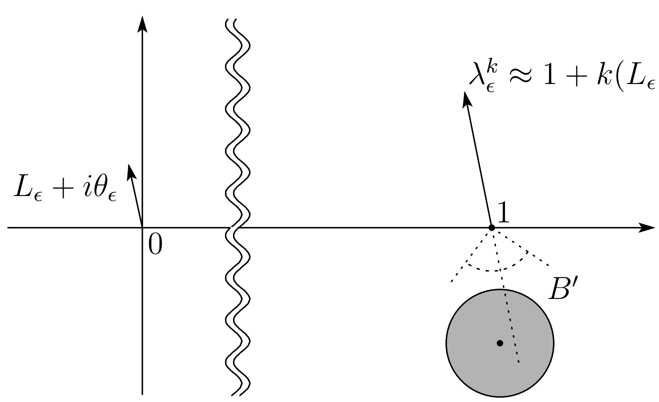

Figure 5: The orbit of $1 = \psi_\epsilon(y)$ near 1

Suppose that $\theta_\epsilon = 0$. Then the orbit of 1 accumulates on 0 along the real axis, and it is disjoint from B'.

Suppose that $\theta_\epsilon \neq 0$. We may assume that $\theta_\epsilon > 0$ because the signature of θ_ϵ determines only the direction of the rotation by the action of $w \mapsto \lambda_\epsilon w$. Then the orbit of 1 returns near the positive real axis by nearly $2\pi/\theta_\epsilon$ times iterations of $w \mapsto \lambda_\epsilon w$. Now we have to handle the case where the order of $\theta_\epsilon \searrow 0$ is lower

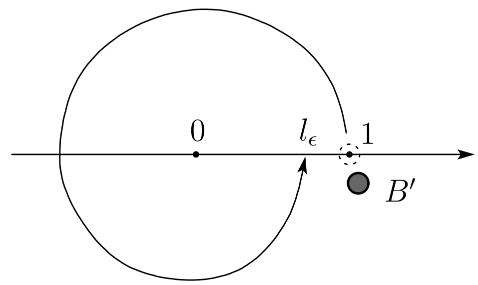

Figure 6: The orbit of 1

than that of $L_\epsilon \nearrow 0$: Then the orbit might touch B'. However, we will show that it cannot occur if $\epsilon \ll 1$.

Now note that the following two facts: when the orbit of 1 returns near the positive real axis, the distance between 0 and the orbit is nearly $l_\epsilon := \exp(2\pi L_\epsilon/\theta_\epsilon)$; on the other hand, by (2.6), B' is contained in a ball centered at 1 with radius $O(|L_\epsilon + i\theta_\epsilon|)$, that is, every point in B' tends to 1 as $\epsilon \to 0$.

By these facts, if $\liminf |L_\epsilon/\theta_\epsilon| \neq 0$, l_ϵ does not tend to 1 and the orbit of 1 never touches B' (Figure 6).

Otherwise we can take a decreasing sequence $\epsilon_n \searrow 0$ such that $L_{\epsilon_n}/\theta_{\epsilon_n} \to 0$. Now $l_{\epsilon_n} \to 1$ as $n \to \infty$. In this case, $|1 - l_{\epsilon_n}| \approx 2\pi|L_{\epsilon_n}|/\theta_{\epsilon_n}$ for $n \gg 0$ thus

$$\frac{O(|L_{\epsilon_n} + i\theta_{\epsilon_n}|)}{|1 - l_{\epsilon_n}|} \;=\; O\big(\big|\theta_{\epsilon_n} + i\theta_{\epsilon_n}^2/|L_{\epsilon_n}|\big|\big) \;\to 0 \quad (\epsilon_n \to 0). \tag{2.7}$$

This means, for any choice of $\{\epsilon_n\}$, every point in B' tends to 1 faster than l_{ϵ_n} does. Note that the order of convergence in (2.7) depends only on the order of $L_\epsilon,\ \theta_\epsilon \to 0$ (not on the choice of $\{\epsilon_n\}$). Hence for $\epsilon \ll 1$, the orbit of 1 is attracted to 0 without entering B'.

Case 2 : $p \geq 2$**.** Now $\phi_\epsilon \to \phi$ with $\phi_\epsilon(a_\epsilon) = \phi(a) = \infty$ conjugates $f_\epsilon \to f$ to

$$T_\epsilon(z) = (\lambda_\epsilon^p z^p + 1)^{1/p} \to T(z) = (z^p + 1)^{1/p} \tag{2.8}$$

on D. As in the case of $p = 1$, we may assume that

$$\phi(f^N(x)) \in \bigcup_{j=0}^{p-1} \left\{ \left| \arg z - \frac{2\pi j}{p} \right| < \frac{\pi}{4p} \right\}$$

for an $N \geq N_1$, and

$$y = \phi_\epsilon(f_\epsilon^N(x)) \in \bigcup_{j=0}^{p-1} \left\{ \left| \arg z - \frac{2\pi j}{p} \right| < \frac{\pi}{4p} \right\}$$

for all $\epsilon \ll 1$.

Let us consider a semiconjugation of T_ϵ by a branched covering $w = \pi(z) = z^p$. Then the dynamics of T_ϵ on D is reduced to the dynamics of $\tilde{T}_\epsilon(w) = \lambda_\epsilon^p w + 1$ on $\pi(D) = \{|w| \geq R^p\}$(Figure 7). Similarly, $\pi(z)$ gives a semiconjugacy from $T(z)$ on D to $\tilde{T}(w) = w + 1$ on $\pi(D)$.

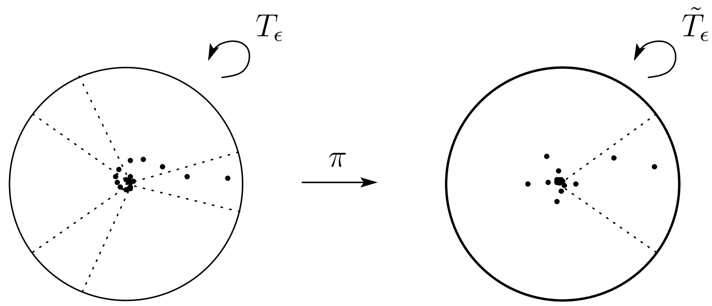

Figure 7: $w = \pi(z) = z^p$

By the same argument as the case of $p = 1$, when $\lambda_\epsilon = 1$, the orbit of $\pi(y)$ tends to $w = \infty$ and never escapes from $\pi(D)$. Similarly, if $|\lambda_\epsilon| \neq 1$, the orbit of $\pi(y)$ tends to an attracting fixed point, which is either $w = \infty$ or $w = 1/(1 - \lambda_\epsilon^p)$, and never escapes from $\pi(D)$. Thus the original orbit of $\phi_\epsilon(f_\epsilon^N(x))$ by T_ϵ never escapes from D. ■

Remark. One can easily check that the same result holds if we replace x with a compact set in the parabolic basin of a. We will use this in the proof of Proposition 3.2.

3 Construction of Ω and Ω_ϵ

In this section, we prepare the ingredients for the construction of the semiconjugacy; Ω, Ω_ϵ and $h_0 : \Omega_\epsilon \to \Omega$.

To simplify the arguments, *from this section to §7, we assume that* $J(f) \neq \hat{\mathbb{C}}$. The case of $J(f) = \hat{\mathbb{C}}$ is treated in §8.

Let us introduce some notation. Let A denote the finite set of all parabolic points of f. We define the sets of all preperiodic critical orbits in the Julia sets by

$$Z := \bigcup_{n=1}^{\infty} f^n(C(f) \cap J(f)), \quad Z_\epsilon := \bigcup_{n=1}^{\infty} f_\epsilon^n(C(f_\epsilon) \cap J(f_\epsilon)).$$

In addition, we set $Z^1 := f^{-1}(Z)$ and $Z_\epsilon^1 := f_\epsilon^{-1}(Z_\epsilon)$. Since $f_\epsilon \to f$ preserves the J-critical relations of f, $\text{card}(Z_\epsilon) \leq \text{card}(Z) < \infty$ in general. The equality holds precisely if none of the parabolic points of f is perturbed into an attracting planet.

3.1 Construction of Ω.

Here we construct a compact set Ω for f.

Proposition 3.1 *There exists a finitely connected compact set $\Omega \subset \hat{\mathbb{C}}$ with the following properties:*

1. $\Omega \cap (P(f) \cup C(f)) = J(f) \cap (P(f) \cup C(f))$. *This set is the union of A and all critical orbits in $J(f)$.*
2. $J(f) \subset \Omega$ *and* $f^{-1}(\Omega) \subset \mathrm{Int}(\Omega) \cup A$.

Proof. To define the compact set Ω, we will construct two open sets F and V which consist of finitely many simply connected components.

Let a be an attracting or parabolic periodic point of f and α the cycle of a. First, we construct F: If α is attracting, we take a small disk neighborhood F_a for each $a \in \alpha$ such that $f(\overline{F_a}) \subset F_{f(a)}$. Here we can take $\{F_a\}$ to be pairwise disjoint. If α is parabolic, we take F_a for each point $a \in \alpha$ to be a small "flower" (that is, a union of attracting petals for each attracting directions of a) such that $f(\overline{F_a} - \{a\}) \subset F_{f(a)}$. Here we can also take $\{F_a\}$ to be pairwise disjoint, and each ∂F_a to be tangent to the repelling directions.

Now we set

$$F := \bigcup_{\alpha} \bigcup_{a \in \alpha} F_a$$

where α ranges over all attracting and parabolic cycles. Note that $f(\overline{F} - A) \subset F$.

Next, we construct V: Let $C(f, \alpha)$ denote the set of all critical points of f whose orbits accumulate on α but never land on it. Now let us set $F_\alpha := \bigcup_{a \in \alpha} F_a$. For each $c \in C(f, \alpha)$, there exists a natural number $N = N(c)$ such that $f^n(c) \in F_\alpha$ for all $n \geq N$. Then we can take a family of open disks $\{V_c^i\}_{i=0}^N$ satisfying the following conditions (See Figure 8):

- V_c^i is a small disk-neighborhood of $f^i(c)$;
- $V_c^i \cap V_c^j = \emptyset$ for $i \neq j$;
- $V_c^N \subset F_\alpha$; and
- $f(\overline{V_c^i}) \subset V_c^{i+1}$ for all $i < N$.

Now we set

$$V := \bigcup_{\alpha} \bigcup_{c \in C(f,\alpha)} \bigcup_{i=0}^{N(c)} V_c^i$$

where α ranges over all attracting and parabolic cycles. Note that $f(\overline{V}) \subset V \cup F$.

Using F and V, we define Ω as $\hat{\mathbb{C}} - (F \cup V)$. Then we can easily check that Ω satisfies the conditions in the statement. ■

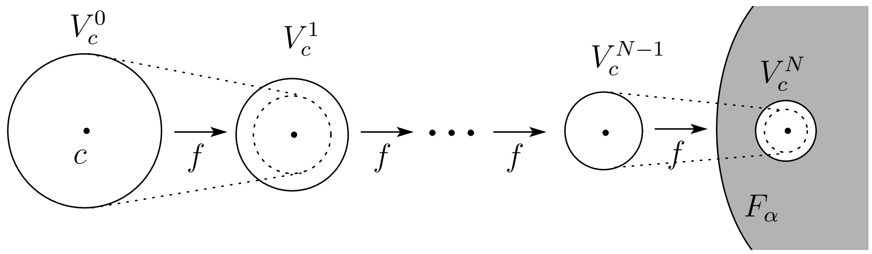

Figure 8: The orbit of c and $\{V_c^i\}$

3.2 Construction of Ω_ϵ and the "0-th" map h_0.

Next we consider a horocyclic perturbation $f_\epsilon \to f$ preserving the J-critical relations of f. For each f_ϵ, we construct a compact set Ω_ϵ corresponding to $\Omega = \hat{\mathbb{C}} - (F \cup V)$, and the correspondence is represented by the map $h_0 (= h_{0,\epsilon}) : \Omega_\epsilon \to \Omega$.

Proposition 3.2 *For each $\epsilon \ll 1$, there exists a compact set $\Omega_\epsilon \subset \hat{\mathbb{C}}$ and a continuous map $h_0 (= h_{0,\epsilon}) : \Omega_\epsilon \to \Omega$ with the following properties:*

1. *$\Omega_\epsilon \cap (P(f_\epsilon) \cup C(f_\epsilon)) = J(f_\epsilon) \cap (P(f_\epsilon) \cup C(f_\epsilon))$, and this set is the union of all parabolic points of f_ϵ and all critical orbits in $J(f_\epsilon)$.*

2. *$J(f_\epsilon) \subset \Omega_\epsilon$ and $f_\epsilon^{-1}(\Omega_\epsilon) \subsetneq \Omega_\epsilon$.*

3. *$h_0 : \Omega_\epsilon \to \Omega$ is surjective.*

4. *If there exists $y \in \Omega$ such that $\mathrm{card}(h_0^{-1}(y)) \geq 2$ then y is a parabolic point and $\mathrm{card}(h_0^{-1}(y)) = p(y)$. Moreover, y is perturbed into an attracting planet and $h_0^{-1}(y)$ is the set of $p(y)$ repelling satellites of the attracting planet.*

5. *For each $b_\epsilon \in Z_\epsilon^1$, there exists a unique $b \in Z^1$ such that $b_\epsilon \to b$, and*
$$h_0(b_\epsilon) = b.$$

Moreover, for any fixed $r > 0$, we can make h_0 satisfy
$$\sup \{d_\sigma(h_0(x), x) : x \in \Omega_\epsilon\} \leq r$$
for all $\epsilon \ll 1$.

For example, suppose that f is hyperbolic; that is, both A and $J(f) \cap C(f)$ are empty. For $\epsilon \ll 1$, f_ϵ is a very small perturbation of f, thus every attracting cycle of f is perturbed into an attracting cycle of f_ϵ. By uniform convergence of $f_\epsilon \to f$, we obtain $f_\epsilon(\overline{F}) \subset F$ for all $\epsilon \ll 1$. Similarly, if $\epsilon \ll 1$, V satisfies $f_\epsilon(\overline{V}) \subset V \cup F$. Hence we can set $\Omega_\epsilon := \Omega = \hat{\mathbb{C}} - (F \cup V)$ and $h_0 := \mathrm{id}$.

For general geometrically finite rational maps, to construct Ω_ϵ for $f_\epsilon \to f$, we need to modify F; in particular, certain parts of the flowers $\{F_a\}_{a\in A}$. We also need additional modification near the critical orbits in the Julia set.

Let us fix an $r > 0$ and set $B_x := B_\sigma(x, r/2)$ for each $x \in A \cup Z^1$. We suppose that r is sufficiently small so that $B_x \cap B_{x'} = \emptyset$ for different $x,\ x' \in A \cup Z^1$ and that $B_x \subset \mathrm{Int}(\Omega)$ for $x \in Z^1 - A$.

Modification of Ω near the parabolics. Fix a parabolic point of f, say $a \in A$. Set $E_a := \Omega \cap \overline{B_a}$. We may assume that E_a is a union of $p(a)$ narrow cusps near the repelling directions.

Lemma 3.1 *For each $\epsilon \ll 1$, there exists a compact set E'_a and a map $h_a : E'_a \to E_a$ with the following conditions:*

- $\partial E_a \cap \partial B_a = \partial E'_a \cap \partial B_a$, *and h_a is the identity on this set.*
- $f_\epsilon^{-1}(E'_{f(a)}) \cap B_a \subset E'_a$.
- $B_a - E'_a \subset F(f_\epsilon)$.
- $h_a : E'_a \to E_a$ *is continuous and surjective.*
- *If $y \in E_a$ and $\mathrm{card}(h_a^{-1}(y)) \geq 2$, then $y = a$. In this case, a is perturbed into an attracting planet a_ϵ and $h_a^{-1}(y)$ is the set of all repelling satellites of a_ϵ.*
- $d_\sigma(h_a(x), x) \leq r$ *for any $x \in E'_a$.*

Proof. For simplicity, here we only treat the case where a is a fixed point with multiplier 1. The case of a with multiplier $\neq 1$ or period $\neq 1$ is similar.

As $f_\epsilon \to f$ horocyclically, suppose that a is perturbed into the planet a_ϵ, a fixed point of f_ϵ.

Let us consider the local dynamics by f^{-1} and f_ϵ^{-1} restricted near B_a. We denote by g (resp. g_ϵ) the branch of f^{-1} (resp. f_ϵ^{-1}) near B_a which fixes a (resp. a_ϵ). Then a is still a parabolic fixed point of g and a_ϵ is a fixed point of g_ϵ with multiplier $1/f'_\epsilon(a_\epsilon)$. Note that $g_\epsilon \to g$ is a locally defined horocyclic perturbation, thus we can apply Lemma 2.1.

Set $p := p(a)$, the petal number of a. The construction of E'_a and h_a breaks into the cases of $p = 1$ and $p \geq 2$.

Case 1 : $p = 1$. In this case, we may assume that a_ϵ is an attracting or parabolic fixed point of g_ϵ. (Here we need not distinguish planet from satellite.)

Now $\partial E_a \cap \partial B_a$ is an arc. Let e_1 and e_2 be its end points. Since r is sufficiently small, we may assume that e_1 and e_2 are enough close to the attracting direction for g, and that their orbits by g accumulate on a within E_a. Then we may apply the argument in Lemma 2.1 to the orbits of e_1 and e_2 by g_ϵ. For $\epsilon \ll 1$, joining the orbits of e_i $(i = 1, 2)$ by g_ϵ contained in B_a, we obtain a piecewise smooth Jordan arcs η_i with the following properties:

- Joining from e_i to a_ϵ.
- $g_\epsilon(\eta_i) \subset \eta_i \subset B_a \cup \{e_i\}$ and $f_\epsilon(\eta_i) - B_a \subset F_a$
- $\eta_1 \cap \eta_2 = \{a_\epsilon\}$.

In fact, joining e_i and $g_\epsilon(e_i)$ by nearly straight curve and taking the union of their forward images by g_ϵ, we obtain such a curve η_i. We define E'_a as the closure of the region in B_a enclosed by η_1, η_2 and $\partial E_a \cap \partial B_a$. Then we see that $f_\epsilon^{-1}(E'_a) \cap B_a \subset E'_a$.

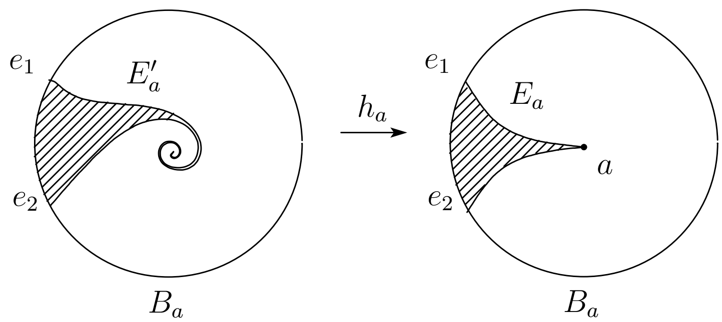

Figure 9: Construction of E'_a

We claim that $B_a - E'_a \subset F(f_\epsilon)$ for $\epsilon \ll 1$. Let us take an arbitrary $x \in B_a - E'_a$.

If the orbit of x never escapes from B_a and is attracted to the parabolic or attracting point of f_ϵ in B_a, then $x \in F(f_\epsilon)$. So we consider the case where the orbit of x escapes from B_a. Then for some $i > 0$, $f_\epsilon^i(x)$ is contained in the compact set $\overline{F_a - B_a} \subset F(f)$.

By the local dynamics in F_a, there exists $N \gg 0$ such that $f^N(\overline{F_a - B_a})$ is contained in B_a and is sufficiently near the attracting direction of a. By uniform convergence of $f_\epsilon \to f$, we may suppose the same holds for $f_\epsilon^N(\overline{F_a - B_a})$. Furthermore, since $f^n(\overline{F_a - B_a})$ converges uniformly to a within B_a as n tends to infinity, we may apply the argument in Lemma 2.1 to the forward images of $f_\epsilon^N(\overline{F_a - B_a})$ by f_ϵ; thus $f_\epsilon^n(\overline{F_a - B_a})$ converges uniformly to the parabolic or attracting point of f_ϵ within B_a. This implies $x \in F(f_\epsilon)$.

Finally we define the map $h_a : E'_a \to E_a$: Let us take a Riemann map $R_\epsilon : \mathrm{Int}(E'_a) \to \mathbb{D}$, here $\mathbb{D}$ is the unit disk. Since the boundary of E'_a is a Jordan curve, R_ϵ is extended to a homeomorphism $R_\epsilon : E'_a \to \overline{\mathbb{D}}$. Similarly, we take an extended Riemann map $R : E_a \to \overline{\mathbb{D}}$. By choosing a suitable topological map $H_\epsilon : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, we obtain $h_a := R^{-1} \circ H_\epsilon \circ R_\epsilon$ such that:

- $h_a : E'_a \to E_a$ is a homeomorphism;

- $h_a|(\partial E'_a \cap \partial B_a) = \mathrm{id}$; and
- $h_a(a_\epsilon) = a$.

Furthermore, since the radius of B_a is $r/2$, we obtain $d_\sigma(h_a(x), x) \leq r$ for any $x \in E'_a$.

Case 2 : $p \geq 2$. Now E_a is the union of p narrow cusps which intersect only at a. We distinguish these p cusps as $\{E_1, \dots, E_p\}$; that is, each E_j is a union of $\{a\}$ and one of the p connected components of $E_a - \{a\}$. Let e_{1j} and e_{2j} be the end points of $\partial E_j \cap \partial B_a$ for $j = 1, \dots, p$.

As in the case of $p = 1$, let us apply the argument in Lemma 2.1. Then we can take g_ϵ-invariant path η_{ij} which joins e_{ij} and a parabolic or attracting point of g_ϵ generated in B_a by the perturbation of a. We define E'_j as the compact set in $\overline{B_a}$ enclosed by η_{1j}, η_{2j}, and $\partial E_j \cap \partial B_a$. Note that we obtain the following three cases:

1. The planet a_ϵ is a parabolic fixed point of f_ϵ, that is, the multiplier $f'_\epsilon(a_\epsilon)$ satisfies $f'_\epsilon(a_\epsilon) = 1$. In this case, each E'_j joins $E_j \cap \partial B_a$ to a_ϵ and $\bigcap_{j=1}^{p} E'_j = \{a_\epsilon\}$.
2. The planet a_ϵ is a repelling fixed point of f_ϵ, that is, the multiplier $f'_\epsilon(a_\epsilon)$ satisfies $|f'_\epsilon(a_\epsilon)| > 1$. In this case, each E'_j joins $E_j \cap \partial B_a$ to a_ϵ and $\bigcap_{j=1}^{p} E'_j = \{a_\epsilon\}$ (Figure 10).
3. The planet a_ϵ is an attracting fixed point of f_ϵ, that is, the multiplier $f'_\epsilon(a_\epsilon)$ satisfies $|f'_\epsilon(a_\epsilon)| < 1$. In this case, each E'_j joins $E_j \cap \partial B_a$ to one of the symmetrically arrayed repelling satellites of a_ϵ and $\bigcap_{j=1}^{p} E'_j = \emptyset$ (Figure 10).

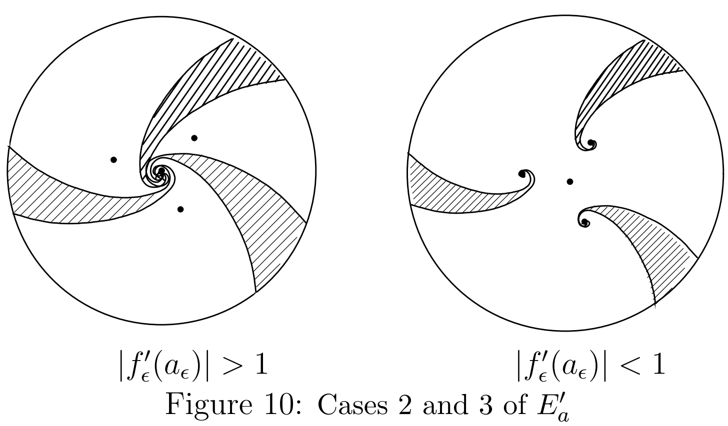

Figure 10: Cases 2 and 3 of E'_a

Now we set $E'_a := \bigcup_{j=0}^{p-1} E'_j$. We can show $B_a - E'_a \subset F(f_\epsilon)$ for $\epsilon \ll 1$ by the same argument as the case of $p = 1$.

For each E'_j, let us take a homeomorphism $h_{a,j} : E'_j \to E_j$ in the same way as h_a for $p = 1$, and define a continuous map $h_a : E'_a \to E_a$ by $h_a|E'_j = h_{a,j}$. Then h_a has the following properties:

- $h_a|(\partial E'_a \cap \partial B_a) = \mathrm{id}$;
- $h_a : E'_a \to E_a$ is surjective; and
- if $y \in E_a$ and $\mathrm{card}(h_a^{-1}(y)) \geq 2$, then $y = a$. Moreover, a is perturbed into the attracting planet a_ϵ, and $h_a^{-1}(y)$ consist of p repelling satellites of a_ϵ.

In particular, we also obtain $d_\sigma(h_a(x), x) \leq r$ for any $x \in E'_a$. ■

Finally let us show the existence of Ω_ϵ.

Proof(Proposition 3.2). For each fixed $\epsilon \ll 1$, set

$$\Omega_\epsilon := \left(\Omega - \bigcup_{a\in A} B_a\right) \cup \bigcup_{a\in A} E'_a.$$

By the construction of E'_a, one can easily check that $J(f_\epsilon) \subset \Omega_\epsilon$ and $f_\epsilon^{-1}(\Omega_\epsilon) \subsetneq \Omega_\epsilon$.

To check that $\Omega_\epsilon \cap (P(f_\epsilon) \cup C(f_\epsilon)) = J(f_\epsilon) \cap (P(f_\epsilon) \cup C(f_\epsilon))$, it is sufficient to show that the critical orbits in the Fatou set never land on Ω_ϵ.

Let us take $c_\epsilon \in C(f_\epsilon) \cap F(f_\epsilon)$. Then there exists $c \in C(f)$ such that $c_\epsilon \to c$ $(\epsilon \to 0)$.

If $c \in J(f)$, by geometric finiteness of f, the orbit of c lands on a parabolic or repelling cycle, say α. Since the J-critical relations of f are preserved, c_ϵ also lands on a cycle. By our assumption that $c_\epsilon \in F(f_\epsilon)$, α must be parabolic and the orbit of c_ϵ must land on an attracting cycle which is generated by the perturbation of α. Thus the orbit of c_ϵ never lands on Ω_ϵ by the definition of $\bigcup_{a\in A} E'_a$.

If $c \in F(f)$, the orbit of c accumulates on a parabolic or attracting cycle. By the construction of Ω, c is not contained in Ω. Similarly, by the definition of Ω_ϵ, we may assume that $c_\epsilon \notin \Omega_\epsilon$. Let us suppose that $f_\epsilon^n(c_\epsilon) \in \Omega_\epsilon$ for some n. Then $c_\epsilon \in f_\epsilon^{-n}(\Omega_\epsilon) \subsetneq \Omega_\epsilon$ and it is a contradiction. Thus $f_\epsilon^n(c_\epsilon) \notin \Omega_\epsilon$ for all n.

Finally we define $h_0 : \Omega_\epsilon \to \Omega$. Since $f_\epsilon \to f$ preserves the J-critical relations of f, we may assume that for any $b \in Z^1 - A$, B_b contains only one point of Z^1_ϵ, say b_ϵ, such that $b_\epsilon \to b$. Recall that $B_b \subset \mathrm{Int}(\Omega)$, by the assumption for r. Let $h_b : B_b \to B_b$ be an arbitrary topological map which satisfies $h_b(b_\epsilon) = b$ and $h_b|\partial B_b = \mathrm{id}$. Then we obtain $d_\sigma(h_b(x), x) \leq r$ for $x \in B_b$.

Let us define $h_0 : \Omega_\epsilon \to \Omega$ by

$$\begin{aligned} h_0 &= h_a \quad \text{on } E'_a \text{ for } a \in A, \\ h_0 &= h_b \quad \text{on } B_b \text{ for } b \in Z^1 - A, \text{and} \\ h_0 &= \mathrm{id} \quad \text{otherwise.} \end{aligned}$$

■

4 Construction of h_n

For Ω_ϵ and Ω constructed in §3, we set

$$\Omega_\epsilon^n := f_\epsilon^{-n}(\Omega_\epsilon) \ \text{ and } \ \Omega^n := f^{-n}(\Omega) \quad (n = 0, 1, 2, \ldots).$$

In addition, we set $U_\epsilon^n := \mathrm{Int}(\Omega_\epsilon^n)$ and $U^n := \mathrm{Int}(\Omega^n)$. By the construction of these sets, $f_\epsilon : \Omega_\epsilon^{n+1} \to \Omega_\epsilon^n$ and $f : \Omega^{n+1} \to \Omega^n$ are branched covering maps, where the critical values are contained in Z_ϵ and Z respectively. Note that $\{\Omega_\epsilon^n\}$ and $\{\Omega^n\}$ form the decreasing sequences as below:

$$\Omega_\epsilon = \Omega_\epsilon^0 \supsetneq \Omega_\epsilon^1 \supsetneq \cdots \supsetneq \Omega_\epsilon^n \supsetneq \Omega_\epsilon^{n+1} \supsetneq \cdots \supsetneq J(f_\epsilon),$$
$$\Omega = \Omega^0 \supsetneq \Omega^1 \supsetneq \cdots \supsetneq \Omega^n \supsetneq \Omega^{n+1} \supsetneq \cdots \supsetneq J(f).$$

In this section, we inductively construct a sequence of lifts of $h_0 : \Omega_\epsilon^0 \to \Omega^0$,

$$\{h_n(= h_{n,\epsilon}) : \Omega_\epsilon^n \to \Omega^n\}_{n=1}^\infty$$

satisfying $f \circ h_{n+1} = h_n \circ f_\epsilon$.

Proposition 4.1 *For an $n \geq 0$, assume that there exists $h_n(= h_{n,\epsilon}) : \Omega_\epsilon^n \to \Omega^n$ satisfying the following properties:*

$(1, n)$ *h_n is continuous and surjective.*

$(2, n)$ *h_n maps U_ϵ^n onto U^n homeomorphically. Moreover, if there exists $y \in \Omega^n$ such that $\mathrm{card}(h_n^{-1}(y)) \geq 2$ then $f^n(y)$ is a parabolic point of f perturbed into an attracting planet and $\mathrm{card}(h_n^{-1}(y)) = \deg(f^n, y) \cdot p(f^n(y))$.*

$(3, n)$ *For any $b_\epsilon \in Z_\epsilon^1$, there exists a unique $b \in Z^1$ such that*

$$h_n(b_\epsilon) = b.$$

Under these assumptions, there exists $h_{n+1}(= h_{n+1,\epsilon}) : \Omega_\epsilon^{n+1} \to \Omega^{n+1}$ satisfying

$$f \circ h_{n+1} = h_n \circ f_\epsilon$$

and properties $(1, n+1)$, $(2, n+1)$ and $(3, n+1)$.

Recall that the map $h_0 : \Omega_\epsilon^0 \to \Omega^0$ has properties (1, 0), (2, 0), and (3, 0). Thus this proposition gives us desired $\{h_n : \Omega_\epsilon^n \to \Omega^n\}_{n=1}^\infty$.

Proof. The proof breaks into 3 steps.

Step 1: Interior correspondence. The first step is to try to construct a homeomorphism between U_ϵ^{n+1} and U^{n+1}. To begin with, we construct h_{n+1} such that the following diagram commutes:

$$\begin{array}{ccc} U_\epsilon^{n+1} - Z_\epsilon^1 & \xrightarrow{h_{n+1}} & U^{n+1} - Z^1 \\ {\scriptstyle f_\epsilon}\downarrow & & \downarrow{\scriptstyle f} \\ U_\epsilon^n - Z_\epsilon & \xrightarrow{h_n} & U^n - Z \end{array}$$

Here $f|(U^{n+1} - Z^1)$ and $f_\epsilon|(U_\epsilon^{n+1} - Z_\epsilon^1)$ are d-sheeted covering maps. Moreover, by properties $(2, n)$ and $(3, n)$, $h_n|(U_\epsilon^n - Z_\epsilon)$ is a homeomorphism. We will construct prospective h_{n+1} in the diagram by lifting this $h_n|(U_\epsilon^n - Z_\epsilon)$. Note that U_ϵ^n and U^n for $n \geq 1$ are either connected or finitely many connected components. (For example, suppose that $J(f)$ is a Cantor set.) Hence we construct h_{n+1} on each connected component of $U_\epsilon^{n+1} - Z_\epsilon^1$.

Let Q_ϵ^1 be a connected component of $U_\epsilon^{n+1} - Z_\epsilon^1$, and take a base point $x_0^1 \in Q_\epsilon^1$. Set $Q_\epsilon := f_\epsilon(Q_\epsilon^1)$, a connected component of U_ϵ^n, and set $x_0 := f_\epsilon(x_0^1) \in Q_\epsilon$. Moreover, set $Q := h_n(Q_\epsilon)$ and $y_0 := h_n(x_0) \in Q$.

Let $y_0^1 \in U^{n+1}$ be the closest point to x_0^1 in $f^{-1}(y_0)$. Such y_0^1 is uniquely determined, since critical values in the Fatou sets stay a bounded distance away from Q_ϵ and Q. Let Q^1 denote a connected component of $f^{-1}(Q)$ containing y_0^1. We will lift h_n to h_{n+1} such that the following diagram commutes:

$$\begin{array}{ccc} (Q_\epsilon^1, x_0^1) & \xrightarrow{h_{n+1}} & (Q^1, y_0^1) \\ {\scriptstyle f_\epsilon}\downarrow & & \downarrow{\scriptstyle f} \\ (Q_\epsilon, x_0) & \xrightarrow{h_n} & (Q, y_0) \end{array}$$

Take a point $x^1 \in Q_\epsilon^1$ and a curve $\eta_\epsilon : [0, 1] \to Q_\epsilon^1$ such that $\eta_\epsilon(0) = x_0^1$ and $\eta_\epsilon(1) = x^1$. Then the curve $h_n(f_\epsilon(\eta_\epsilon))$ has the initial point y_0. We lift this curve to $\eta : [0, 1] \to Q^1$ with the initial point y_0^1, and define $h_{n+1}(x^1)$ as its end point $\eta(1)$.

Since $h_n|Q_\epsilon$ is a homeomorphism and the J-critical relations of f are preserved, for the fundamental groups $\pi_1(Q_\epsilon^1, x_0^1)$ and $\pi_1(Q^1, y_0^1)$,

$$(h_n)_* : (f_\epsilon)_*\pi_1(Q_\epsilon^1, x_0^1) \to f_*\pi_1(Q^1, y_0^1)$$

is a group isomorphism. Hence the above definition of $h_{n+1}(x^1)$ gives the homeomorphism $h_{n+1} : (Q_\epsilon^1, x_0^1) \to (Q^1, y_0^1)$ as a lift of $h_n : (Q_\epsilon, x_0) \to (Q, y_0)$ (See [11, Ch.III]).

Now we have a homeomorphism $h_{n+1} : U_\epsilon^{n+1} - Z_\epsilon^1 \to U^{n+1} - Z^1$. For $x \in U_\epsilon \cap Z_\epsilon^1$, let us set $h_{n+1}(x) := h_n(x)$. Then we obtain a homeomorphism $h_{n+1} : U_\epsilon^{n+1} \to U^{n+1}$ as a natural lift of $h_n : U_\epsilon^n \to U^n$.

Step 2: Boundary correspondence. The second step is to extend h_{n+1} defined on U_ϵ^{n+1} to the boundary $\partial U_\epsilon^{n+1} = \partial\Omega_\epsilon^{n+1}$, in a natural way. Here we should be careful about the boundary correspondence near the preimages of a parabolic point which is perturbed into an attracting planet. Note that the injectivity of h_n has already been broken at some of these points.

To construct $h_{n+1}|\partial\Omega_\epsilon^{n+1}$, it suffices to construct $h_{n+1}|\partial Q_\epsilon^1$ for each Q_ϵ^1 in Step 1. For $x_0^1 \in Q_\epsilon^1$ and $x^1 \in \partial Q_\epsilon^1$, take a curve $\eta_\epsilon : [0,1] \to Q_\epsilon^1 \cup \{x^1\}$ with $\eta_\epsilon(0) = x_0^1$ and $\eta_\epsilon(1) = x^1$. Now the value of h_{n+1} at x^1 is defined by

$$h_{n+1}(x^1) := \lim_{t\to 1} h_{n+1}(\eta_\epsilon(t)) \in \partial Q^1.$$

One can easily check that this value does not depend on the choice of η_ϵ.

By this definition, if $a \in \partial Q^1$ is a parabolic point with $p \geq 2$ petals and is perturbed into an attracting planet, then $h_{n+1}^{-1}(a)$ is p distinct points in ∂Q_ϵ^1 corresponding to p distinct accesses to a in E_a. The case of k-th preimages of a with $k \leq n+1$ is similar. Moreover, note that $h_{n+1}(x^1) = h_n(x^1)$ if $x^1 \in \partial Q_\epsilon^1 \cap Z_\epsilon^1$.

Step 3: Checking the properties. Now we have already defined a continuous map $h_{n+1} : \Omega_\epsilon \to \Omega$. For the last step, we check that h_{n+1} has properties $(1, n+1)$, $(2, n+1)$ and $(3, n+1)$.

Note that $h_{n+1}|Q_\epsilon^1$ is a homeomorphism and $h_{n+1}|\overline{Q_\epsilon^1}$ is continuous. Thus bijectivity of h_{n+1} may break only at the boundary points. For a boundary point y^1 of Q^1, take a curve $\eta : [0,1] \to Q^1 \cup \{y^1\}$ such that $\eta(0) = y_0^1$ and $\eta(1) = y^1$. Then the limit of $h_{n+1}^{-1}(\eta(t))$ as $t \to 1$ determines an element of $h_{n+1}^{-1}(y^1)$ which is contained in the boundary of Q_ϵ^1. Hence $h_{n+1}|\partial Q_\epsilon^1$ is surjective and we obtain property $(1, n+1)$.

Next, suppose that $q := \mathrm{card}(h_{n+1}^{-1}(y^1)) \geq 2$. Note that η determines an access to y^1 within Q^1 and an element of $h_{n+1}^{-1}(y^1)$. Thus $q \geq 2$ means that there are two or more distinct accesses to y^1 (more precisely, there are two or more distinct prime ends of Q^1 at y^1). By the definition of Ω^{n+1}, $f^{n+1}(y^1)$ must be a parabolic point with $p \geq 1$ petals such that $q = p \cdot \deg(f^{n+1}, y^1) \geq 2$. By the definition of Ω_ϵ^{n+1}, such a must be perturbed into an attracting planet, since otherwise all possible η determines the same element of $h_{n+1}^{-1}(y^1)$. Thus we obtain property $(2, n+1)$.

Finally, we obtain property $(3, n+1)$ by the fact that $h_{n+1}(x^1) = h_n(x^1)$ if $x^1 \in Z_\epsilon^1$. ■

5 Contracting property of f^{-1}

By the construction above, h_n is one of the branches of $f^{-n} \circ h_0 \circ f_\epsilon^n$. This implies, to obtain the convergence of $\{h_n\}$ on $J(f)$, it is necessary to use some kind of contracting property of the branches of f^{-1} (in other words, some kind of expanding property of f) near the Julia set. In this section, to obtain such a property of f, we follow [16, *Step* 2-5] with brief sketches of the proofs. The idea is originally due to A. Douady and J. H. Hubbard[4, Exposé No.X].

5.1 Branched covering of Ω

There exists a function $v : \Omega \to \mathbb{N}$ such that $v(x)$ is the multiple of $v(y) \cdot \deg(f, y)$ for each $y \in f^{-1}(x)$. For example,

$$v(x) = \prod_{f^n(y)=x} \deg(f, y)$$

satisfies this condition. Here we take v as the function which takes minimal possible values. Note that $Z = \{x \in \Omega : v(x) \geq 2\}$.

Let O be an open δ-neighborhood of Ω with $\delta \ll 1$. Then O contains a neighborhood of each $a \in A$. For $x \in O - \Omega$, set $v(x) = 1$. Let us take an N-sheeted branched covering $q : O^* \to O$ such that:

- O^* is connected;
- there are $N/v(x)$ points over $x \in O$; and
- for any $y \in q^{-1}(x)$, $\deg(q, y) = v(x)$.

Now set $U := \mathrm{Int}(\Omega)$, $U^* := q^{-1}(U)$ and $\Omega^* := q^{-1}(\Omega)$. For U^* let us take the universal covering $\pi : \mathbb{D} \to U^*$, where $\mathbb{D}$ is the unit disk. Then we obtain a branched covering $p := q \circ \pi : \mathbb{D} \to U$.

Let Γ be the fundamental group of U^* and $\Lambda(\Gamma)$ the limit set of Γ. By lifting paths in Ω^* terminating at boundary points, we can continuously extend π to the ideal boundary, $\pi|(\partial\mathbb{D} - \Lambda(\Gamma)) \to \partial\Omega^*$. Thus we obtain a branched covering $p : \overline{\mathbb{D}} - \Lambda(\Gamma) \to \Omega$.

Remark. For a parabolic point a of f with multiple petals, every component of $E_a - \{a\}$ defines a different access to a. For such accesses, corresponding ideal boundary points of $\partial\mathbb{D} - \Lambda(\Gamma)$ over a are distinct.

5.2 Lifting f^{-1}

Next, we lift f^{-1} to the branched covering $\overline{\mathbb{D}} - \Lambda(\Gamma)$ of Ω.

Proposition 5.1 *There is a holomorphic map $g : \mathbb{D} \to \mathbb{D}$ such that $f \circ p \circ g = p$. Moreover, g can be extended to $g : \overline{\mathbb{D}} - \Lambda(\Gamma) \to \overline{\mathbb{D}} - \Lambda(\Gamma)$ continuously.*

Sketch of the proof. For $x \in \Omega$, we take a small disk neighborhood B_x. Let G be one of the components of $q^{-1}(B_x)$, and H that of $(f \circ q)^{-1}(B_x)$. Then there exists a unique y such that $\{y\} = f^{-1}(x) \cap q(H)$. By taking suitable local coordinates, $q|G \to B_x$ and $(f \circ q)|H \to B_x$ are represented as $z \mapsto z^{v(x)}$ and $z \mapsto z^{v(y) \deg(f,y)}$ respectively. Thus we can define the unique map $g_{GH} : G \to H$ which has the form

$$z \mapsto z^{v(x)/(v(y) \deg(f,y))}$$

as a branch of $(f \circ q)^{-1} \circ q$.

Let us fix $x_0 \in \Omega - Z$ and $\tilde{x}_0 \in p^{-1}(x_0)$. Let η be a curve $\eta : [0,1] \to \Omega^*$ with $\eta(0) = \pi(\tilde{x}_0)$ and $\eta((0,1)) \subset U^*$, and η' be the unique lifting of η by π with $\tilde{\eta}(0) = \tilde{x}_0$. Now we consider analytic continuation of the function elements $\{g_{GH}\}$ along $\tilde{\eta}$. Let $g_{G_0H_0}$ be a function element at $\pi(\tilde{x}_0)$. Since $\overline{\mathbb{D}} - \Lambda(\Gamma)$ is simply connected, the analytic continuation of $g_{G_0H_0}$ along $\tilde{\eta}$ determines a unique function element at $\tilde{\eta}(1)$. Next, by ranging over all possible η, we obtain $g : \overline{\mathbb{D}} - \Lambda(\Gamma) \to \overline{\mathbb{D}} - \Lambda(\Gamma)$. It is clear that $g|\mathbb{D}$ is holomorphic.■

5.3 The metric ρ

Proposition 5.2 *There exists a piecewise continuous metric ρ with the following properties:*

- *ρ is defined on $U - Z$ and small disk neighborhoods for each parabolic point of f.*
- *For every C^1 curve $\eta \subset f^{-1}(\Omega) = \Omega^1$,*

$$\text{length}_\rho(f \circ \eta) > \text{length}_\rho(\eta).$$

So f is expanding for ρ in the sense of this inequality.

Sketch of the proof. Let $\rho_0 = u_0(z)|dz|$ be a metric of $U - Z$ induced from the Poincaré metric of $\mathbb{D}$ by the branched covering $p : \mathbb{D} \to U$. Note that $u_0(z) \asymp |z - b|^{-1+1/v(b)}$ near $b \in Z$. Thus any rectifiable curve $\eta : [0,1] \to U$ passing through Z has finite length with respect to ρ_0.

However, any curve in $f^{-1}(\Omega)$ terminating at A has infinite length with respect to ρ_0. So we try to modify ρ_0 so that such a curve has finite length.

For a sufficiently small $\delta > 0$ and for each $a \in A$, set $\mathcal{D}_a := B_\sigma(a, \delta)$ and $\mathcal{D} := \bigcup_{a \in A} \mathcal{D}_a$. Note that $\Omega \cap \mathcal{D}$ is a finite union of narrow cusps near the repelling directions. Thus on each $\mathcal{D}_a$, we can take a suitable local coordinate ζ_a such that f is strictly expanding from the metric $|d\zeta_a|$ to the metric $|d\zeta_{f(a)}|$ on any compact subset of $f^{-1}(\Omega \cap \mathcal{D}_{f(a)}) \cap \mathcal{D}_a - \{a\}$. Furthermore, we take a sufficiently large $M > 0$ so that for any $a \in A$, f is expanding from ρ_0 to $M|d\zeta_a|$ on a relatively compact set $f^{-1}(\Omega \cap \mathcal{D}_a - Z) - \mathcal{D}$. Set $u_a(z)|dz| := |d\zeta_a|$. Then we define the metric $\rho = u(z)|dz|$ on $U \cup \mathcal{D} - Z$ by $u(z) := \min\{u_0(z), Mu_a(z)\}$ for $z \in \mathcal{D}_a$, and by $u(z) := u_0(z)$ otherwise.

By construction, it is not difficult to show

$$u(f(z))|f'(z)| > u(z)$$

for $z \in f^{-1}(\Omega - Z) - A$. This implies

$$\text{length}_\rho(f \circ \eta) > \text{length}_\rho(\eta).$$

for every C^1 curve $\eta \subset f^{-1}(\Omega)$. ■

5.4 Continuous modulus

Let $\tilde{\rho}$ be the lifting of ρ on $p^{-1}(U - Z)$. Since $f^{-1}(\Omega) = \Omega^1$ has one or more connected components, $p^{-1}(\Omega^1)$ is either connected or has countably many connected components. Take one of the components of $p^{-1}(\Omega^1)$, say Q, and take $x,\ y \in Q$. We define the distance by

$$d_{\tilde{\rho}}(x, y) := \inf_{\tilde{\eta}} \text{length}_{\tilde{\rho}}(\tilde{\eta}),$$

where $\tilde{\eta}$ ranges over all rectifiable curves such that

$$\tilde{\eta} : [0, 1] \to p^{-1}(\Omega^1),\ \tilde{\eta}(0) = x,\ \text{and } \tilde{\eta}(1) = y.$$

Note that such $\tilde{\eta}$ has finite length with respect to $\tilde{\rho}$. Now $(Q, d_{\tilde{\rho}})$ is a complete metric space. For different components Q and Q' of $p^{-1}(\Omega^1)$, we formally define $d_{\tilde{\rho}}(x, y) := \infty$ if $x \in Q$ and $y \in Q'$.

For g, a lifting of f^{-1}, we define a function $\tau_g : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tau_g(s) := \sup\left\{d_{\tilde{\rho}}(g(x), g(y)) : x,\ y \in p^{-1}(\Omega^1),\ d_{\tilde{\rho}}(x, y) \leq s\right\}.$$

Furthermore, we define $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tau(s) := \sup\left\{\tau_g(s) : g \text{ a lifting of } f^{-1}\right\}.$$

Then we obtain:

Proposition 5.3 *τ has the following properties:*

(i) τ is an increasing and right-continuous function;

(ii) $s > \tau(s)$ for any s;

(iii) the function $s \mapsto s - \tau(s)$ is also increasing; and

(iv) For any $x,\ y \in p^{-1}(\Omega^1)$ and any lifting g of f^{-1},

$$d_{\tilde{\rho}}(g(x), g(y)) \leq \tau(d_{\tilde{\rho}}(x, y)).$$

Sketch of the proof. If we replace τ by τ_g, then (i), (ii) and (iv) are almost clear by definition. (iii) follows from the fact that $\tau_g(s_1+s_2) \leq \tau_g(s_1)+\tau_g(s_2)$. A calculation shows that there exist d distinct liftings of f^{-1}, say $g_1, \ldots, g_d$, such that any τ_g coincide with one of $\tau_{g_1}, \ldots, \tau_{g_d}$. Thus

$$\tau(s) = \sup\{\tau_{g_i}(s) : 1 \leq i \leq d\},$$

and satisfies properties (i)-(iv). ■

6 Convergence of h_n

In this section, we give the proof of the convergence of the sequence $\{h_n : \Omega_\epsilon^n \to \Omega^n\}_{n=0}^\infty$. Here the expanding property of f with respect to ρ plays an important role. For instance, we can easily show the convergence when f is hyperbolic:

Proposition 6.1 *Suppose that f is hyperbolic. For $\epsilon \ll 1$, the sequence h_n converges uniformly to the limit h_ϵ on $J(f_\epsilon)$ which satisfies $f \circ h_\epsilon = h_\epsilon \circ f_\epsilon$.*

Proof. Since f has no parabolic point nor critical point in $J(f)$, the metric ρ in Proposition 5.2 is the Poincaré metric on U. Now $\Omega^1 \subset U$ thus there is a constant C such that $f^*\rho/\rho \geq C > 1$ on Ω^1.

Note that the constant

$$M := \sup\left\{d_\rho(h_0(x), h_1(x)) : x \in \Omega_\epsilon^1\right\}$$

is finite since $h_0(\Omega_\epsilon^1) \subset U$. For any $x \in \Omega_\epsilon^2$, we obtain

$$\begin{aligned} & C d_\rho(h_1(x), h_2(x)) \\ \leq & d_\rho(f(h_1(x)), f(h_2(x))) = d_\rho(h_0(f_\epsilon(x)), h_1(f_\epsilon(x))) \\ \leq & M, \end{aligned}$$

thus $d_\rho(h_1(x), h_2(x)) \leq M/C$. Similarly, for any $x \in J(f_\epsilon)$, we obtain

$$d_\rho(h_n(x), h_{n+1}(x)) \leq M/C^n \to 0 \quad (n \to \infty).$$

(Recall that $J(f_\epsilon) \subset \Omega_\epsilon^n$ and thus $h_n|J(f_\epsilon)$ are defined for any $n \geq 0$.) Hence h_n converges uniformly and rapidly to the limit h_ϵ on $J(f_\epsilon)$. The relation $f \circ h_\epsilon = h_\epsilon \circ f_\epsilon$ follows from $f \circ h_{n+1} = h_n \circ f_\epsilon$. ■

Let us consider the general case. When f has parabolic points, it is not uniformly expanding on Ω^1. However, since it is uniformly expanding on each compact subset of Ω^1 with respect to the metric ρ, h_n converges slowly to the limit:

Proposition 6.2 *For $\epsilon \ll 1$, the sequence h_n converges uniformly to the limit h_ϵ on $J(f_\epsilon)$ which satisfies $f \circ h_\epsilon = h_\epsilon \circ f_\epsilon$. Moreover, h_ϵ can be arbitrarily close to the identity map: That is, for arbitrarily small $r > 0$, if $\epsilon \ll 1$, h_ϵ satisfies*

$$\sup\{d_\sigma(h_\epsilon(x), x) : x \in J(f_\epsilon)\} < r.$$

Proof. Let us fix an arbitrary $L > 0$. Then we may assume that

$$d_\rho(h_0(x), h_1(x)) < L - \tau(L)$$

for any $x \in J(f_\epsilon)$. In fact, by the construction of h_0 and h_1, if $\epsilon \ll 1$, $d_\rho(h_0(x), h_1(x))$ can be arbitrarily small for any $x \in J(f_\epsilon)$.

We claim that $d_\rho(h_0(x), h_n(x)) < L$ for any $n \geq 1$ and any $x \in J(f_\epsilon)$. If $n = 1$, $d_\rho(h_0(x), h_1(x)) < L - \tau(L) < L$. For $n = k$, let us assume that $d_\rho(h_0(x), h_k(x)) < L$ for any $x \in J(f_\epsilon)$. We first show that

$$d_\rho(h_1(x), h_{k+1}(x)) < \tau(L).$$

By assumption, we can take a rectifiable curve $\eta : [0, 1] \to \Omega^1$ such that

- $\eta(0) = h_0(f_\epsilon(x))$ and $\eta(1) = h_k(f_\epsilon(x))$;
- $\eta \cap Z = \emptyset$; and
- $L > \text{length}_\rho(\eta)$.

Fix $z_0 \in p^{-1}(h_0(f_\epsilon(x)))$, and let $\tilde{\eta}$ be the lifting of η by p whose initial point is z_0. Then the end point over $h_k(f_\epsilon(x))$ is uniquely determined, say z_1, and

$$\begin{aligned} L > \text{length}_\rho(\eta) &= \text{length}_{\tilde{\rho}}(\tilde{\eta}) \\ &> d_{\tilde{\rho}}(z_0, z_1). \end{aligned}$$

By using the function τ,

$$\tau(L) > \tau(d_{\tilde{\rho}}(z_0, z_1)) \geq d_{\tilde{\rho}}(g(z_0), g(z_1)),$$

where g is a lifting of f^{-1} such that $p \circ g(z_0) = h_1(x)$. Then we can take a curve $\tilde{\eta}' : [0, 1] \to \overline{\mathbb{D}} - \Lambda(\Gamma)$ such that

- $\tilde{\eta}'(0) = g(z_0)$ and $\tilde{\eta}'(1) = g(z_1)$;
- $\tilde{\eta}' \cap p^{-1}(Z) = \emptyset$; and
- $\tau(L) > \text{length}_{\tilde{\rho}}(\tilde{\eta}')$.

Hence

$$\begin{aligned} \tau(L) > \text{length}_{\tilde{\rho}}(\tilde{\eta}') &= \text{length}_\rho(p \circ \tilde{\eta}') \\ &> d_\rho(p(g(z_0)), p(g(z_1))) = d_\rho(h_1(x), h_{k+1}(x)). \end{aligned}$$

Then for $n = k + 1$ and for any $x \in J(f_\epsilon)$,

$$\begin{aligned} d_\rho(h_0(x), h_{k+1}(x)) &\leq d_\rho(h_0(x), h_1(x)) + d_\rho(h_1(x), h_{k+1}(x)) \\ &< L - \tau(L) + \tau(L) = L. \end{aligned}$$

Thus we have shown the claim by induction on n.

Let us show the convergence. By the same argument as above, for sufficiently large integer $l,\ m$,

$$\begin{aligned} d_\rho(h_l(x), h_{m+l}(x)) &< \tau^l\big(d_\rho(h_0(f_\epsilon^l(x)), h_m(f_\epsilon^l(x)))\big) \\ &< \tau^l(L) \to 0 \qquad (l \to \infty). \end{aligned}$$

Because we can take arbitrary $x \in J(f_\epsilon)$, h_n converges uniformly on $J(f_\epsilon)$ with respect to the distance d_ρ. Since the topology of Ω^n defined by d_ρ is equivalent to the topology defined by the spherical distance d_σ, h_n also converges uniformly on $J(f_\epsilon)$ with respect to d_σ. By continuity of each h_n, the limit h_ϵ is also continuous. The relation $f \circ h_\epsilon = h_\epsilon \circ f_\epsilon$ follows from $f \circ h_{n+1} = h_n \circ f_\epsilon$.

Finally we show the last part of the statement. Let us fix any $r > 0$ and suppose that $\epsilon \ll 1$. Then we can take h_0 such that $d_\sigma(x, h_0(x)) < r/2$ for any $x \in J(f_\epsilon)$. On the other hand, by the claim above, we obtain $d_\rho(h_0(x), h_\epsilon(x)) \le L$ for arbitrarily small L. Since we may also suppose that L is sufficiently small such that $d_\sigma(h_0(x), h_\epsilon(x)) < r/2$ for any $x \in J(f_\epsilon)$, we obtain

$$d_\sigma(x, h_\epsilon(x)) \le d_\sigma(x, h_0(x)) + d_\sigma(h_0(x), h_\epsilon(x)) < r.$$

■

7 Almost bijectivity and uniqueness of h_ϵ

In this section, we prove that the continuous map h_ϵ in Proposition 6.2 maps $J(f_\epsilon)$ onto $J(f)$ "almost bijectively"; that is, there are at most countably many points in $J(f)$ where h_ϵ is not one-to-one. Furthermore we prove the uniqueness of such an h_ϵ.

First we show:

Proposition 7.1 *h_ϵ maps $J(f_\epsilon)$ to $J(f)$.*

Proof. Let X denote the set of all repelling periodic points of f_ϵ. Since $h_\epsilon \circ f_\epsilon^n = f^n \circ h_\epsilon$ for any n, h_ϵ maps X to a set of periodic points of f in Ω, which must be a subset of $J(f)$. Since h_ϵ is continuous and $J(f_\epsilon) = \overline{X}$, h_ϵ maps $J(f_\epsilon)$ into $J(f)$. ■

Next, we complete the proof of Theorem 1.1 under the assumption that $J(f) \neq \hat{\mathbb{C}}$. For fixed ϵ, let $A_- = A_{-,\epsilon} \subset A$ be the set of all parabolic points of f which are perturbed into attracting planets of f_ϵ.

Proposition 7.2 *If $\epsilon \ll 1$, $h_\epsilon : J(f_\epsilon) \to J(f)$ has the following properties:*

- *(Surjectivity) h_ϵ is surjective.*

- *(Almost injectivity) If $h_\epsilon(x) = h_\epsilon(x')$ for distinct $x,\ x' \in J(f_\epsilon)$, then there exists an integer N such that $f_\epsilon^N(x)$ and $f_\epsilon^N(x')$ are repelling satellites of an attracting planet a_ϵ generated by the perturbation of a point in A_-.*

- *(Uniqueness) h_ϵ is the unique semiconjugacy between f_ϵ and f on their respective Julia sets which satisfies properties 1 and 2 in Theorem 1.1.*

By the almost injectivity above, we obtain the precise condition for h_ϵ to be a topological conjugacy.

Corollary 7.1 *h_ϵ is a topological conjugacy if and only if $A_- = \emptyset$; that is, none of the parabolic points of f is perturbed into an attracting planet.*

Proof of Proposition 7.2: Surjectivity. Fix any $y \in J(f)$. By surjectivity of h_n, there is a sequence $x_n \in \Omega_\epsilon^n \subset \Omega_\epsilon$ such that $h_n(x_n) = y$. Since Ω_ϵ is compact, $\{x_n\}$ has an accumulation point $x \in \Omega_\epsilon$ and we can choose a subsequence x_{n_k} so that $x_{n_k} \to x$ $(k \to \infty)$. Now we claim that $x \in J(f_\epsilon)$. If $x \in F(f_\epsilon)$, $f_\epsilon^n(x)$ is attracted to an attracting or parabolic cycle as $n \to \infty$. Thus there exists an N and a small disk neighborhood D such that $f_\epsilon^n(D)$ is outside of Ω_ϵ for all $n \geq N$. On the other hand, for all $k \gg 0$, we have $n_k \geq N$, $x_{n_k} \in D$, and $f_\epsilon^{n_k}(x_{n_k}) \in \Omega_\epsilon$. This is a contradiction.

Since $h_n \to h_\epsilon$ uniformly and the family $\{h_n\}$ is clearly equicontinuous, the inequality

$$d_\rho(y, h_\epsilon(x)) \leq d_\rho(h_{n_k}(x_{n_k}), h_{n_k}(x)) + d_\rho(h_{n_k}(x), h_\epsilon(x))$$

implies $y = h_\epsilon(x)$. Thus h_ϵ is surjective.

Preliminary to the almost injectivity and uniqueness. Since f is geometrically finite and the assumption that $J(f) \neq \hat{\mathbb{C}}$, f has at least one critical point in the Fatou set, and so does f_ϵ. Now we take suitable conjugations of $f_\epsilon \to f_0 = f$ by rotations of $\hat{\mathbb{C}}$ so that $\infty \in C(f_\epsilon) \cap F(f_\epsilon)$. By the construction of Ω_ϵ, there exist $R \gg 0$ such that $D(R) := \hat{\mathbb{C}} - \{|z| \leq R\}$ is a disk neighborhood of ∞ which is not contained in Ω_ϵ for all $0 \leq \epsilon \ll 1$. Then Ω_ϵ and $J(f_\epsilon)$ are bounded sets in the complex plane.

For $\delta > 0$ and $x \in \mathbb{C}$, we set

$$B(x, \delta) := \{z \in \mathbb{C} : |z - x| < \delta\},$$

which is an open Euclidean ball. Now we fix δ to be sufficiently small so that the set

$$\mathcal{B} := \bigcup_{x \in A \cup Z^1} B(x, \delta)$$

is a disjoint union of balls satisfying the following conditions:

- if an $x \in A \cup Z^1$ is periodic, then there exists a local chart on $B(x, \delta)$ as (2.2) or (2.4); and
- for $x \in Z^1 - A$, $P(f) \cap B(x, \delta) = \{x\}$.

Set $\tilde{s} := d(P(f), J(f) - \mathcal{B})$, where $d(\cdot, \cdot)$ is the distance between sets measured by Euclidean distance. Since f is geometrically finite, every critical orbit either accumulates on an attracting or parabolic cycle, or is already contained in Z^1. Hence we obtain $0 < \tilde{s} \le \delta$.

Now we claim that $d(P(f_\epsilon), J(f_\epsilon) - \mathcal{B}) > \tilde{s}/2$ for all $\epsilon \ll 1$. It suffices to restrict our attention to the perturbation of the critical orbits accumulating on A or Z^1. First, take a parabolic cycle $\alpha \subset A$ and a critical orbit accumulating to α. By horocyclicity of $f_\epsilon \to f$, we may apply Lemma 2.1. That is, for $\epsilon \ll 1$, the corresponding perturbed critical orbit of f_ϵ is contained in $\cup_{a \in \alpha} B(a, \delta) \subset \mathcal{B}$ except finitely many points in the orbit. Since $f_\epsilon \to f$ uniformly, such finitely many points are very close to the original ones. On the other hand, $J(f_\epsilon)$ is very close to $J(f)$ with respect to the Hausdorff topology, since h_ϵ maps $J(f_\epsilon)$ onto $J(f)$ and r-neighborhood of $J(f)$ with respect to the spherical distance contains $h_\epsilon^{-1}(J(f)) = J(f_\epsilon)$. (Recall that r is fixed and arbitrarily small for $\epsilon \ll 1$.) Thus such finitely many points stay away from $J(f_\epsilon) - \mathcal{B}$ for $\epsilon \ll 1$, and the distance can be at least $\tilde{s}/2$. Next, take $b \in Z^1$. Since $f_\epsilon \to f$ preserves the J-critical relations of f, for all $\epsilon \ll 1$, we may suppose that there exists a unique $b_\epsilon \in f_\epsilon^{-1}(P(f_\epsilon))$ such that $|b - b_\epsilon| < \delta/2$. For such b_ϵ, $d(b_\epsilon, J(f_\epsilon) - \mathcal{B}) \ge \delta/2 \ge \tilde{s}/2$. Thus we conclude the claim.

Replacing f_ϵ (resp. f) by its suitable iteration, we may consider the extreme case where every point in $h_0^{-1}(A) \cup Z_\epsilon$ (resp. $A \cup Z$) is a fixed point of f_ϵ (resp. f), and the multipliers of all parabolic points are 1. Then Z_ϵ and Z are the sets of all critical values of f_ϵ and f on their respective Julia sets.

Set $\mathit{\Gamma}_- = \mathit{\Gamma}_{-,\epsilon} := h_0^{-1}(A_-)$, the set of all repelling satellites generated by the perturbation of parabolic points in A_-. Note that now every element in A_- or $\mathit{\Gamma}_-$ is a fixed point of f or f_ϵ respectively. Also, note that $\mathit{\Gamma}_-$ and Z_ϵ are disjoint.

Almost injectivity. Now let us start the discussion on the almost injectivity of h_ϵ. We suppose that $h_\epsilon(x) = h_\epsilon(x')$ for distinct $x,\ x' \in J(f_\epsilon)$. Set $x_n := f_\epsilon^n(x)$ and $x'_n := f_\epsilon^n(x')$. Then $h_\epsilon(x_n) = h_\epsilon(x'_n)$ because $f^n \circ h_\epsilon = h_\epsilon \circ f_\epsilon^n$. Recall that $d_\sigma(x, h_\epsilon(x)) < r$ for any $x \in J(f_\epsilon)$. Thus we obtain

$$d_\sigma(x_n, x'_n) \le d_\sigma(x_n, h_\epsilon(x_n)) + d_\sigma(h_\epsilon(x'_n), x'_n) < 2r$$

and it implies $|x_n - x'_n| = O(r)$. Indeed, since the Julia set is contained in $\hat{\mathbb{C}} - D(R)$, there exists a constant $M \approx 1 + R^2$ such that $|x_n - x'_n| \le Mr$ for sufficiently small r. Now we set

$$\tilde{r} := \sup_n |x_n - x'_n| \ \ (\le Mr).$$

Then we may suppose that r is sufficiently small such that $\tilde{r} \le Mr < \tilde{s}/2$ for $\epsilon \ll 1$. Note that $\tilde{r} \le Mr < \delta/2$ also holds.

For the orbit of the x and x', we consider the following three cases:

1. Both x_n and x'_n land on Γ_-.
2. x_n lands on Γ_- but x'_n never lands on Γ_-.
3. Both x_n and x'_n never land on Γ_-.

Case 1: Suppose that x_n lands on $h_\epsilon^{-1}(a)$ for some $a \in A_-$ when $n = N$. Here $h_\epsilon^{-1}(a) \subset \Gamma_-$ is a set of repelling fixed points contained in $B_\sigma(a, r)$. By the facts that

$$B_\sigma(a, r) \subset B(a, Mr) \subset B(a, \delta/2)$$

and $\tilde{r} < \delta/2$, x'_n must be contained in $B(a, \delta)$ for all $n \ge N$. If $x'_N \notin h_\epsilon^{-1}(a)$, by the local dynamics of f_ϵ on $B(a, \delta)$ in the form (2.4), x'_n goes out of $B(a, \delta)$. Thus $x'_N \in h_\epsilon^{-1}(a)$; that is, x_n and x'_n simultaneously land on repelling satellites in $h_\epsilon^{-1}(a)$, when $n = N$.

Hence we need to show that the other cases cannot occur.

Case 2: We suppose again that x_n lands on $h_\epsilon^{-1}(a)$ for some $a \in A_-$ when $n = N$. By the same argument as Case 1, x'_n must be contained in $B(a, \delta)$ for all $n \ge N$. However, $x'_n \notin h_\epsilon^{-1}(a) \subset \Gamma_-$, and thus by the local dynamics of f_ϵ on $B(a, \delta)$ in the form (2.4), x'_n goes out of $B(a, \delta)$. This is a contradiction.

Case 3: Furthermore we need to consider the following three cases:

I. x_n lands on Z_ϵ but x'_n never lands on Z_ϵ.

II. Both x_n and x'_n land on Z_ϵ.

III. Both x_n and x'_n never land on Z_ϵ.

Case 3-I: Suppose that x_n lands on $h_\epsilon^{-1}(b)$ for some $b \in Z$ when $n = N$. Here $h_\epsilon^{-1}(b) \subset Z_\epsilon$ is a repelling or parabolic fixed point of f_ϵ contained in $B_\sigma(b, r)$. By the same argument as above, x'_n must be contained in $B(b, \delta)$ for $n \ge N$. Now x'_n never lands on Z_ϵ. This implies, by the local dynamics of f_ϵ on $B(b, \delta)$ in the form (2.2) or (2.4), x'_n goes out of $B(b, \delta)$. This is also a contradiction.

Case 3-II: Since $\tilde{r} < \delta/2$ and all elements of Z_ϵ^1 remain at leaset δ apart, the orbits of x and x' have merged before landing on Z_ϵ: That is, there exist two integers N_1 and N_2 with $N_1 < N_2$ such that

- $x_{N_1} \ne x'_{N_1}$ and $x_{N_1+1} = x'_{N_1+1}$, and

- $x_{N_2} = x'_{N_2} \in Z^1_\epsilon$ and $x_{N_2+1} = x'_{N_2+1} \in Z_\epsilon$.

Set $w := x_{N_1+1} = x'_{N_1+1}$. Since w is not contained in Z_ϵ, which is the set of critical values, the inverse image $f_\epsilon^{-1}(w)$ consists of d distinct points. (Recall that d is the degree of f.) Similarly, by the construction of h_ϵ, $h_\epsilon(w) =: z$ is not contained in Z and $f^{-1}(z)$ also consists of d distinct points. Moreover, since h_ϵ is surjective, $h_\epsilon^{-1}(f^{-1}(z))$ must consist of at least d points.

Note that $f_\epsilon^{-1}(w) \subset h_\epsilon^{-1}(f^{-1}(z))$. Since $h_\epsilon(x_{N_1}) = h_\epsilon(x'_{N_1})$ for distinct x_{N_1}, $x'_{N_1} \in f_\epsilon^{-1}(w)$, there exists an $x'' \in h_\epsilon^{-1}(f^{-1}(z)) - f_\epsilon^{-1}(w)$ which satisfies $f_\epsilon(x'') \neq w$ and $h_\epsilon(f_\epsilon(x'')) = z$. Setting $w' := f_\epsilon(x'')$, we obtain $h_\epsilon(w) = h_\epsilon(w')$ for $w \neq w'$. Let us replace x and x' by w and w' respectively. This reduces Case 3-II with $x_{N_2} \in Z^1_\epsilon$ to Case 3-I or 3-II with $x_{N_2-N_1-1} \in Z^1_\epsilon$.

However, as we have seen, Case 3-I implies a contradiction. In Case 3-II, we can repeat the argument above. Hence we eventually consider the case where $h_\epsilon(x) = h_\epsilon(x')$ for $x \neq x'$ with $x \in Z^1_\epsilon$.

Suppose that $f_\epsilon(x) = h_\epsilon^{-1}(b)$ for some $b \in Z$. Then $f_\epsilon(x)$ is contained in $B_\sigma(b, r)$ and is a repelling or parabolic fixed point. On the other hand, since the elements of Z^1_ϵ remain separated, $x \neq x'$ implies $x' \notin Z^1_\epsilon$, and thus $f_\epsilon(x') \notin Z_\epsilon$. By the local dynamics of f_ϵ on $B(b, \delta)$ in the form (2.2) or (2.4), $f_\epsilon(x')$ is not a fixed point and goes out of $B(b, \delta)$. This is a contradiction.

Case 3-III: If either x_n or x'_n lands in $\mathcal{B}$, it goes out of $\mathcal{B}$ by finitely many iterations of f_ϵ. Now we take a subsequence $\{n_k\}$ of $\{n\}$ so that each x_{n_k} is never contained in $\mathcal{B}$; that is, $x_{n_k} \in J(f_\epsilon) - \mathcal{B}$. Recall that $d(P(f_\epsilon), J(f_\epsilon) - \mathcal{B}) > \tilde{s}/2$. For any s satisfying $\tilde{r} < s < \tilde{s}/2$ and for any $k \gg 0$, there exists a branch g_{n_k} of $f_\epsilon^{-n_k}$ on $B(x_{n_k}, s)$ which is univalent and $g_{n_k}(x_{n_k}) = x$. Set $V_{n_k} := g_{n_k}(B(x_{n_k}, \tilde{r}))$. Then V_{n_k} contains x and x'. By applying the Koebe distortion theorem to g_{n_k} on $B(x_{n_k}, s)$, we obtain

$$\operatorname{diam} V_{n_k} = O(|g'_{n_k}(x_{n_k})|) = O(1/|(f_\epsilon^{n_k})'(x)|).$$

If $|P(f_\epsilon)| < 3$, f_ϵ is conjugate to $z \mapsto z^{\pm d}$, and thus it is hyperbolic. On the Julia set, $|(f_\epsilon^{n_k})'(x)| \to \infty$ as $k \to \infty$ hence $\lim(\operatorname{diam} V_{n_k}) = 0$. It contradicts $x \neq x'$.

If $|P(f_\epsilon)| \geq 3$, let ρ_ϵ be the Poincaré metric of $\hat{\mathbb{C}} - P(f_\epsilon)$. By [13, Theorem 3.6], since $x_n \notin P(f_\epsilon)$ for any n, we obtain

$$\|(f_\epsilon^n)'(x)\|_{\rho_\epsilon} = \frac{\rho_\epsilon(f_\epsilon^n(x))|(f_\epsilon^n)'(x)|}{\rho_\epsilon(x)} \to \infty \quad (n \to \infty).$$

Now recall again that $d(P(f_\epsilon), J(f_\epsilon) - \mathcal{B}) > \tilde{s}/2$. Since x_{n_k} and x stay away from $P(f_\epsilon)$, $\rho_\epsilon(f_\epsilon^{n_k}(x))$ and $\rho_\epsilon(x)$ are bounded. Hence $|(f_\epsilon^{n_k})'(x)| \to \infty$ as $k \to \infty$, then $\lim(\operatorname{diam} V_{n_k}) = 0$. It contradicts $x \neq x'$ again.

Uniqueness. From Proposition 6.2, Proposition 7.1 and the proof of the almost bijectivity above, it is easy to check that h_ϵ satisfies properties in Theorem 1.1. In particular, we obtain property 1 in Theorem 1.1 from the almost injectivity discussed above and property $(2, n)$ of h_n in Proposition 4.1.

Let h'_ϵ be another semiconjugacy between f_ϵ and f on their respective Julia sets with properties 1 and 2 in Theorem 1.1. Take a repelling periodic point x of f_ϵ which has period more than one. By our assumption that $h_\epsilon^{-1}(A) \cup Z_\epsilon$ is a set of fixed points, x does not belong to $\Gamma_- \cup Z_\epsilon$. By surjectivity of h'_ϵ, there exists an $x' \in J(f_\epsilon)$ such that

$$h_\epsilon(x) = h'_\epsilon(x').$$

It is easy to see that $h_\epsilon(x)$ and x' are also repelling periodic points with the same period as x.

Set $x_n := f_\epsilon^n(x)$ and $x'_n := f_\epsilon^n(x')$. Then $h_\epsilon(x_n) = h_\epsilon(x'_n)$ because h_ϵ and h'_ϵ are semiconjugacies. Moreover, we obtain $d_\sigma(x_n, x'_n) < 2r$ from property 2 in Theorem 1.1. Thus

$$|x_n - x'_n| \leq Mr < \delta/2$$

for all n and we may suppose that x'_n belongs to $\Gamma_- \cup Z_\epsilon$ as well as x_n.

Now we can apply the same argument as Case 3-III of the proof of the almost injectivity, and we conclude that $x = x'$. This means that $h_\epsilon = h'_\epsilon$ on the dense subset of $J(f_\epsilon)$, because repelling periodic points are dense in the Julia set. Since h_ϵ and h'_ϵ are continuous, h'_ϵ must coincide with h_ϵ on $J(f_\epsilon)$. ■

8 Geometrically finite maps with the empty Fatou set

In this section, we prove Theorem 1.1 for a geometrically finite rational map f with $J(f) = \hat{\mathbb{C}}$ by using the same idea as in the case of $J(f) \neq \hat{\mathbb{C}}$.

Now f has no parabolic or (super)attracting periodic point. Moreover, by the geometric finiteness, every critical point of f is preperiodic; that is, f is postcritically finite. Then we can consider the orbifold $\mathcal{O}_f$ with base space $\hat{\mathbb{C}}$ which is parabolic or hyperbolic type[13, §A]. This $\mathcal{O}_f$ has an orbifold metric $\rho = \rho(z)|dz|$ which is induced from the Euclidean or hyperbolic metric of the universal covering. In both cases, there exists a constant $C > 1$ such that

$$\|f'\|_\rho := \frac{f^*\rho}{\rho} \geq C.$$

(See the argument in [13, Theorem A.6]). Note that ρ has singularity at $b \in P(f)$ as $|d(z-b)^{1/v(b)}|$.

Let us consider a horocyclic perturbation $f_\epsilon \to f$ preserving the J-critical relations of f. Since f has no parabolic point, horocyclicity is trivial. By the J-critical relations of f, f_ϵ is also postcritically finite. Since f has no attracting or superattracting periodic point, f_ϵ has no superattracting periodic point: This implies $J(f_\epsilon)$ is also the whole sphere (See [13, Theorem A.6] again).

Now let us begin the construction of h_ϵ.

Proof of Theorem 1.1 in the case of $J(f) = \hat{\mathbb{C}}$. First, set $\Omega := \hat{\mathbb{C}}$ and $\Omega_\epsilon := \hat{\mathbb{C}}$. We take $h_0 : \Omega_\epsilon \to \Omega$ as a homeomorphism which satisfies condition 5 of Proposition 3.2. For any fixed $r > 0$, if $\epsilon \ll 1$, such h_0 satisfies $d_\sigma(h_0(x), x) < r$ for all $x \in \hat{\mathbb{C}}$.

Next, we lift h_0 to the family of homeomorphism $\{h_n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}_{n=1}^{\infty}$ as in Proposition 4.1. We can show that h_n converges to the limit h_ϵ in the same way as Proposition 6.1. In fact, we may replace the Poincaré metric in the proof of Proposition 6.1 with the orbifold metric ρ of $\mathcal{O}_f$. Furthermore, we can also lift h_0^{-1} to the uniformly convergent sequence of homeomorphisms $\{h_n^{-1}\}$. The limit must be surjective and thus $h_\epsilon : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a homeomorphism.

Finally, we show the uniqueness in the same way as Proposition 7.2: Let h'_ϵ be another conjugacy with property 2 in Theorem 1.1, and x be a repelling periodic point of f_ϵ which does not belong to $P(f)$. Since h'_ϵ is a homeomorphism, there exists a unique x' such that $h_\epsilon(x) = h'_\epsilon(x')$. Set $x_n := f_\epsilon^n(x)$ and $x'_n := f_\epsilon^n(x')$. By using the uniformly expanding property of f_ϵ with respect to the orbifold metric ρ_ϵ of $\mathcal{O}_{f_\epsilon}$, $d_{\rho_\epsilon}(x, x')$ is bounded by $d_{\rho_\epsilon}(x_n, x'_n)/C_\epsilon^n$ with $C_\epsilon > 1$. This implies $x = x'$. Thus $h_\epsilon = h'_\epsilon$ on a dense subset of the sphere, which is a set of repelling periodic points. By continuity of h_ϵ and h'_ϵ, we obtain $h_\epsilon = h'_\epsilon$ on the whole sphere. ■

Remark. If the orbifold $\mathcal{O}_f$ does *not* have signature $(2, 2, 2, 2)$, by Thurston's theorem([5], [13, Theorem B.2]), h_ϵ is a Möbius transformation which conjugates f_ϵ to f. Here we gave a general construction of the conjugacy h_ϵ including such a particular case of signature (2,2,2,2).

Acknowledgements. I would like to thank H. Shiga, M. Shishikura, H. Sumi, M. Taniguchi and the referee for their helpful advice and encouragement. This paper was revised during my stay in SUNY at Stony Brook. I also would like to thank M. Lyubich and S. Sutherland for their hospitality and helpful comments.

References

[1] A.F. Beardon. *Iteration of Rational Functions*. Springer-Verlag, 1991.

[2] L. Carleson and T. Gamelin. *Complex Dynamics* . Springer-Verlag, 1993.

[3] G. Cui. Geometrically finite rational maps with given combinatorics. *preprint*, 1997.

[4] A. Douady and J. H. Hubbard. *Etude dynamique des polynômes complexes I & II*. Pub. Math. d'Orsay 84–02, 85–05, 1984/85.

[5] A. Douady and J. H. Hubbard. A proof of Thurston's topological characterization of rational maps. *Acta Math.* **171**(1993), 263–297.

[6] L.R. Goldberg and J. Milnor. Fixed points of polynomial maps, Part II: Fixed point portraits. *Ann. Sci. Éc. Norm. Sup.* **26**(1993), 51–98.

[7] P. Haïsssinsky. Déformation J-équivalence de polynômes géométriquement finis. *Fund. Math.* **163**(2000), no.2, 131–141.

[8] T. Kawahira. On continuity of the Julia sets in parabolic bifurcations (in Japanese). Master's thesis, University of Tokyo, 2000.

[9] T. Kawahira. On dynamical stability of the Julia sets of parabolic rational maps. *preprint*, 2000.

[10] T. Kawahira. Semiconjugacies between the Julia sets of geometrically finite rational maps II. *preprint*, 2003.

[11] B. Maskit. *Kleinian Groups*. Springer-Verlag, 1988.

[12] C. McMullen. Hausdorff dimension and conformal dynamics II: Geometrically finite rational maps. *Comm. Math. Helv.* **75**(2000), no.4, 535–593

[13] C. McMullen. *Complex Dynamics and Renormalization*. Annals of Math Studies 135, Princeton University Press, 1994.

[14] J. Milnor. *Dynamics in one complex variable: Introductory lectures*. vieweg, 1999.

[15] R. Mañé, P. Sad and D. Sullivan. On the dynamics of rational maps. *Ann. Sci. Éc. Norm. Sup.* **16**(1983), 193–217.

[16] Tan Lei and Yin Y. Local connectivity of the Julia set for geometrically finite rational maps. *Science in China A* **39**(1996), 39–47.