A proof of simultaneous linearization with a polylog estimate

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Abstract

In this note we give an alternative proof of simultaneous linearization recently shown by T.Ueda, which connects the Schröder equation and the Abel equation analytically. Indeed, we generalize Ueda's original result so that we may apply it to the parabolic fixed points with multiple petals.

1 Perturbation of parabolics

Let f be an analytic map defined on a neighborhood of 0 in $\bar{\mathbb{C}}$ which is tangent to identity at 0. That is, f near 0 is of the form

$$f(w) \ = \ w + Aw^{m+1} + O(w^{m+2})$$

where $A \neq 0$ and $m \in \mathbb{N}$. By taking a linear coordinate change $w \mapsto A^{1/m}w$, we may assume that $A = 1$. In the theory of complex dynamics such a germ appears when we consider iteration of local dynamics near the parabolic periodic points, and plays very important roles. (See [Mi] and [Sh] for example.) Now we consider a perturbation $f_\epsilon \to f$ of the form

$$f_\epsilon(w) \ = \ \mathit{\Lambda}_\epsilon w \left(1 + w^m + O(w^{m+1})\right)$$

with $\mathit{\Lambda}_\epsilon \to 1$ as $\epsilon \to 0$. By taking branched coordinate changes $z = -\mathit{\Lambda}_\epsilon^m/(mw^m)$ and setting $\tau_\epsilon := \mathit{\Lambda}_\epsilon^{-m}$, we have

$$\begin{aligned} f_\epsilon(z) \ &= \ \tau_\epsilon z + 1 + O(|z|^{-1/m}) \\ \longrightarrow f_0(z) \ &= \ z + 1 + O(|z|^{-1/m}) \end{aligned}$$

uniformly near $w = \infty$ on the Riemann sphere $\hat{\mathbb{C}}$. The aim of this note is to give partially linearizing coordinates of f_ϵ that depend continuously on ϵ under some condition of $\tau_\epsilon \to 1$.

2 Simultaneous linearization

We first formalize radial accesses to 1 in the complex plane: For a variable $\tau \in \mathbb{C}$ converging to 1, we say $\tau \to 1$ *radially* (or more precisely, α*-radially*) if τ satisfies $|\arg(\tau - 1)| \le \alpha$ for some fixed $\alpha \in [0, \pi/2)$.

Ueda's modulus. Let us consider a continuous family of complex numbers $\{\tau_\epsilon\}$ with $\epsilon \in [0,1]$ such that $|\tau_\epsilon| \ge 1$ and $\tau_\epsilon \to 1$ α-radially as $\epsilon \to 0$. For simplicity we assume that $\tau_\epsilon = 1$ iff $\epsilon = 0$. Set $\ell_\epsilon(z) := \tau_\epsilon z + 1$, which is an isomorphism of the Riemann sphere $\hat{\mathbb{C}}$. If $\epsilon > 0$, then $b_\epsilon := 1/(1-\tau_\epsilon)$ is the repelling fixed point of ℓ_ϵ with $\ell_\epsilon(z) - b_\epsilon = \tau_\epsilon(z - b_\epsilon)$. Thus the function

$$N_\epsilon(z) \ := \ |z - b_\epsilon| - |b_\epsilon|$$

satisfies the uniformly increasing property

$$N(\ell_\epsilon(z)) \ \ge \ |\tau_\epsilon| N(z) + \cos\alpha.$$

Similarly, if $\epsilon = 0$, the function

$$N_0(z) \ := \ \sup\left\{\mathrm{Re}\,(e^{i\theta} z) \ : \ |\theta| < \alpha\right\}$$

also has the corresponding property

$$N_0(\ell_0(z)) \ \ge \ N_0(z) + \cos\alpha.$$

In both cases, set

$$\mathbb{V}_\epsilon(R) \ := \ \{z \in \mathbb{C} : N_\epsilon(z) \ge R\}$$

for $R > 0$. One can check that $N_\epsilon(z) \le |z|$ and $\mathbb{V}_\epsilon(R) \subset \mathbb{B}(R) := \{z \in \mathbb{C} : |z| \ge R\}$ for all $\epsilon \in [0,1]$.

We establish:

Theorem 2.1 (Simultaneous linearization) *Let $\{f_\epsilon : \epsilon \in [0,1]\}$ be a family of holomorphic maps on $\mathbb{B}(R)$ such that as $\epsilon \to 0$ we have the uniform convergence on compact sets of the form*

$$\begin{aligned} f_\epsilon(z) \ &= \ \tau_\epsilon z + 1 + O(1/|z|^\sigma) \\ \longrightarrow f_0(z) \ &= \ z + 1 + O(1/|z|^\sigma) \end{aligned}$$

for some $\sigma \in (0,1]$ and $\tau_\epsilon \to 1$ radially. If $R \gg 0$, then:

(1) For any $\epsilon \in [0,1]$ there exists a holomorphic map $u_\epsilon : \mathbb{V}_\epsilon(R) \to \mathbb{C}$ such that

$$u_\epsilon(f_\epsilon(z)) \ = \ \tau_\epsilon u_\epsilon(z) + 1.$$

(2) For any compact set K contained in $\mathbb{V}_\epsilon(R)$ for all $\epsilon \in [0,1]$, $u_\epsilon \to u_0$ uniformly on K.

This theorem is a mild generalization of Ueda's theorem in [Ue] that deals with the case of $\sigma = 1$. This plays a crucial role to show the continuity of tessellation of the filled Julia set for hyperbolic and parabolic quadratic maps. See [Ka].

A remark on the domain of convergence. We can take such a compact subset K above in

$$\begin{aligned}\Pi(R) &:= \mathbb{C} - \left\{e^{\theta i}z : \mathrm{Re}\, z < R,\ |\theta| \le \alpha\right\} \\ &= \{z \in \mathbb{C} : \mathrm{Re}\,(z - R') \ge |z - R'| \sin\alpha\},\end{aligned}$$

which is a closed sector at $z = R' = R/(\cos\alpha) > 0$. In fact, for any $R > 0$ and $\epsilon \in [0,1]$, $\Pi(R)$ is contained in $\mathbb{V}_\epsilon(R)$. One can check it as follows: Now the complement of $\mathbb{V}_\epsilon(R)$ is contained in $\left\{e^{\theta i}z : \mathrm{Re}\, z < R,\ \theta = \arg(-b_\epsilon)\right\}$. Since $|\arg(-b_\epsilon)| \le \alpha$, we have the claim.

In the next section we give a proof of this theorem that is also an alternative proof of Ueda's simultaneous linearization when $\sigma = 1$. His original proof given in [Ue] uses a technical difference equation but it makes the proof beautiful and the statement a little more detailed. Here we present a simplified proof based on the argument of [Mi, Lemma 10.10] and an estimate on polylogarithm functions given in §4.

3 Proof of the theorem

Let us start with a couple of lemmas. Set $\delta := (\cos\alpha)/2 > 0$. We first check:

Lemma 3.1 *If $R \gg 0$, there exists $M > 0$ such that $|f_\epsilon(z) - (\tau_\epsilon z + 1)| \le M/|z|^\sigma$ on $\mathbb{B}(R)$ and $N_\epsilon(f_\epsilon(z)) \ge N_\epsilon(z) + \delta$ on $\mathbb{V}_\epsilon(R)$ for any $\epsilon \in [0,1]$.*

Proof. The first inequality and the existence of M is obvious. By replacing R by a larger one, we have $|f_\epsilon(z) - (\tau_\epsilon z + 1)| \le M/R^\sigma < \delta$ on $\mathbb{B}(R)$. Then

$$N_\epsilon(f_\epsilon(z)) \ \ge\ N_\epsilon(\ell_\epsilon(z)) - \delta \ \ge\ N_\epsilon(z) + \delta.$$

■

Let us fix such an $R \gg 0$. Then the lemma above implies that $f_\epsilon(\mathbb{V}_\epsilon(R)) \subset \mathbb{V}_\epsilon(R)$. Moreover, since $N_\epsilon(z) \le |z|$, we have

$$|f_\epsilon^n(z)| \ \ge\ N_\epsilon(f_\epsilon^n(z)) \ \ge\ N_\epsilon(z) + n\delta \ \ge\ R + n\delta \ \to\ \infty. \tag{2.1}$$

Thus $\mathbb{V}_\epsilon(R)$ is contained in the basin of infinity and uniformly attracted to ∞ in spherical metric of $\hat{\mathbb{C}}$. In particular, this convergence to ∞ is uniform on $\Pi(R)$ for any ϵ.

Next we show a key lemma for the theorem:

Lemma 3.2 *There exists $C > 0$ such that for any $\epsilon \in [0,1]$ and $z_1,\ z_2 \in \mathbb{B}(2S)$ with $S > R$, we have:*

$$\left|\frac{f_\epsilon(z_2) - f_\epsilon(z_1)}{z_2 - z_1} - \tau_\epsilon\right| \ \le\ \frac{C}{S^{1+\sigma}}.$$

Proof. Set $g_\epsilon(z) := f_\epsilon(z) - (\tau_\epsilon z + 1)$. For any $z \in \mathbb{B}(2S)$ and $w \in \mathbb{D}(z, S) := \{w : |w - z| < S\}$, we have $|w| > S$. This implies $|g_\epsilon(w)| \le M/|w|^\sigma < M/S^\sigma$ and thus g_ϵ maps $\mathbb{D}(z, S)$ into $\mathbb{D}(0, M/S^\sigma)$. By the Cauchy integral formula (or the Schwarz lemma), it follows that $|g'_\epsilon(z)| \le (M/S^\sigma)/S = M/S^{1+\sigma}$ on $\mathbb{B}(2S)$.

Let $[z_1, z_2]$ denote the oriented line segment from z_1 to z_2. If $[z_1, z_2]$ is contained in $\mathbb{B}(2S)$, the inequality easily follows by

$$|g_\epsilon(z_2) - g_\epsilon(z_1)| \;=\; \left|\int_{[z_1,z_2]} g'_\epsilon(z)dz\right| \;\le\; \int_{[z_1,z_2]} |g'_\epsilon(z)||dz| \;\le\; \frac{M}{S^{1+\sigma}}|z_2 - z_1|$$

and by taking $C := M$. Otherwise we have to take a roundabout way to get the estimate. Now $[z_1, z_2]$ and $\partial\mathbb{B}(2S)$ have two crossing points w_1 and w_2 such that w_1 is closer to z_1 than w_2. (When z_1 or z_2 is on the boundary of $\mathbb{B}(2S)$, we set $w_1 := z_1$ or $w_2 := z_2$.) Without loss of generality, we may assume that $z_1 < 0$ and $\operatorname{Im} z_2 \ge 0$. Let $\{z_1, z_2\}$ denote the semicircle over the segment $[z_1, z_2]$ that is contained in the upper half space (Figure 1). Let $H = \operatorname{hull}\{z_1, z_2\}$ be its convex hull in the plane. By the same way, we take semicircles $\{z_1, w_2\}$ and $\{w_1, w_2\}$ over the segments $[z_1, w_2]$ and $[w_1, w_2]$ such that they are contained in H. Now we have

$$H - \mathbb{B}(2S) \;\subset\; \operatorname{hull}\{w_1, w_2\} \;\subset\; \operatorname{hull}\{z_1, w_2\} \;\subset\; H$$

thus $\{z_1, z_2\}$ is contained in $\mathbb{B}(2S)$. Hence

$$|g_\epsilon(z_2) - g_\epsilon(z_1)| \;=\; \left|\int_{\{z_1,z_2\}} g'_\epsilon(z)dz\right| \;\le\; \int_{\{z_1,z_2\}} |g'_\epsilon(z)||dz| \;\le\; \frac{M}{S^{1+\sigma}} \cdot \frac{\pi}{2}|z_2 - z_1|$$

and the lemma holds by setting $C := M\pi/2$ $(> M)$ for any $z_1,\ z_2 \in \mathbb{B}(2S)$. ■

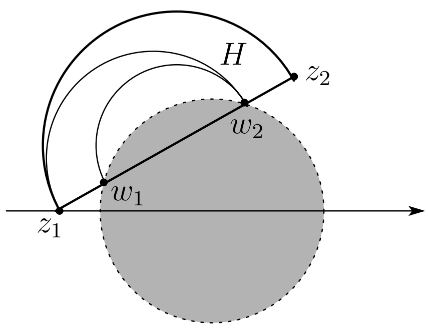

Figure 1: A roundabout way.

Proof of Theorem 2.1. Set $z_n := f_\epsilon^n(z)$ for $z \in \mathbb{V}_\epsilon(2R)$. Note that such z_n satisfies $|z_n| \ge N_\epsilon(z_n) \ge 2R + n\delta$ by (2.1). Now we fix $a \in \mathbb{V}_\epsilon(2R)$ and define $\phi_{n,\epsilon} = \phi_n : \mathbb{V}_\epsilon(2R) \to \mathbb{C}$ $(n \ge 0)$ by

$$\phi_n(z) \;:=\; \frac{z_n - a_n}{\tau_\epsilon^n}.$$

For example, one can take such an a in $\Pi(2R)$ independently of ϵ. Then we have

$$\left|\frac{\phi_{n+1}(z)}{\phi_n(z)} - 1\right| \;=\; \left|\frac{z_{n+1} - a_{n+1}}{\tau_\epsilon(z_n - a_n)} - 1\right| \;=\; \frac{1}{|\tau_\epsilon|} \cdot \left|\frac{f_\epsilon(z_n) - f_\epsilon(a_n)}{z_n - a_n} - \tau_\epsilon\right|.$$

We apply Lemma 3.2 with $2S = 2R + n\delta$. Since $z_n,\ a_n \in \mathbb{V}_\epsilon(2S) \subset \mathbb{B}(2S)$, we have

$$\left|\frac{\phi_{n+1}(z)}{\phi_n(z)} - 1\right| \;\le\; \frac{C}{|\tau_\epsilon|(R + n\delta/2)^{1+\sigma}} \;\le\; \frac{C'}{(n+1)^{1+\sigma}},$$

where $C' = 2^{1+\sigma}C/\delta^{1+\sigma}$ and we may assume $R > \delta/2$. Set $P := \prod_{n\ge 1}(1 + C'/n^{1+\sigma})$. Since $|\phi_{n+1}(z)/\phi_n(z)| \le 1 + C'/(n+1)^{1+\sigma}$, we have

$$|\phi_n(z)| \;=\; \left|\frac{\phi_n(z)}{\phi_{n-1}(z)}\right| \cdots \left|\frac{\phi_1(z)}{\phi_0(z)}\right| \cdot |\phi_0(z)| \;\le\; P|z - a|.$$

Hence

$$|\phi_{n+1}(z) - \phi_n(z)| \;=\; \left|\frac{\phi_{n+1}(z)}{\phi_n(z)} - 1\right| \cdot |\phi_n(z)| \;\le\; \frac{C'P}{(n+1)^{1+\sigma}} \cdot |z - a|.$$

This implies that $\phi_\epsilon = \phi_0 + (\phi_1 - \phi_0) + \cdots = \lim \phi_n$ converges uniformly on compact subsets of $\mathbb{V}_\epsilon(2R)$ and for all $\epsilon \in [0, 1]$. The univalence of ϕ_ϵ is shown in the same way as [Mi, Lemma 10.10].

Next we claim that $\phi_\epsilon(f_\epsilon(z)) = \tau_\epsilon \phi_\epsilon(z) + B_\epsilon$ with $B_\epsilon \to 1$ as $\epsilon \to 0$. One can easily check that $\phi_n(f_\epsilon(z)) = \tau_\epsilon \phi_{n+1}(z) + B_n$ where

$$B_n \;=\; \frac{a_{n+1} - a_n}{\tau_\epsilon^n} \;=\; \frac{(\tau_\epsilon - 1)a_n}{\tau_\epsilon^n} + \frac{1 + g_\epsilon(a_n)}{\tau_\epsilon^n}.$$

When $\tau_\epsilon = 1$, B_n tends to 1 since

$$|g_\epsilon(a_n)| \;\le\; \frac{M}{|a_n|^\sigma} \;\le\; \frac{M}{(2R + n\delta)^\sigma} \;\le\; \frac{M}{(n\delta)^\sigma} \;\to\; 0.$$

When $|\tau_\epsilon| > 1$, the last term of the equation on B_n above tends to 0. For $n \ge 1$, we have

$$a_n \;=\; \tau_\epsilon^n a \;+\; \frac{\tau_\epsilon^n - 1}{\tau_\epsilon - 1} \;+\; \sum_{k=0}^{n-1} \tau_\epsilon^{n-1-k} g_\epsilon(a_k).$$

Thus

$$\frac{(\tau_\epsilon - 1)a_n}{\tau_\epsilon^n} \;=\; (\tau_\epsilon - 1)\left(a + \frac{g_\epsilon(a)}{\tau_\epsilon} + \sum_{k=1}^{n-1} \frac{g_\epsilon(a_k)}{\tau_\epsilon^{k+1}}\right) + 1 - \frac{1}{\tau_\epsilon^n}.$$

By the inequality on $|g_\epsilon(a_n)|$ above, we have

$$\left|(\tau_\epsilon - 1)\sum_{k=1}^{n-1} \frac{g_\epsilon(a_k)}{\tau_\epsilon^{k+1}}\right| \;\le\; \frac{M}{\delta^\sigma}\frac{|\tau_\epsilon - 1|}{|\tau_\epsilon|}\sum_{k=1}^{n-1} \frac{1}{k^\sigma |\tau_\epsilon|^k} \;\le\; \frac{M}{2\delta^{1+\sigma}}(1 - \frac{1}{|\tau_\epsilon|})\mathrm{Li}_\sigma(\frac{1}{|\tau_\epsilon|})$$

where we used the inequality

$$|\tau_\epsilon - 1| \;\le\; \frac{\operatorname{Re}\tau_\epsilon - 1}{\cos\alpha} \;\le\; \frac{|\tau_\epsilon| - 1}{2\delta}$$

that comes from the radial convergence. By Proposition 4.1 in the next section, B_n converges to some B_ϵ. More precisely, if we set $|\tau_\epsilon| = e^L$, then $\tau_\epsilon - 1 = O(L)$ and one can check that $B_\epsilon = 1 + O(L^{\sigma/(1+\sigma)})$.

Finally, $u_\epsilon(z) := \phi_\epsilon(z)/B_\epsilon$ gives a desired holomorphic map (with R in the statement replaced by $2R$). ■

Remarks.

- When $\sigma = 1$, we have

$$\left|\sum_{k=1}^{n-1} \frac{g_\epsilon(a_k)}{\tau_\epsilon^{k+1}}\right| \;\le\; \frac{M}{\delta|\tau_\epsilon|}\sum_{k=1}^{n-1}\frac{1}{k|\tau_\epsilon|^k} \;\le\; -\frac{M}{\delta}\log(1 - \frac{1}{|\tau_\epsilon|})$$

  and this implies that $B_\epsilon = 1 + O(L\log L)$ if we set $|\tau_\epsilon| = e^L$. This fact is consistent with the result in [Ue].

- By this proof, if $\{f_\epsilon(z)\}$ analytically depends on ϵ, then $\{B_\epsilon\}$ and $\{u_\epsilon(z)\}$ do the same for fixed a in $\Pi(2R)$.

- It is not difficult to check that $u_0(z) = z(1 + o(1))$ as $\operatorname{Re} z \to \infty$. Indeed, it is well-known that if $f_0(z) = z + 1 + a_0/z + \cdots$ then the Fatou coordinate is of the form $u_0(z) = z - a_0 \log z + O(1)$. See [Sh] for example.

4 An estimate on polylogarithm functions

We define the *polylogarithm function* of exponent $s \in \mathbb{C}$ by

$$\operatorname{Li}_s(z) \;:=\; \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$

This function makes sense when $|z| < 1$ and $\sigma := \operatorname{Re} s > 0$ and it is a holomorphic function of z. In particular, if $\operatorname{Re} s > 1$ the function tends to $\zeta(s)$ as $z \to 1$ within the unit disk. In the following we consider the behavior of $\operatorname{Li}_s(z)$ as $z \to 1$ within the unit disk when $0 < \sigma \le 1$. We claim:

Proposition 4.1 *Suppose $0 < \operatorname{Re} s = \sigma \le 1$ and $z \to 1$ with $|z| < 1$. Set $\epsilon := 1 - |z|$. Then there exists a uniform constant C independent of s such that*

$$|\operatorname{Li}_s(z)| \;\le\; C\epsilon^{-\frac{1}{1+\sigma}}$$

as $z \to 1$. In particular, we have

$$|(z-1)\operatorname{Li}_s(z)| \;\le\; C\epsilon^{\frac{\sigma}{1+\sigma}} \;\to\; 0$$

if $z \to 1 - 0$ along the real axis.

Proof. Clearly

$$|\mathrm{Li}_s(z)| \;\le\; \sum_{n=1}^{\infty} \frac{|z|^n}{n^\sigma}$$

so it is enough to consider the sum

$$S \;:=\; \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \cdot \lambda^n$$

where $\lambda := |z| = 1 - \epsilon$. Let S_n be the partial sum to the nth term. By the Hölder inequality, we have

$$S_n \;\le\; \left(\sum_{k=1}^{n} \frac{1}{k^{\sigma p}} \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} \lambda^{kq} \right)^{\frac{1}{q}}$$

for any $p, q > 1$ with $1/p + 1/q = 1$. Now let us set $p := 1/\sigma + 1 \ge 2$ (then $1 < q = 1 + \sigma \le 2$). Since $\sigma p = 1 + \sigma > 1$, the first sum is uniformly bounded as follows:

$$\sum_{k=1}^{n} \frac{1}{k^{\sigma p}} \;\le\; 1 + \int_1^\infty \frac{1}{x^{1+\sigma}} dx \;=\; 1 + \frac{1}{\sigma} = p.$$

On the other hand, for the second sum, we still have $0 < \lambda^q < 1$ and thus

$$\sum_{k=1}^{n} \lambda^{kq} \;\le\; \frac{\lambda^q}{1-\lambda^q} \;=\; \frac{1}{q\epsilon}(1 + o(1)) \;\le\; \frac{2}{q\epsilon}$$

when $\epsilon \ll 1$. Hence we have the following uniform bound:

$$S_n \;\le\; p^{\frac{1}{p}} \left(\frac{2}{q\epsilon} \right)^{\frac{1}{q}} \;\le\; 2 \left(\frac{p^{\frac{1}{p}}}{q^{\frac{1}{q}}} \right) \epsilon^{-\frac{1}{q}}.$$

One can easily check that $1 \le x^{\frac{1}{x}} \le e^{\frac{1}{e}} = 1.44467\cdots$ for $x \ge 1$. Thus

$$S \;\le\; 2e^{\frac{1}{e}} \epsilon^{-\frac{1}{q}} \;=\; 2e^{\frac{1}{e}} \epsilon^{-\frac{1}{1+\sigma}}$$

when $\epsilon \ll 1$ and we have the desired estimate with $C = 2e^{\frac{1}{e}} < 3$. The last inequality of the statement follows by:

$$|(z-1)\,\mathrm{Li}_s(z)| \;\le\; C\epsilon^{1-\frac{1}{q}} \;=\; C\epsilon^{\frac{1}{p}} \;=\; C\epsilon^{\frac{\sigma}{1+\sigma}}.$$

(Indeed, $|(z-1)\,\mathrm{Li}_s(z)| = O(\epsilon^{\frac{\sigma}{1+\sigma}})$ if $z \to 1$ radially.) ■

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