Tessellation and Lyubich-Minsky laminations associated with quadratic maps I: Pinching semiconjugacies

Tomoki Kawahira

Graduate School of Mathematics, Nagoya University

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Abstract

We introduce *tessellation* of the filled Julia sets for hyperbolic and parabolic quadratic maps. Then the dynamics inside their Julia sets are organized by tiles which work like external rays outside. We also construct continuous families of pinching semiconjugacies associated with hyperblic-to-parabolic degenerations *without* using quasiconformal deformation. Instead we use tessellation and investigation on the hyperbolic-to-parabolic degeneration of linearizing coordinates inside the Julia set.

1 Introduction

After the works by Douady and Hubbard, dynamics of quadratic map $f = f_c : z \mapsto z^2 + c$ with an attracting or parabolic cycle has been investigated in detail, because such parameters c of f_c are contained in the Mandelbrot set and they are very important elements that determine the topology of the Mandelbrot set. (See [DH] or [Mi2].)

The aim of this paper is to provide a new method to describe combinatorial changes of dynamics when the parameter c moves from one hyperbolic component to another via a "parabolic parameter" (i.e., c of f_c with a parabolic cycle).

For example, the simplest case is the motion in the Mandelbrot set along a path joining $c = 0$ and the center $c_{p/q}$ of the p/q-satellite component of the main cardioid via the root of p/q-limb. In particular, we join them by the two segments characterized as follows:

(s1) c of f_c which has a fixed point of multiplier $re^{2\pi i p/q}$ with $0 < r \le 1$; and

(s2) c of f_c which has a q-periodic cycle of multiplier $1 \ge r > 0$.

Note that we avoid the hyperbolic centers (i.e., c of f_c with superattracting cycle) because we regard them as non-generic special cases far away from parabolic bifurcations.

In the magnified box of Figure 1, segments (s1) and (s2) for $p/q = 1/3$ is drawn in the Mandelbrot set. By the Douady-Hubbard theory, the change of dynamics of $f = f_c$ on and outside the Julia set is described by external rays $R_f(\theta)$ with $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and their landing points $\gamma_f(\theta)$ satisfying $f(R_f(\theta)) = R_f(2\theta)$ and $f(\gamma_f(\theta)) = \gamma_f(2\theta)$. For example, as c moves from (s1) to (s2), the map $\gamma_f : \mathbb{T} \to J_f$ looses the injectivity at a dense subset Θ_f of $\mathbb{T}$ consisting of the countably many angles that eventually land on $\{1/7, 2/7, 4/7\}$ by angle doubling $\delta : \theta \mapsto 2\theta$.

On the other hand, the dynamics inside the filled Julia set K_f has no particular method to describe degeneration and bifurcation like external rays. However, as the pictures of filled Julia sets in Figure 1 (with equipotential curves drawn in) indicate, the interior of K_f preserves a certain combinatorial structure along (s1) and (s2).
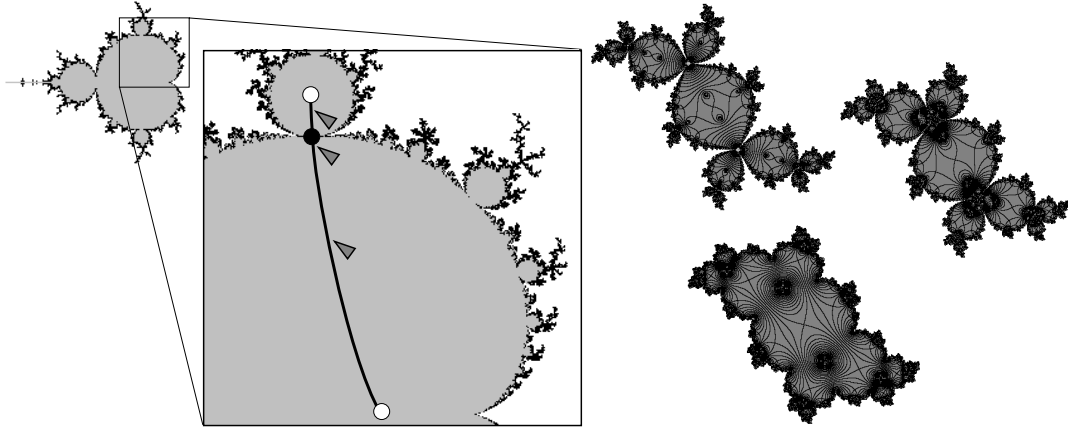
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Figure 1: Chubby rabbits

Degeneration pairs and tessellation. In this paper, we introduce *tessellation* of the interior K_f° of K_f to detect hyperbolic-to-parabolic degeneration or parabolic-to-hyperbolic bifurcation of quadratic maps.

Let X be a hyperbolic component of the Mandelbrot set. By a theorem due to Douady and Hubbard [Mi2, Theorem 6.5], there exists the conformal map λ_X from $\mathbb{D}$ onto X that parameterize the multiplier of the attracting cycle of $f = f_c$ for $c \in X$. Moreover, the map λ_X has the homeomorphic extension $\lambda_X : \bar{\mathbb{D}} \to \bar{X}$ such that $\lambda_X(e^{2\pi i p/q})$ is a parabolic parameter for all $p, q \in \mathbb{N}$. A *degeneration pair* $(f \to g)$ is a pair of hyperbolic $f = f_c$ and parabolic $g = f_\sigma$ where $(c, \sigma) = (\lambda_X(re^{2\pi i p/q}), \lambda_X(e^{2\pi i p/q}))$ for some $0 < r < 1$ and coprime $p, q \in \mathbb{N}$. By letting $r \to 1$, the map f converges uniformly to g on $\bar{\mathbb{C}}$ and we have a path which generalize segment (s1) or (s2). For a degeneration pair, we have the associated tessellations which have the same combinatorics:

Theorem 1.1 (Tessellation) *Let $(f \to g)$ be a degeneration pair. There exist families* $\mathrm{Tess}(f)$ *and* $\mathrm{Tess}(g)$ *of simply connected sets with the following properties:*

(1) *Each element of* $\mathrm{Tess}(f)$ *is called a* tile *and identified by an* angle θ *in* $\mathbb{Q}/\mathbb{Z}$, *a level* m *in* $\mathbb{Z}$, *and a signature* $* = +$ *or* $-$.

(2) *Let* $T_f(\theta, m, *)$ *be such a tile in* $\mathrm{Tess}(f)$. *Then* $f(T_f(\theta, m, *)) = T_f(2\theta, m+1, *)$.

(3) *The interiors of tiles in* $\mathrm{Tess}(f)$ *are disjoint topological disks. Tiles with the same signature are univalently mapped each other by a branch of* $f^{-i} \circ f^j$ *for some* i, $j > 0$;

(4) *Let* $\Pi_f(\theta, *)$ *denote the union of tiles with angle θ and signature $*$. Then its interior* $\Pi_f(\theta, *)^\circ$ *is also a topological disk and its boundary contains the landing point* $\gamma_f(\theta)$ *of* $R_f(\theta)$. *In particular,* $f(\Pi_f(\theta, *)) = \Pi_f(2\theta, *)$.

The properties above holds if we replace f by g. Moreover:

(5) *There exists an f-invariant family I_f of star-like graphs such that the union of tiles in* $\mathrm{Tess}(f)$ *is* $K_f^\circ - I_f$. *On the other hand, the union of tiles in* $\mathrm{Tess}(g)$ *is* K_g°.

(6) *The boundaries of $T_f(\theta, m, *)$ and $T_f(\theta', m', *')$ in $K_f^\circ - I_f$ intersect iff so do the boundaries of $T_g(\theta, m, *)$ and $T_g(\theta', m', *')$ in K_g°.*
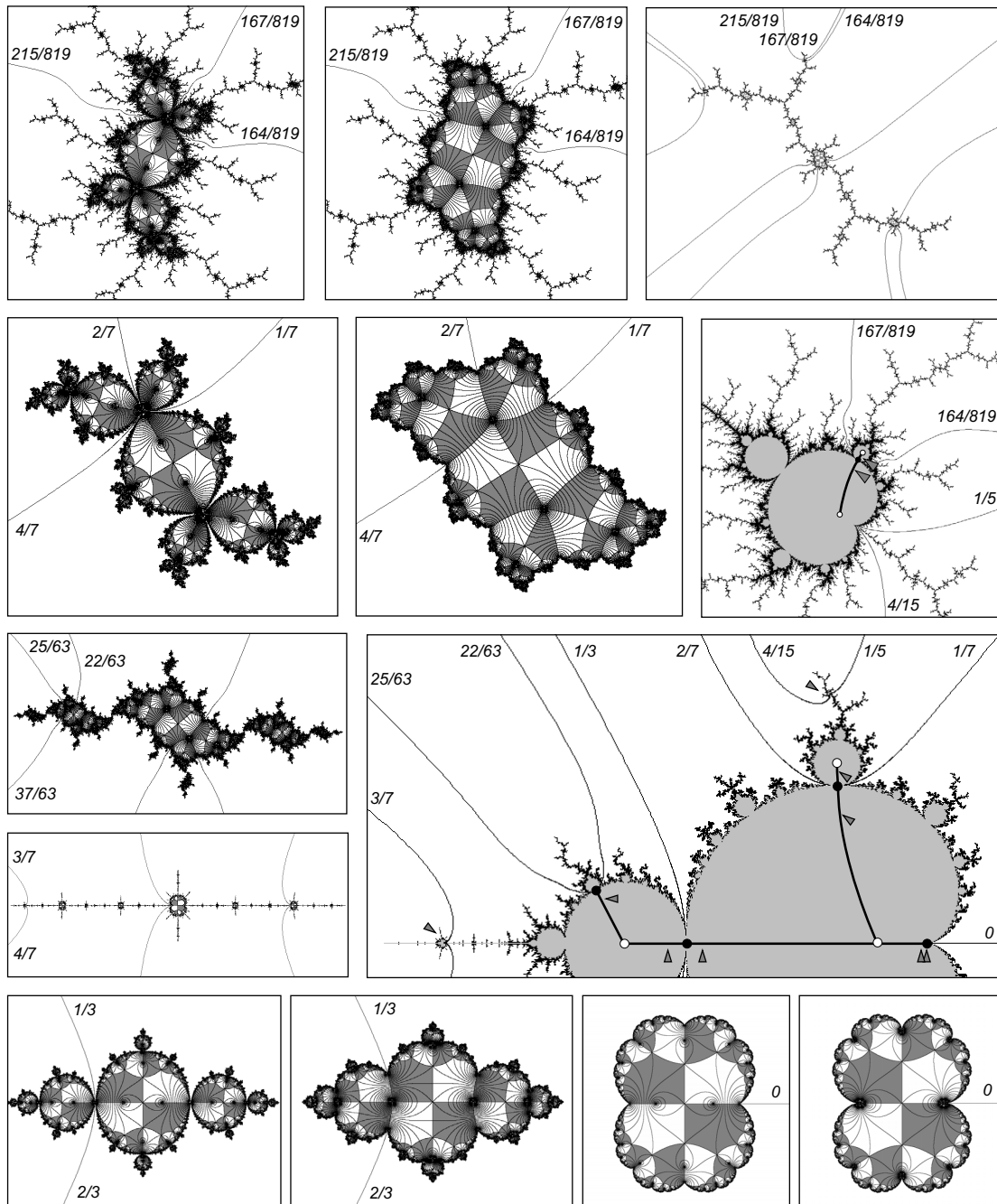
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Figure 2: Samples of tessellation. For the two figures at the upper left, parameters are taken from period 12 and 4 hyperbolic components of the Mandelbrot set as indicated in the figure of a small Mandelbrot set.

Here angles of tiles must be the angles of external rays which eventually land on the parabolic cycle of g. (For example, if $(f \to g)$ are on (s1) or (s2) in Figure 1, the set of angles of tiles coincides with Θ_f.) See Sections 2 and 3 for construction of tessellation and Figure 2 for examples. One can find that the combinatorics of tessellations are preserved along (s1) and (s2). (This is justified in Section 4 more generally.) Since $f_c \in X - \{\lambda_X(0)\}$ is structurally stable, we have the tessellation of K_{f_c} with the same properties of Tess(f).

Pinching semiconjugacy. As an application of tessellation, we show that there exists a pinching semiconjugacy from f to g for the degeneration pair $(f \to g)$. In Sections 4 and 5 we will establish:

Theorem 1.2 (Pinching semiconjugacy) *Let $(f \to g)$ be a degeneration pair. There exists a semiconjugacy $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ from f to g such that:*

(1) h only pinches I_f to the grand orbit of the parabolic cycle of g.

(2) h sends all possible $T_f(\theta, m, *)$ to $T_g(\theta, m, *)$, $R_f(\theta)$ to $R_g(\theta)$, and $\gamma_f(\theta)$ to $\gamma_g(\theta)$.

(3) h tends to the identity as f tends to g.

One may easily imagine the situation by seeing the figures of tessellation. As a corollary, we have convergence of the tiles when f of $(f \to g)$ tends to g (Corollary 5.2).

We first prove the existence of h with properties (1) and (2) in Section 4 (Theorem 4.1) by using combinatorial properties of tessellation. Property (3) is proved in Section 5 (Theorem 5.1) by means of the continuity results about the extended Böttcher coordinates (Theorem 5.4) on and outside the Julia sets and the linearizing coordinates (i.e., the Königs and Fatou coordinates) inside the Julia sets associated with $(f \to g)$ (Theorem 5.5).

In Appendix, we will give some useful results on perturbation of parabolics used for the proof.

Notes.

1. For any $f_c \in X - \{\lambda_X(0)\}$, we have a semiconjugacy h_c which has similar properties to (1) and (2) by structural stability. By results of Cui ([Cu]), Haïssinsky and Tan Lei([Ha2], [HT]), it is already known that such a semiconjugacy exists. Their method is based on the quasiconformal deformation theory and works even for some geometrically infinite rational maps. On the other hand, our method is faithful to the quadratic dynamics and the semiconjugacy is constructed in a more explicit way without using quasiconformal deformation. It is possible to extend our results to some class of higher degree polynomials or rational maps but it is out of our scope.

2. This paper is the first part of works on Lyubich-Minsky laminations. In [LM], they introduced the hyperbolic 3-laminations associated with rational maps as an analogue of the hyperbolic 3-manifolds associated with Kleinian groups. In the second part of this paper [Ka3], we will investigate combinatorial and topological change of 3-laminations associated with hyperbolic-to-parabolic degeneration of quadratic maps by means of tessellation and pinching semiconjugacies.

3. The most recent version of this paper and author's other articles are available at:
   http://www.math.nagoya-u.ac.jp/~kawahira

2 Degeneration pair and degenerating arc system

Segments (s1) and (s2) in the previous section are considered as hyperbolic-to-parabolic degeneration processes of two distinct directions. Degeneration pairs generalize all of such processes in the quadratic family. The aim of this section is to give a dichotomous classification of the degeneration pairs $\{(f \to g)\}$ and to define invariant families of star-like graphs (*degenerating arc systems*) for each f of $(f \to g)$.

Classification of degeneration pairs. We first fix some notation used throughout this paper. Let p and q be relatively prime positive integers, and set $\omega := \exp(2\pi i p/q)$. (We allow the case of $p = q = 1$.) Take an r from the interval $(0, 1)$ and set $\lambda := r\omega$. As in the previous section, we take a hyperbolic component X of the Mandelbrot set. Then we have a *degeneration pair* $(f \to g)$ that is a pair of hyperbolic $f = f_c$ and parabolic $g = f_\sigma$ where $(c, \sigma) = (\lambda_X(re^{2\pi ip/q}), \lambda_X(e^{2\pi ip/q}))$.

For the degeneration pair $(f \to g)$, let $O_f := \{\alpha_1, \ldots, \alpha_l\}$ be the attracting cycle of f with multiplier $\lambda = r\omega$ and $f(\alpha_j) = \alpha_{j+1}$ (taking subscripts modulo l). Similarly, let $O_g := \{\beta_1, \ldots, \beta_{l'}\}$ be the parabolic cycle of g with $g(\beta_{j'}) = \beta_{j'+1}$ (taking subscripts modulo l'). Let $\omega' = e^{2\pi ip'/q'}$ denote the multiplier of O_g with relatively prime positive integers p' and q'. (Then O_g is a parabolic cycle with q' repelling petals.)

Our fundamental classification is described by the following proposition:

Proposition 2.1 *Any degeneration pair* $(f \to g)$ *satisfies either*

> **Case (a):** $q = q'$ *and* $l = l'$; *or*

> **Case (b):** $q = 1 < q'$ *and* $l = l'q'$.

For both cases, we have $lq = l'q'$.

The proof is given by summing up results in sections 2, 4 and 6 of [Mi2]. For example, a degeneration pair $(f \to g)$ on segment (s1) (resp. (s2)) with $q > 1$ is a Case (a) (resp. Case(b)) above. Degeneration pairs $(f_c \to f_\sigma)$ with $\sigma = 1/4$ or $\sigma = -7/4$ satisfy $q = q' = 1$ and thus Case (a).

Note on terminology. According to [Mi2], a parabolic g with $q' = 1$ is called *primitive*. The parabolic $g = f_\sigma$ with $\sigma = 1/4$ is also called *trivial*. For these g's any degeneration pair $(f \to g)$ is automatically Case (a) by the proposition above. When we define tessellation for non-trivial primitive $(f \to g)$, a little extra care will be required.

Perturbation of O_g and degenerating arcs. For a degeneration pair $(f \to g)$ with $r \approx 1$, the parabolic cycle O_g is approximated by an attracting or repelling cycle O'_f with the same period l' and multiplier $\lambda' \approx e^{2\pi ip'/q'}$. Let α'_1 be the point in O'_f with $\alpha'_1 \to \beta_1$ as $r \to 1$. (cf. [Mi2, §4])

In Case (a), the cycle O'_f is attracting (thus $O'_f = O_f$) and there are q' symmetrically arrayed repelling periodic points around $\alpha_1 = \alpha'_1$. Then we will show that there exits an $f^{l'}$-invariant star-like graph $I(\alpha'_1)$ that joins α'_1 and the repelling periodic points by q' arcs. In Case (b), the cycle O'_f is repelling and there are $q' = l/l'$ symmetrically arrayed

attracting periodic points around α_1'. Then we will show that there exits an $f^{l'}$-invariant star-like graph $I(\alpha_1')$ that joins α_1' and the attracting periodic points by q' arcs.

In both cases, we define *degenerating arc system* I_f by

$$I_f := \bigcup_{n \geq 0} f^{-n}(I(\alpha_1')).$$

The rest of this section is mainly devoted for the detailed construction of I_f, which has a role of parabolic cycle and its preimages. It would be helpful to see Figure 3 first, showing what we aim to have.

2.1 External and internal landing

First we consider the parameter c on segment (s1) such that $f = f_c$ has an attracting fixed point $O_f = \{\alpha_1\}$ of multiplier $\lambda = r\omega$, thus $c = \lambda/2 - \lambda^2/4$. When r tends to 1, the hyperbolic map f tends to a parabolic g which has a parabolic fixed point $O_g = \{\beta_1\}$ with multiplier $\omega = e^{2\pi i p/q}$. (Note that $q = q'$ and $l = l'(= 1)$, thus Case (a) by Proposition 2.1.) It is known that the Julia set J_f of f is a quasicircle, and the dynamics on J_f is topologically the same as that of $f_0(z) = z^2$ on the unit circle. Since J_f is locally connected, for any angle $\theta \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$ its external ray $R_f(\theta)$ has a unique landing point $\gamma_f(\theta)$. The same is true for $R_g(\theta)$, since J_g is also locally connected. (See [DH, Exposé No.X].)

External landing. By [Mi1, Theorem 18.11] due to Douady, there is at least one external ray with rational angle lands at β_1. Now [GM, Lemma 2.2] and the local dynamics of β_1 insure:

Lemma 2.2 *In the dynamics of g, there exist exactly q external rays of angles $\theta_1, \ldots, \theta_q$ with $0 \leq \theta_1 < \cdots < \theta_q < 1$ which land on β_1. Moreover, the map g sends $R_g(\theta_j)$ onto $R_g(\theta_k)$ univalently (equivalently, $\theta_k = 2\theta_j$ modulo 1) iff $k \equiv j + p \mod q$.*

In particular, such angles are determined uniquely by the value $p/q \in \mathbb{Q}/\mathbb{Z}$. We take the subscripts of $\{\theta_j\}$ modulo q. For these angles, we call $p/q \in \mathbb{Q}/\mathbb{Z}$ the *(combinatorial) rotation number*. Note that the external rays $\{R_g(\theta_j)\}$ divide the complex plane into q open pieces, called *sectors* based at β_1.

Internal landing lemma. On the other hand, the set of landing points $\{\gamma_f(\theta_j)\}$ of $\{R_f(\theta_j)\}$ is a repelling cycle of period q and their corresponding rays do not divide the plane. However, they continuously extend and penetrate though the Julia set, and land at the attracting fixed point:

Lemma 2.3 (Internal landing) *For $\theta_1, \ldots, \theta_q$ as above, there exist q open arcs $I(\theta_1), \ldots, I(\theta_q)$ such that:*

- *For each j modulo q, the arc $I(\theta_j)$ joins α_1 and $\gamma_f(\theta_j)$.*

- *f maps $I(\theta_j)$ onto $I(\theta_k)$ univalently iff $k \equiv j + p \mod q$.*

An important fact is, the set $\{I(\theta_j) \cup \gamma_f(\theta_j) \cup R_f(\theta_j)\}_{j=1}^{q}$ divide the plane into q sectors based at α_1. This is topologically the same situation as g. Indeed, as r tends to 1, we may consider that the arcs $\{I(\theta_j)\}$ constructed as above degenerate to the parabolic β_1.

Sketch of the proof. (See [Ka2, Lem. 3.1] for the detailed proof.) Let $w = \phi_f(z)$ be a linearizing coordinate near α_1, where f near α_1 is viewed as $w \mapsto \lambda w$. We can extend

it to $\phi_f : K_f^\circ \to \mathbb{C}$ and normalize it so that $\phi_f(0) = 1$ [Mi1, §8]. Now we pull-back the q-th root of the negative real axis in w-plane, which are q symmetrically arrayed invariant radial rays, to the original dynamics. Then we can show that the pulled-back arcs land at a unique repelling cycle with external angles determined by the rotation number p/q. In particular, they are disjoint from the critical orbit. ∎

Degenerating arcs.   Note that in the construction of $\{I(\theta_j)\}$ above we make a particular choice of such arcs so that they are laid opposite to the critical orbit in w-plane. We call these arcs *degenerating arcs*.

2.2 Degenerating arc system

Let us return to a general degeneration pair $(f \to g)$ as we first defined.

Renormalization.   Let B_1 be the Fatou component containing the critical value c. We may assume that B_1 is the immediate basin of α_1 for f^l. Then it is known that there exists a topological disk U containing B_1 such that f^l maps U over itself properly by degree two. That is, the map $f^l : U \to f(U)$ is a quadratic-like map which is a renormalization of f. See [Mi2, §8] or [Ha1, Partie 1]. In particular, the map $f^l : U \to f(U)$ is hybrid equivalent to $f_1(z) = z^2 + c_1$ with $c_1 = \lambda/2 - \lambda^2/4$ in segment (s1), which we dealt with above. More precisely, the dynamics of f^l near B_1 (resp. on B_1) is topologically (resp. conformally) identified as that of f_1 near K_{f_1} (resp. on $K_{f_1}^\circ$).

Degenerating arc system.   In K_{f_1}, we have q degenerating arcs associated with the attracting fixed point f_1. By pulling them back to the closure of B_1 with respect to the conformal identification above, we have q open arcs $\{I_j\}_{j=1}^q$ which are cyclic under f^l.

When $q = q'$ and $l = l'$, thus in Case (a), the arcs $\{I_j\}_{j=1}^q$ join q' repelling points (cyclic under $f^l = f^{l'}$) and $\alpha_1 = \alpha_1'$. In this case we define $I(\alpha_1')$ by the closure of the union of $\{I_j\}_{j=1}^q$. When $1 = q < q' = l/l'$, thus in Case (b), we only have I_1 that joins the repelling point α_1' (fixed under $f^{l'}$) and α_1. In this case we define $I(\alpha_1')$ by the closure of the union of $\left\{ f^{kl'}(I_1) \right\}_{k=0}^{q'-1}$. In both cases, we have $I(\alpha_1')$ as desired. Now we define the *degenerating arc system* of f by

$$ I_f := \bigcup_{n>0} f^{-n}(I(\alpha_1')). $$

For $z \in I_f$, it is useful to denote the connected component of I_f containing z by $I(z)$.

For later usage, we define the set of all points that eventually land on the attracting cycle O_f, by $\alpha_f := \bigcup_{n>0} f^{-n}(\alpha_1)$. Note that I_f and α_f are forward and backward invariant, and disjoint from the critical orbit. In particular, for any $z \in I_f$, the components $I(z)$ and $I(\alpha_1')$ are homeomorphic. In Case (a), the points in α_f and the connected components of I_f have one-to-one correspondence. In Case (b), however, they are q'-to-one correspondence. See Figure 3 and Proposition 2.5.

Correspondingly, for g of degeneration pair $(f \to g)$ and one of its parabolic point $\beta_1 \in O_g$, we define

$$ I_g := \bigcup_{n>0} g^{-n}(\beta_1). $$

We will see that this naturally corresponds to I_f rather than α_f.

Types.   After [GM], we define the *type* $\Theta(z)$ of z in J_f (or J_g) by the set of all angles of external rays which land on z. Let $\delta : \mathbb{T} \to \mathbb{T}$ be the angle doubling map. Since J_f has
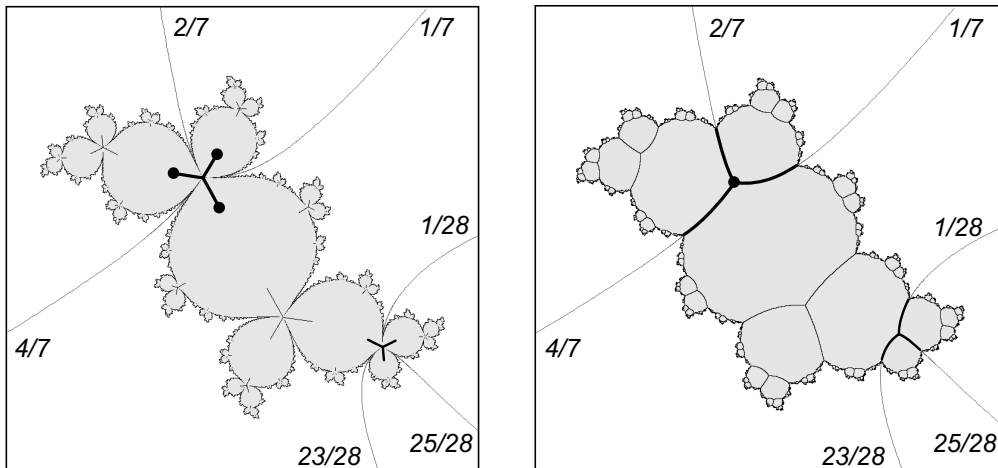
Figure 3: Left, the Julia set of an f in segment (s2) for $p/q = 1/3$, and right, one in segment (s1), with their degenerating arc system roughly drawn in. Attracting cycles are shown in heavy dots. Degenerating arcs with types $\{1/7, 2/7/4/7\}$ and $\{1/28, 23/28, 25/28\}$ are emphasized.

no critical points, one can easily see that $\delta(\Theta(z))$ coincides with $\Theta(f(z))$. The same holds for g. Now a fact originally due to Thurston implies:

Lemma 2.4 *For any point z in J_f or J_g, the set $\Theta(z)$ consists of finitely many angles.*

See [Ki] for a generalized statement and the proof.

We abuse the notation $\Theta(\cdot)$ like this: For any subset E of the filled Julia set, its *type* $\Theta(E)$ is the set of angles of the external rays that land on E. For each ζ in α_f, we formally define the type of ζ by $\Theta(\zeta) := \Theta(I(\zeta))$. Then one can easily see that $\delta^n(\Theta(\zeta)) = \Theta(\alpha_1)$ for some $n > 0$. We also set $\Theta_f := \Theta(I_f)$ and $\Theta_g := \Theta(I_g)$. We will show that Θ_f equals to Θ_g in the next proposition.

Valence. For any $\zeta \in \alpha_f$, the component $I(\zeta)$ of I_f is univalently mapped onto $I(\alpha_1)$ by iteration of f. Thus the value $\mathrm{val}(f) := \mathrm{card}(\Theta(\zeta))$ is a constant for f. Similarly, since a small neighborhood of ξ in I_g is sent univalently over β_1 by iteration of g, the value $\mathrm{val}(g) := \mathrm{card}(\Theta(\xi))$ is constant for g. Now we claim:

Proposition 2.5 *For any degeneration pair $(f \to g)$, we have $\Theta_f = \Theta_g$ and $\mathrm{val}(f) = \mathrm{val}(g)$. Moreover,*

- *if $q = q' = 1$ and $l = l' > 1$ (thus Case (a) and non-trivial primitive), then $\mathrm{val}(g) = 2$.*

- *Otherwise $\mathrm{val}(g) = q'$.*

We call $\mathrm{val}(f) = \mathrm{val}(g)$ the *valence* of $(f \to g)$. Note that the valence depends only on g.

Proof. The two possibilities of $\mathrm{val}(g)$ above is shown in [Mi2, Lemma 2.7, §6]. If we show that $\Theta(\alpha_1') = \Theta(\beta_1)$, then $\Theta_f = \Theta_g$ and $\mathrm{val}(f) = \mathrm{val}(g)$ automatically follow.

Case (a): $q = q'$. (Recall that in this case we have $l = l'$ and $\alpha_1 = \alpha_1'$.) First we consider the case of $q = q' = 1$. In this case by the argument of [Mi2, Theorem 4.1] there exists a repelling cycle $\{\gamma_1, \dots, \gamma_{l'}\}$ of f satisfying $\gamma_{j'} \to \beta_{j'}$ as $f \to g$ and $\Theta(\gamma_{j'}) = \Theta(\beta_{j'})$ for $j' = 1, \dots, l'$. Take the degenerating arc $\{I_1\}$ in the construction of I_f. Then I_1 joins $\alpha_1(= \alpha_1')$ and γ_1 thus $\Theta(\alpha_1') = \Theta(I(\alpha_1)) = \Theta(\gamma_1) = \Theta(\beta_1)$.

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Next we consider the case of $q = q' > 1$. When f is in segment (s1), the identity $\Theta(\alpha_1') = \Theta(\beta_1)$ is clear by Lemma 2.3. In the general case, we use renormalization.

Let us take a path η in the parameter space joining c to σ according to the motion as $r \to 1$. By [Ha1, Théorème 1], there is an analytic family of quadratic-like maps $\{f_{c'}^l : U_{c'} \to f_{c'}^l(U_{c'})\}$ over a neighborhood of η such that the straightening maps are continuous and they give one-to-one correspondence between η and (s1).

Let $\alpha_1 \in O_f$ and $\beta_1 \in O_g$ be the attracting and parabolic fixed points of $f^l = f_c^l : U_c \to f_c^l(U_c)$ and $g^l = f_\sigma^l : U_\sigma \to f_\sigma^l(U_\sigma)$ respectively, satisfying $\alpha_1 \to \beta_1$ as $f \to g$. By Lemma 2.2, we can find q external rays landing at β_1 in the original dynamics of g, which is cyclic under g^l. In particular, there are no more rays landing at β_1 since such rays must be cyclic of period q under g^l and this contradicts [Mi2, Lemma 2.7]. Similarly in the dynamics of f, by Lemma 2.3 and continuity of the straightening, there are exactly q external rays of angles in $\Theta(\beta_1)$ landing at q ends of $I(\alpha_1) = I(\alpha_1')$. In fact, if there is another ray of angle $t \notin \Theta(\beta_1)$ landing on such an end, then $R_g(t)$ must land on β_1 by orbit forcing ([Mi2, Lemma 7.1]). This is a contradiction. Thus $\Theta(\alpha_1) = \Theta(\alpha_1') = \Theta(\beta_1)$.

Case (b): $q = 1 < q'$. By the argument of [Mi2, Theorem 4.1], the repelling points $O_f' = \{\alpha_1', \ldots, \alpha_{l'}'\}$ must satisfy $\Theta(\alpha_{j'}') = \Theta(\beta_{j'})$ for $j' = 1, \ldots, l'$. ∎

In both Cases (a) and (b), it is convinient to assume that $\alpha_{j'}, \alpha_{j'+l'}, \ldots, \alpha_{j'+(q'-1)l'}$ has the same types as that of $\beta_{j'}$ for each $j' = 1, \ldots, l'$. Equivalently, we assume that

$$ I(\alpha_{j'}) \;=\; I(\alpha_{j'+l'}) \;=\; \cdots \;=\; I(\alpha_{j'+(q'-1)l'}) $$

throughout this papar.

2.3 Critical sectors

For ξ in I_g, the external rays of angles in $\Theta(\xi)$ cut the plane up into $\mathrm{val}(g)$ open regions, called *sectors* based at ξ. Similarly, for ζ in α_f, the union of the external rays of angles in $\Theta(\zeta)$ and $I(\zeta)$ cut the plane up into $\mathrm{val}(f) = \mathrm{val}(g)$ open regions. We abuse the term *sectors based at $I(\zeta)$* for these regions.

Let B_0 be the Fatou component of g that contains the critical point $z = 0$. We may ssume that $\beta_0 = \beta_{l'}$ is on the boundary of B_0. Now one of the sectors based at β_0 contains the critical point 0, which is called the *critical sector*. For later usage, let θ_0^+, $\theta_0^- \in \mathbb{R}/\mathbb{Z}$ denote the angles of external rays bounding the critical sector such that if we take representatives $\theta_0^+ < \theta_0^- \leq \theta_0^+ + 1$ the external ray of angle θ with $\theta_0^+ < \theta < \theta_0^-$ is contained in the critical sector. For example, we define $\theta_0^+ := 4/7$ and $\theta_0^- := 1/7$ in the case of Figure 3. In the case of Figure 8, we define $\theta_0^+ := 5/7$ and $\theta_0^- := 2/7$. We also define the *critical sector* based at $I(\alpha_0)$ by one of the sector bounded by $I(\alpha_0)$ and $R_f(\theta_0^\pm)$.

3 Tessellation

In this section, we develop (and compactify) the method in [Ka2], and construct *tessellation* of the interior of the filled Julia sets for a degeneration pair $(f \to g)$.

For each $\theta \in \Theta_f = \Theta_g$ and some $m \in \mathbb{Z}$ (with a condition depending on θ), we will define the tiles $T_f(\theta, m, \pm)$ and $T_g(\theta, m, \pm)$ with the properties listed in Theorem 1.1. The idea of tessellation is so simple as one can see in Figure 2, but we need to construct them precisely to figure out their detailed combinatorial structure.

3.1 Fundamental model of tessellation

Take an $R \in (0, 1)$ and let us consider the affine maps $F(W) = RW + 1$ and $G(W) = W + 1$ on $\mathbb{C}$ as the W-plane. The map F has a fixed point $a = 1/(1 - R)$ and one can see the action by the relation $F(W) - a = R(W - a)$.

Tiles for F. Set $I := [a, \infty)$, a half line invariant under F. For each $\mu \in \mathbb{Z}$, we define "tiles" of level μ for F by:

$$A_\mu(+) := \left\{ W \in \mathbb{C} - I : R^{\mu+1}a \leq |W - a| \leq R^\mu a, \ \mathrm{Im}\, W \geq 0 \right\}$$
$$A_\mu(-) := \left\{ W \in \mathbb{C} - I : R^{\mu+1}a \leq |W - a| \leq R^\mu a, \ \mathrm{Im}\, W \leq 0 \right\}.$$

Then one can check that $F(A_\mu(*)) = A_{\mu+1}(*)$ where $* \in \{+, -\}$. For the boundary of each $A_\mu(*)$, we define

- the *circular edges* by the intersection with $A_{\mu \pm 1}(*)$;

- the *degenerating edge* by $\overline{A_\mu(*)} \cap I$; and

- the *critical edge* by the intersection with $(-\infty, a)$.

Note that the degenerating edge is *not* contained in $A_\mu(*)$.

Tiles for G. Correspondingly, for each $\mu \in \mathbb{Z}$, we define "tiles" of level μ for G by:

$$C_\mu(+) := \{ W \in \mathbb{C} : \mu \leq \mathrm{Re}\, W \leq \mu + 1, \ \mathrm{Im}\, W \geq 0 \}$$
$$C_\mu(-) := \{ W \in \mathbb{C} : \mu \leq \mathrm{Re}\, W \leq \mu + 1, \ \mathrm{Im}\, W \leq 0 \}.$$

Then one can check that $G(C_\mu(*)) = C_{\mu+1}(*)$. For the boundary of each $C_\mu(*)$, we define

- the *circular edges* by the intersection with $C_{\mu \pm 1}(*)$, which are vertical half lines;

- the *critical edge* by the intersection with $(-\infty, \infty)$.

Note that there is no degenerating edge for $C_\mu(*)$. One can consider $C_\mu(*)$ the limit of $A_\mu(*)$ as $R \to 1$.

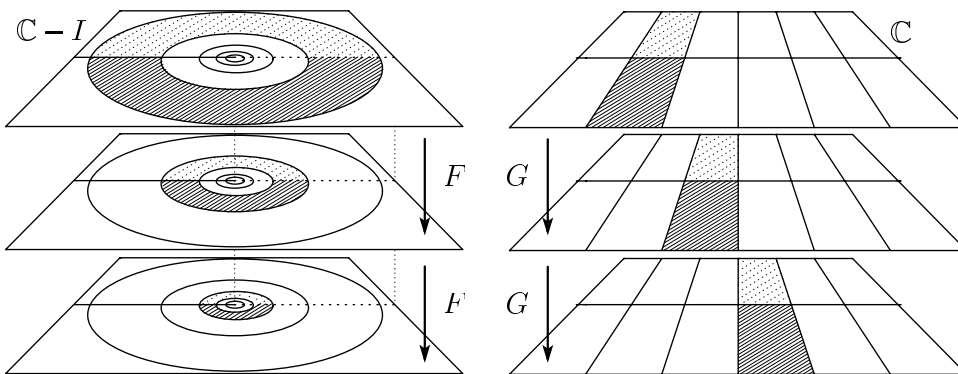Figure 4: The fundamental model

3.2 Tessellation for f and g.

First we reduce the dynamics of $f|_{K_f^\circ}$ and $g|_{K_g^\circ}$ to the dynamics of F and G on $\mathbb{C}$.

From f to F. Let B_0 be the Fatou component of f containing 0. We may assume that $\alpha_0 = \alpha_l \in B_0$. There exists a unique extended linearizing coordinate $\phi_f : B_0 \to \mathbb{C}$ such that $\phi_f(\alpha_0) = \phi_f(0) - 1 = 0$ and $\phi_f(f^l(z)) = \lambda\phi_f(z)$ [Mi1, §8]. Set $w := \phi_f(z)$ and $R := \lambda^q = r^q$. Then $f^{lq}|_{B_0}$ is semiconjugate to $w \mapsto Rw$. To reduce this situation to our fundamental model, first we take a branched covering $W = w^q$. Then $f^l|_{B_0}$ and $f^{lq}|_{B_0}$ are semiconjugate to $W \mapsto RW$ and $W \mapsto R^q W$ respectively. Next, we take a linear conjugation by $W \mapsto a(1 - W)$. Then $f^l|_{B_0}$ and $f^{lq}|_{B_0}$ are finally semiconjugate to F and F^q in the fundamental model respectively. Let Φ_f denote this final semiconjugation. Now we have $\Phi_f(0) = 0$ and $\Phi_f(B_0 \cap I_f) = I$. (The second equality comes from the construction of the degenerating arcs in Lemma 2.3.) In particular, the map Φ_f branches at $z \in B_0$ if and only if either $f^{ln}(z) = 0$ for some $n \geq 0$, or $q > 1$ and $f^{ln}(z) = \alpha_0$ for some $n \geq 0$.

From g to G. Let B_0' be the Fatou component of g containing 0. We may assume that $\beta_0 = \beta_{l'} \in \partial B_0'$. There exists a unique extended Fatou coordinate $\phi_g : B_0' \to \mathbb{C}$ such that $\phi_g(0) = 0$ and $\phi_g(g^{lq}(z)) = \phi_g(z) + 1$ [Mi1, §10]. Set $w := \phi_g(z)$, then $g^{lq}|_{B_0}$ is semiconjugate to $w \mapsto w + 1$. To adjust the situation to that of f, we take an additional conjugacy by $w \mapsto W = qw$. Then $g^{lq}|_{B_0}$ is semiconjugate to $G^q(W) = W + q$. We denote this semiconjugation $z \mapsto w \mapsto W$ by Φ_g. Note that $\Phi_g(0) = 0$, and Φ_g branches at $z \in B_0'$ iff $g^{lqn}(z) = 0$ for some $n \geq 0$.
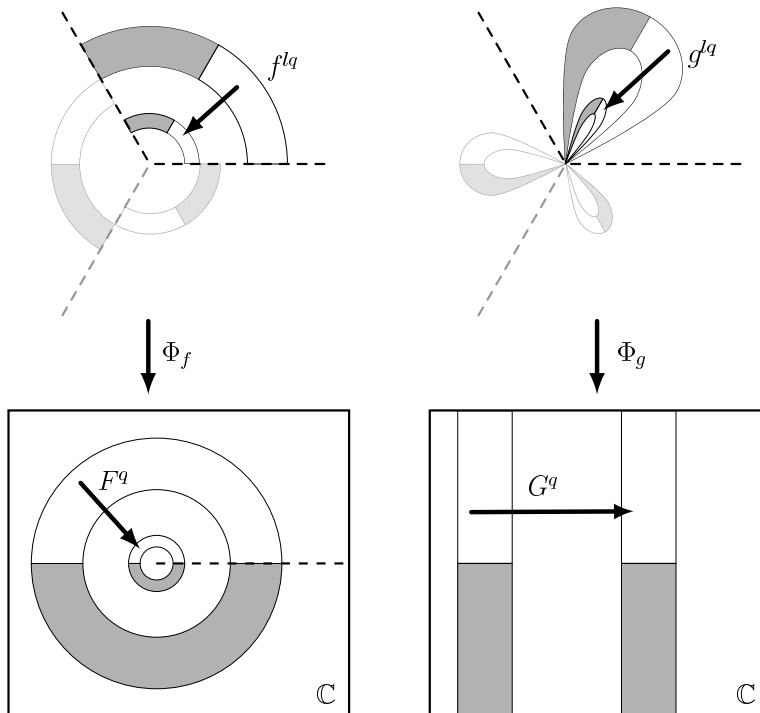

Figure 5: f^{lq} and g^{lq} are semiconjugate to F^q and G^q.

Let us summarize these reduction steps. Now $\Phi_f : B_0 \to \mathbb{C}$ semiconjugates the action of $f^{lq} : B_0 - I_f \to B_0 - I_f$ to $F^q : \mathbb{C} - I \to \mathbb{C} - I$. Similarly, the map $\Phi_g : B_0' \to \mathbb{C}$ semiconjugates the action of $g^{lq} : B_0' \to B_0'$ to $G^q : \mathbb{C} \to \mathbb{C}$ (Figure 5). In addition, we

have one important property as follows:

Proposition 3.1 *The branched linearization Φ_f do not ramify over $\mathbb{C} - (-\infty, 0]$ or $\mathbb{C} - (-\infty, 0] \cup \{a\}$ according to $q = 1$ or $q > 1$. Similarly, the branched linearization Φ_g do not ramify over $\mathbb{C} - (-\infty, 0]$. In particular, both Φ_f and Φ_g do not ramify over tiles of level $\mu > 0$.*

See Theorem 5.5 for another important property of Φ_f and Φ_g.

Definition of tiles. A subset $T \subset K_f^\circ$ is a *tile* for f if there exist $n \in \mathbb{N}$ and $\mu \in \mathbb{Z}$ such that $f^n(T)$ is contained in B_0 and $\Phi_f \circ f^n$ maps T homeomorphically onto $A_\mu(+)$ or $A_\mu(-)$. We define *circular, degenerating, and critical edges* for T by their corresponding edges of $A_\mu(\pm)$. We call the collection of such tiles the *tessellation* of $K_f^\circ - I_f$, and denote it by $\mathrm{Tess}(f)$. In fact, one can easily check that

$$K_f^\circ - I_f \;=\; \bigcup_{T \in \mathrm{Tess}(f)} T$$

and each $z \in K_f^\circ - I_f$ is either in the interior of an unique $T \in \mathrm{Tess}(f)$; a vertex shared by four or eight tiles in $\mathrm{Tess}(f)$ if $f^m(z) = f^n(0)$ for some $n, m > 0$; or on an edge shared by two tiles in $\mathrm{Tess}(f)$ otherwise.

Tiles for g and tessellation of $K_g^\circ - I_g = K_g^\circ$ are also defined by replacing f, B_0, and $A_\mu(\pm)$ by g, B_0', and $C_\mu(\pm)$ respectively.

Addresses. Each tile is identified by an *address*, which consists of an *angle*, a *level*, and a *signature* defined as followings:

Level and signature. For $T \in \mathrm{Tess}(f)$ above, i.e., $f^n(T) \subset B_0$ and $\Phi_f \circ f^n(T) = A_\mu(*)$ with $* = +$ or $-$, we say that T has *level* $m = \mu l - n$ and *signature* $*$. Then the critical point $z = 0$ is a vertex of eight tiles of level 0 and $-l$.

For a tile $T' \in \mathrm{Tess}(g)$, its level and signature is defined in the same way.

Angles. For $T \in \mathrm{Tess}(f)$, there exists ζ in α_f such that $I(\zeta)$ contains the degenerating edge of T. Then there are $\mathrm{val}(f) = v \geq 1$ rays landing on $I(\zeta)$, and the rays and $I(\zeta)$ divide the plane into v sectors. (In the case of $v = 1$, equivalently $g(z) = z^2 + 1/4$, we consider the sector as the plane with a slit.) Take two angles $\theta_+ < \theta_- (\leq \theta_+ + 1)$ of external rays bounding the sector containing T. (That is, any external ray of angle θ with $\theta_+ < \theta < \theta_-$ is contained in the sector.) Now we define the *angle* of T by θ_* where $*$ is the signature of T. (See Figure 6)

For a tile $T' \in \mathrm{Tess}(g)$, one can check that there exists a unique point $\beta' \in I_g \cap \partial T'$. Since there are v rays land on β' and they divide the plane into v sectors as in the case of $T \in \mathrm{Tess}(f)$, we define the angle of T' in the same way as above.

We denote such tiles by $T = T_f(\theta_*, m, *)$ and $T' = T_g(\theta_*, m, *)$, and we call the triple $(\theta_*, m, *)$ the *address* of the tiles. For example, Figure 7 shows the structure of addresses for the two tessellations at the lower left of Figure 2.

Now one can easily check the desired property

$$f(T_f(\theta, m, *)) \;=\; T_f(2\theta, m+1, *).$$

The same holds if we replace f by g. One can also check properties (1) to (5) of Theorem 1.1 easily.
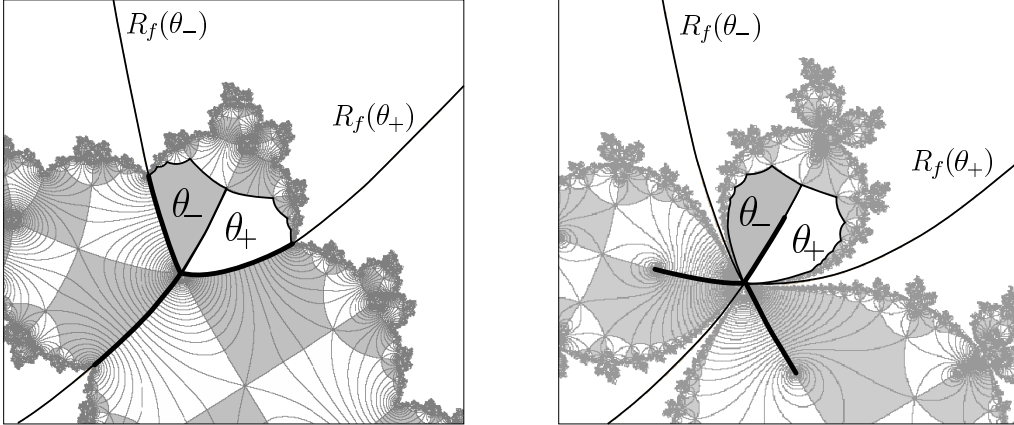
Remarks on angles and levels.

Figure 6: Angles of tiles in Case (a) (left) and Case (b) (right) with $q' = 3$. The thick arcs show degenerating arcs.

- We make an exception for non-trivial primitives ($q = q'$ and $l = l' > 1$). If ($f \to g$) is non-trivial primitive, then $v = 2$ and only tiles of addresses $(\theta_\pm, m, \pm)$ are defined. However, we formally define tiles of addresses $(\theta_\pm, m, \mp)$ by tiles of addresses $(\theta_\mp, m, \mp)$ respectively. (See Figure 8.)

- For a degeneration pair ($f \to g$), the space of possible addresses of tiles is not equal to $\Theta_f \times \mathbb{Z} \times \{+, -\}$ in general. For both f and g, all possible addresses are realized when $l = 1$. But when $l > 1$, the address $(\theta, m, \pm)$ is realized iff $m + n \equiv 0 \mod l$ for some $n > 0$ with $2^n \theta = \theta_0^\pm$. In any case, note that $T_f(\theta, m, *)$ exists iff $T_g(\theta, m, *)$ exists.

3.3 Edge sharing

Let us investigate the combinatorics of tiles in $\mathrm{Tess}(f)$ and $\mathrm{Tess}(g)$. We will show the following proposition that is a detailed version of Theorem 1.1(6):

Proposition 3.2 *For $\theta \in \Theta_f = \Theta_g$ and $* \in \{+, -\}$, let us take an $m \in \mathbb{Z}$ such that $T = T_f(\theta, m, *)$ and $S = T_g(\theta, m, *)$ exists. Then:*

1. *The circular edges of T and S are shared by $T_f(\theta, m \pm l, *)$ and $T_g(\theta, m \pm l, *)$ respectively.*

2. *The degenerating edge of T is contained in $I(\zeta)$ with $\zeta \in \alpha_f$ of type $\Theta(\zeta)$ iff S attaches at $\xi \in I_g$ of type $\Theta(\xi) = \Theta(\zeta)$. Moreover, the degenerating edge of T is shared with $T_f(\theta, m, \bar{*})$, where $\bar{*}$ is the opposite signature of $*$.*

3. *T shares its critical edge with $T_f(\theta', m', *')$ iff S does the same with $T_g(\theta', m', *')$. In this case, we have $m' = m$ and $*' = \bar{*}$.*

Thus the combinatorics of $\mathrm{Tess}(f)$ and $\mathrm{Tess}(g)$ are the same.

Proof. (1) Circular edges: By Proposition 3.1, for any $n \geq 0$, the inverse map $f^{-n} \circ \Phi_f^{-1}$ over $\mathbb{C} - (-\infty, a]$ is a multi-valued function with univalent branches. Now it follows that the property "$A_\mu(*)$ shares its circular edges with $A_{\mu \pm 1}(*)$" is translated to "$T(\theta, m, *)$ shares its circular edges with $T(\theta, m \pm l, *)$" by one of such univalent branches. The same argument works for $\Phi_g : B_0' \to \mathbb{C}$, which does not ramify over $\mathbb{C} - (-\infty, 0]$.
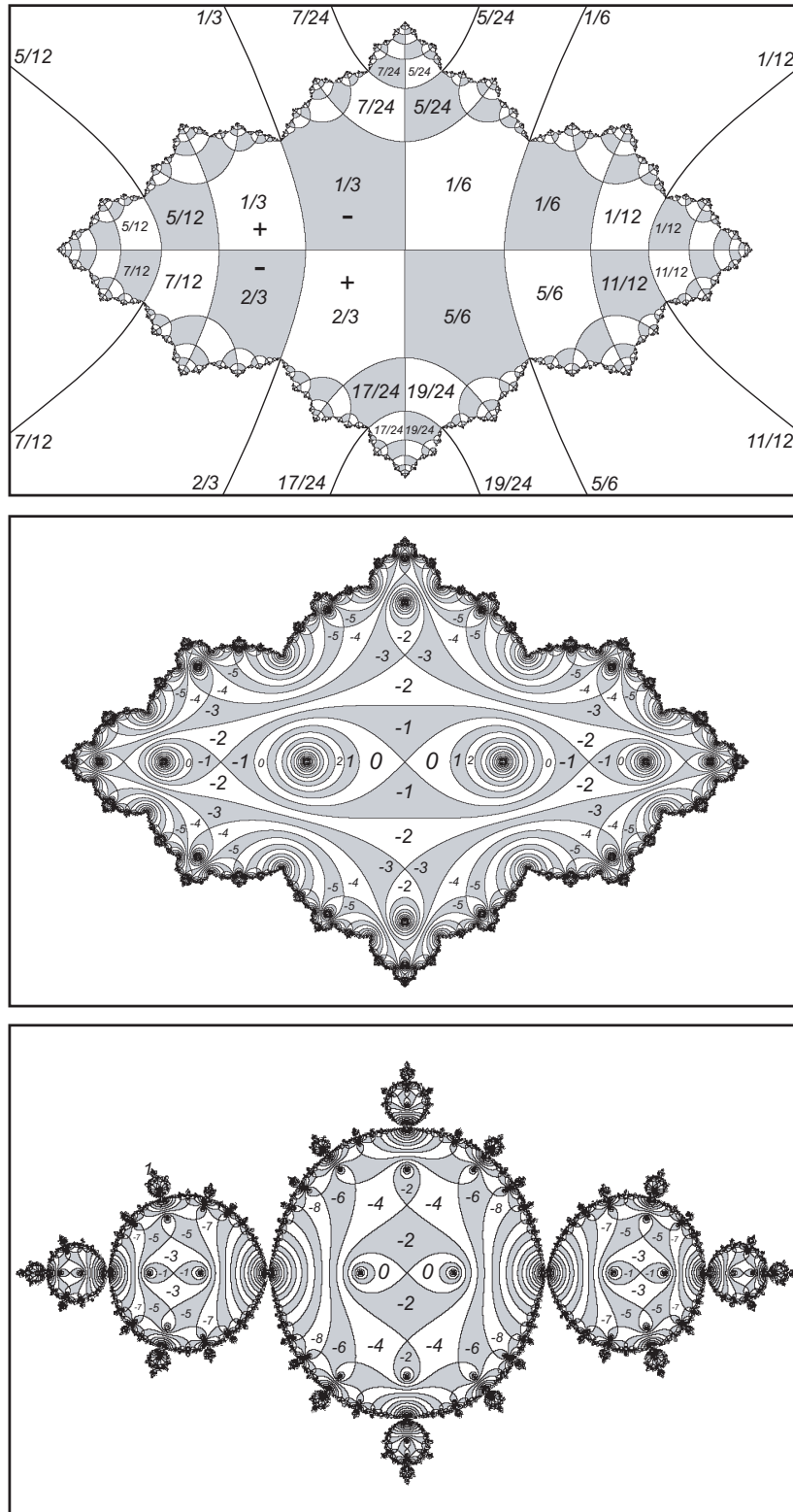
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Figure 7: "Checkerboard" and "Zebras", showing the structure of the addresses of tiles. "Checkerboard", with some external rays drawn in, shows the relation between the external angles and the angles of tiles. The invariant regions colored in white and gray correspond to tiles of signature $+$ and $-$ respectively. "Zebras" show the levels of tiles for f_c with $-1 < c < 0$. Levels get higher near the preimages of the attracting periodic points.
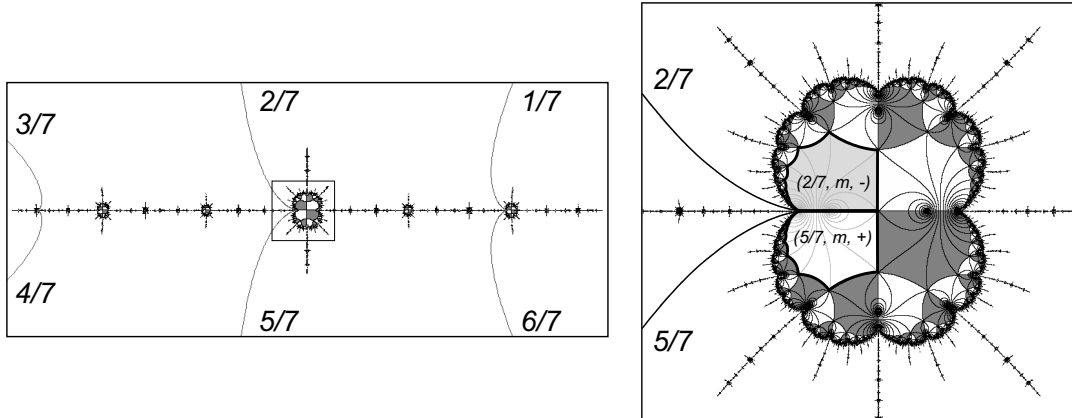
14

Figure 8: A non-trivial primitive $(f \to g)$ with $g(z) = z^2 - 7/4$. For example, we define $T_f(2/7, m, +)$ by $T_f(5/7, m, +)$.

(2) Degenerating edges: The statement is clear by definition of tiles and addresses.

(3) Critical edges: The combinatorics of tiles are essentially determined by the connection of critical edges. They are organized as follows.

In the fundamental model, we consider a family of curves

$$\overline{A_\mu(*)} \cap \left\{ |W - a| = R^{\mu+1/2} \right\}$$
$$C_\mu(*) \cap \{ \operatorname{Re} W = \mu R + 1/2 \}$$

for $\mu \in \mathbb{Z}$ and we call the *essential curves* of $A_\mu(*)$ and $C_\mu(*)$. Since $\Phi_f \circ f^n$ and $\Phi_g \circ g^n$ do not ramify over these essential curves, their pulled-back images in the original dynamics form "equipotential curves" in K_f° and K_g°. The *essential curve* of a tile is the intersection with such equipotential curves.

Let us consider a general tile $T \in \operatorname{Tess}(f)$ as in the statement. By taking suitable $n \gg 0$, we may assume that $f^n(T)$ is a tile in B_0 with angle t in $\{\theta_0^+, \theta_0^-\} \subset \Theta(\alpha_0)$ and level μl for some $\mu \geq 0$. In particular, we may assume that $f^n(T)$ is in the critical sector based at $I(\alpha_0)$. Then for S in the statement, we can take the same n and μ as T such that $g^n(S)$ is a tile in B_0' with angle t in $\Theta(\beta_0) = \Theta(\alpha_0)$ and level μl.

Case (a): $q = q'$. Let η_0 be the union of essential curves of tiles of the form $T_f(t, \mu l, *)$ with t in $\Theta(\alpha_0)$. Then η_0 forms an equipotential curve around α_0, since $\Phi_f|_{\eta_0}$ is a q-fold covering over the circle $\left\{ |W - a| = R^{1/2+\mu} \right\}$. For $n > 0$, set $\eta_{-n} = f^{-n}(\eta_0)$. Then η_{-n} is a disjoint union of simple closed curves passing through tiles of level $\mu l - n$ and angles in $\delta^{-n}(\Theta(\alpha_0))$. In particular, each curve crosses degenerating edges and critical edges alternatively. More precisely, let η be a connected component of η_{-n}. Then the degree of $f^n : \eta \to \eta_0$ varies according to how many curves in $\{f^k(\eta)\}_{k=1}^n$ enclose the critical point $z = 0$. One can check the degree by counting the number of points of $f^{-n}(\alpha_0)$ inside η. Let $\zeta_1, \ldots, \zeta_N$ be such points. Then η crosses each $I(\zeta_i)$, and thus η crosses the tiles of level $-n$ with angles in $\Theta(\zeta_1) \cup \cdots \cup \Theta(\zeta_N) \subset \mathbb{T}$ in cyclic order, and with signatures switching as crossing the edges of tiles. This observation gives us how critical and degenerating edges are shared among tiles along η.

Now we can take η passing though T. This observation concludes that: if T shares its critical edge with $T_f(\theta', m', *')$, then $m' = m$, and $*' = \bar{*}$; and if T shares its degenerating edge with $T_f(\theta', m', *')$, then $\theta' = \theta$, $m' = m$, and $*' = \bar{*}$.

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For S, consider a circle around β_0 which is so small that the circle and the essential curves of tiles with angle $\theta \in \Theta(\beta_0)$ and level μl bound a flower-like disk (Figure 9). Let us denote the boundary of the disk by η_0', which works as η_0. Since the combinatorics of pulled-back sectors based at β_0 and $I(\alpha_0)$ is the same, the observation of $g^{-n}(\eta_0') = \eta_{-n}'$ must be the same as that of η_{-n}. This concludes the statement.
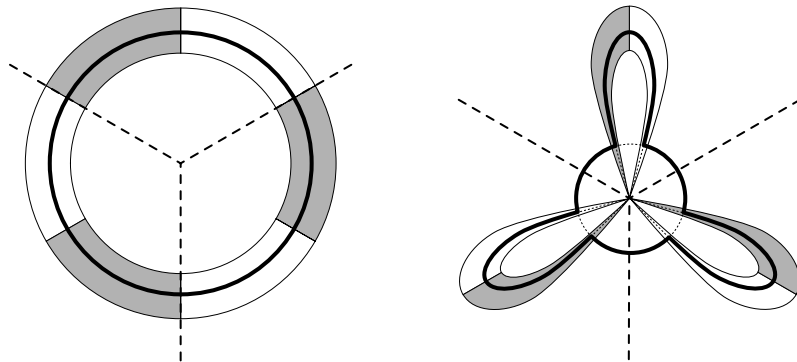

Figure 9: The thick curves show η_0 and η_0' in Case (a) with $q = q' = 3$. The dashed lines indicate the degenerating arcs or external rays.

Case (b): $\boldsymbol{q = 1 < q'}$. (Recall that in this case O_g is perturbed into the repelling cycle $O_f' = \{\alpha_1', \ldots, \alpha_{l'}' = \alpha_0'\}$ with $\alpha_0' \to \beta_0$ as $f \to g$.) The argument above works if we take η_0 and η_0' as following: First in the fundamental model, take $\epsilon \ll 1$ and two radial half-lines from a with argument $\pm\epsilon$. Then there are univalently pulled-back arcs of two lines in the critical sector which joins α_0 and α_0'. Next we take simple closed curves around α_0 and α_0'. For α_0, we take the essential curves along tiles of address $(\theta_0^\pm, \mu l, \pm)$. For α_0', we take just a small circle around α_0'. Then the two arcs and two simple closed curves bound a dumbbell-like topological disk. We define η_0 by its boundary curve.
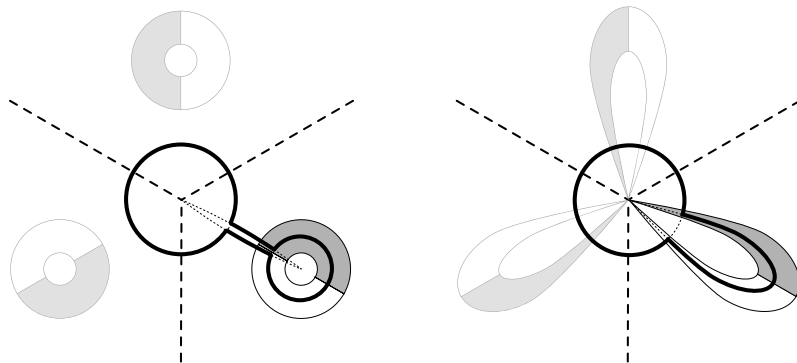

Figure 10: η_0 and η_0' in Case (b) with $q' = 3$.

Correspondingly, for g, the essential curves of tiles of address $(\theta_0^\pm, \mu l, \pm)$ and a small circle around β_0 bounds a topological disk. We take η_0' as its boundary (Figure 10). ∎

3.4 Tiles and panels with small diameters

Next we show that the diameter of tiles are controlled by their angles. For θ in $\Theta_f = \Theta_g$ and $* \in \{+, -\}$, let $\Pi_f(\theta, *)$ and $\Pi_g(\theta, *)$ be the union of tiles with angle θ and signature $*$ in $\mathrm{Tess}(f)$ and $\mathrm{Tess}(g)$ respectively. We call them *panels* of angle θ and signature $*$.

(For later usage, by $\Pi_f(\theta)$ we denote $\Pi_f(\theta,+) \cup \Pi_f(\theta,-)$.) The *depth* of angle θ is the minimal $n \geq 0$ such that $2^n\theta = \theta_0^+$, where $\theta_0^+ \in \Theta(\alpha_0) = \Theta(\beta_0)$ is defined in Section 3. (Note that $\Pi_f(\theta_0^+) = \Pi_f(\theta_0^-)$ when $(f \to g)$ is non-trivial primitive.) We denote such an n by $\mathrm{depth}(\theta)$. Here we show the following:

Proposition 3.3 *For any fixed degeneration pair $(f \to g)$ and any $\epsilon > 0$, there exists $N = N(\epsilon, f, g)$ such that*

$$\mathrm{diam}\,\Pi_f(\theta, *) < \epsilon \quad \text{and} \quad \mathrm{diam}\,\Pi_g(\theta, *) < \epsilon$$

for any signature $*$ and any $\theta \in \Theta_f$ with $\mathrm{depth}(\theta) \geq N$.

Proof. We first work with f and signature $+$. One can easily check that the interior Π of $\Pi_f(\theta_0^+, +)$ is a topological disk. For any $\theta \in \Theta_f$, the panel $\Pi_f(\theta, +)^\circ$ is sent univalently onto Π by f^n with $n = \mathrm{depth}(\theta)$. Let F_θ be the univalent branch of f^{-n} which sends Π to $\Pi_f(\theta, +)^\circ$. Since the family $\{F_\theta : \theta \in \Theta_f\}$ on Π avoids the values outside the Julia set, it is normal.

Now we claim: $\mathrm{diam}\,\Pi_f(\theta, +) \to 0$ as $\mathrm{depth}(\theta) \to \infty$. Otherwise one can find a sequence $\{\theta_k\}_{k>0}$ with depth $n_k \to \infty$ and $\delta > 0$ such that $\mathrm{diam}\,\Pi_f(\theta_k, +) > \delta$ for any k. By passing through a subsequence, we may assume that $F_k := F_{\theta_k}$ has a non-constant limit ϕ. Fix any point $z \in \Pi$, and set $\zeta := \phi(z) = \lim F_k(z)$. Since ϕ is holomorphic and thus is an open map, there exists a neighborhood V of ζ such that $V \subset \phi(\Pi)$ and $V \subset F_k(\Pi)$ for all $k \gg 0$. Since $f^{n_k}(V) \subset \Pi \subset K_f^\circ$, any point in V are attracted to the cycle O_f. However, by univalence of F_k, there exists a neighborhood W of z with $W \subset F_k^{-1}(V) = f^{n_k}(V)$ for all $k \gg 0$. This is a contradiction.

Finally we arrange the angles of Θ_f in a sequence $\{\theta_i\}_{i>0}$ such that $\mathrm{depth}(\theta_n)$ is non-decreasing. Note that for any integer n, there are only finitely many angles with depth n. Thus there exists an integer $N = N(\epsilon, f, +)$ such that $\Pi_f(\theta, +)$ has diameter less than ϵ if $\mathrm{depth}(\theta) \geq N$.

This argument works if we switch the map (from f to g) or the signature. Then we have four distinct N as above. Now we can take $N(\epsilon, f, g)$ as their maximum. ∎

Indeed, as depth tends to infinity we have uniformly small panels for $f \approx g$ (Proposition 5.6).

4 Pinching semiconjugacy

In this section we construct a semiconjugacy $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ associated with $(f \to g)$ by gluing tile-to-tile homeomorphisms inside the Julia sets and the topological conjugacy induced from the Böttcher coordinates outside the Julia sets.

Theorem 4.1 *For a degeneration pair $(f \to g)$, there exists a semiconjugacy $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ from f to g such that*

(1) h maps $\bar{\mathbb{C}} - I_f$ to $\bar{\mathbb{C}} - I_g$ homeomorphically and is a topological conjugacy between $f|_{\bar{\mathbb{C}}-I_f}$ and $g|_{\bar{\mathbb{C}}-I_g}$;

(2) For each $\zeta \in \alpha_f$ with type $\Theta(\zeta)$, h maps $I(\zeta)$ onto a point $\xi \in I_g$ with type $\Theta(\xi) = \Theta(\zeta)$.

(3) h sends all possible $T_f(\theta, m, *)$ to $T_g(\theta, m, *)$, $R_f(\theta)$ to $R_g(\theta)$, and $\gamma_f(\theta)$ to $\gamma_g(\theta)$.

This theorem emphasizes the combinatorial property of h. In the next section we will show that $h \to \mathrm{id}$ as f uniformly tends to g.

Trans-component partial conjugacy and subdivision of tessellation. Let $(f_1 \to g_1)$ and $(f_2 \to g_2)$ be distinct satellite degeneration pair with $g_1 = g_2$. More precisely, we consider $(f_1 \to g_1)$ and $(f_2 \to g_2)$ are tuned copy of degeneration pairs in segment (s1) and (s2) with $q > 1$ by the same tuning operator. By composing homeomorphic parts of the conjugacies associated with $(f_1 \to g_1)$ and $(f_2 \to g_2)$, we have:

Corollary 4.2 *There exists a topological conjugacy* $\kappa = \kappa_{f_1,f_2} : \bar{\mathbb{C}} - I_{f_1} \to \bar{\mathbb{C}} - I_{f_2}$ *from* f_1 *to* f_2.

For example, the panel $\Pi_{f_1}(\theta, *)$ is mapped to the panel $\Pi_{f_2}(\theta, *)$. Now we can compare $\mathrm{Tess}(f_1)$ and $\mathrm{Tess}(f_2)$ via $\mathrm{Tess}(g_i)$. By comparing $\mathrm{Tess}(g_1)$ and $\mathrm{Tess}(g_2)$, one can easily check that

$$T_{g_2}(\theta, \mu, *) \;=\; \bigcup_{j=0}^{q-1} T_{g_1}(\theta, \; \mu + lj, \; *)$$

for any $T_{g_2}(\theta, \mu, *) \in \mathrm{Tess}(g_2)$. Thus $\mathrm{Tess}(g_1)$ is just a subdivision of $\mathrm{Tess}(g_2)$.

Take a tile $T_{f_1}(\theta, m, *) \in \mathrm{Tess}(f_1)$. Then there is a homeomorphic image $T_2'(\theta, m, *) := \kappa(T_{f_1}(\theta, m, *))$ in $K_{f_2}^\circ$. We say the family

$$\mathrm{Tess}'(f_2) := \{\kappa(T) : T \in \mathrm{Tess}(f_1)\}$$

is the *subdivided tessellation* of $K_{f_2}^\circ - I_{f_2}$. Since $\mathrm{Tess}(f_1)$ and $\mathrm{Tess}(f_2)$ have the same combinatorics as $\mathrm{Tess}(g_1)$ and $\mathrm{Tess}(g_2)$ respectively,

$$T_{f_2}(\theta, \mu, *) \;=\; \bigcup_{j=0}^{q-1} T_{f_2}'(\theta, \; \mu + lj, \; *)$$

for any $T_{f_2}(\theta, \mu, *) \in \mathrm{Tess}(f_2)$. Now we have a natural tile-to-tile correspondence between $\mathrm{Tess}(f_1)$, $\mathrm{Tess}(g_1)$ and $\mathrm{Tess}'(f_2)$. In other word, combinatorial property of tessellation is preserved under the degeneration from f_1 to g and the bifurcation from g to f_2.

In Part II of this paper, we will use this property to investigate the structure of the Lyubich-Minsky hyperbolic 3-laminations associated with f_1, g, and f_2.

4.1   Proof of Theorem 4.1

The rest of this section is devoted to the proof of the theorem. The proof breaks into five steps.

1.   Conjugacy on the fundamental model.   First we make a topological map $H : \mathbb{C} - I \to \mathbb{C}$ which maps $A_\mu(\pm)$ to $C_\mu(\pm)$ homeomorphically. For $W \in \mathbb{C} - I$, set $W := a + \rho e^{it}$ where $\rho > 0$ and $0 < t < 2\pi$. We define the map H by

$$H(W) := \frac{\log \rho - \log a}{\log R} + 2ai \tan \frac{\pi - t}{2} \in \mathbb{C}.$$

Then one can check that H conjugates the action of F on $\mathbb{C} - I$ to that of G on $\mathbb{C}$ and H maps $A_\mu(\pm)$ homeomorphically onto $C_\mu(\pm)$.

2. Tile-to-tile conjugation.   First we consider the critical sectors of f and g. Let Π_0 and Π_0' denote the union of tiles of addresses $(\theta_0^\pm, \mu l, \pm)$ with $\mu > 0$ in $\mathrm{Tess}(f)$ and $\mathrm{Tess}(g)$ respectively.
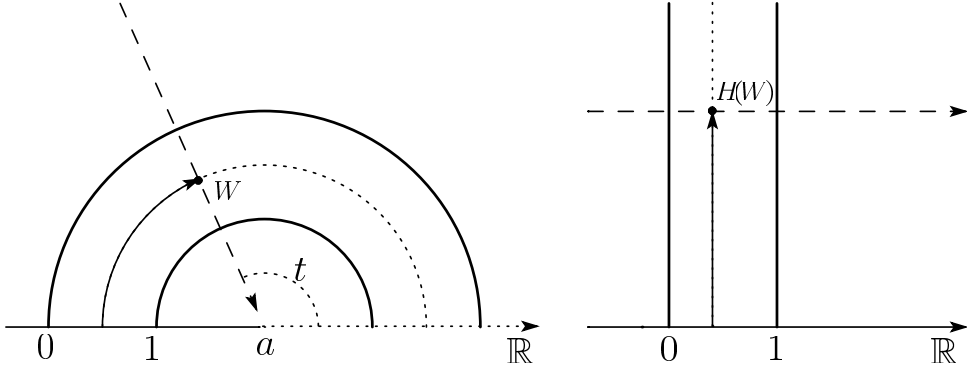
Figure 11: H maps $A_0(+)$ to $C_0(+)$.

By Proposition 3.1, the map $\Phi_f : \Pi_0 \to \mathbb{C}$ is univalent and we can choose a univalent branch Ψ_g of Φ_g^{-1} which sends $\{W : \operatorname{Re} W \geq 1\}$ to Π_0'. For each point in Π_0, we define $h := \Psi_g \circ H \circ \Phi_f |_{\Pi_0}$. Then we have a conjugacy between $f^{lq}|_{\Pi_0}$ and $g^{lq}|_{\Pi_0'}$. Any tile eventually lands on tiles in Π_0 or Π_0'. According to the combinatorics of tiles determined by pulling-back essential curves in Π_0 and Π_0', we can pull-back h over $K_f^\circ - I_f$ and $h : K_f^\circ - I_f \to K_g^\circ$ is a conjugacy.

3. Continuous extension to degenerating arc system. Take any $\zeta \in \alpha_f$. For any point z in $I(\zeta)$, we define $h(z)$ by the unique $\xi \in I_g$ with $\Theta(\xi) = \Theta(\zeta)$.

Now we show the continuity of $h : K_f^\circ \cup I_f \to K_g^\circ \cup I_g$ which we have defined. Take any z in $I(\zeta)$. We claim that any sequence $z_n \in K_f^\circ \cup I_f$ converging to z satisfies $h(z_n) \to \xi$.

First when z is neither ζ nor one of the endpoints of $I(\zeta)$, it is enough to consider the case of $z_n \in K_f^\circ - I_f$ for all n. Now z is on the degenerating edges of at most four tiles. Let $T = T_f(\theta, m, +)$ be one of such tiles. The subsequence z_{n_i} of z_n contained in T is mapped to $T_g(\theta, m, +)$. In the fundamental model, the sequence $h(z_{n_i})$ corresponds to a sequence whose imaginary part is getting higher. Thus $h(z_{n_i})$ converges to ξ with type containing θ, which must coincide with $\Theta(\zeta)$. By changing the choice of T, we have $h(z_n) \to \xi$ with $\Theta(\xi) = \Theta(\zeta)$.

Next, if z is ζ or one of the endpoints of $I(\zeta)$, it is an attracting or repelling periodic point. If z is attracting, the levels of tiles containing z_n go to $+\infty$. According to the fundamental model, we have $h(z_n) \to \xi$.

The last case is when z is repelling, thus in the Julia set. We deal with this case in the next paragraph.

4. Continuous extension to the Julia set. Take any $z \in J_f$ and any sequence $z_n \in K_f^\circ \cup I_f$ converging to z. Then we take a sequence $\theta_n \in \Theta_f$ such that $z_n \in \Pi_f(\theta_n)$. After passing to a subsequence we may assume θ_n and $h(z_n)$ converge to some $\theta \in \mathbb{T}$ and $w \in K_g$ respectively.

We first claim that $z = \gamma_f(\theta)$, that is, $\theta \in \Theta(z)$. If the depth of θ_n is bounded, then $\theta_n = \theta \in \Theta_f$ for all $n \gg 0$. This implies $z_n \in \Pi_f(\theta)$ for all $n \gg 0$ and it follows that $z \in \overline{\Pi_f(\theta)} \cap J_f$. Thus $z = \gamma_f(\theta)$ by definition of $\Pi_f(\theta)$. If the depth of θ_n is unbounded, it is enough to consider the subsequences with the depth of θ_n monotonously increasing. Take any $\epsilon > 0$. For $n \gg 0$, we have $|\gamma_f(\theta_n) - z_n| < \epsilon$ by Proposition 3.3, and we also have $|\gamma_f(\theta_n) - \gamma_f(\theta)| < \epsilon$ by continuity of $\gamma_f : \mathbb{T} \to J_f$. Finally $|z - z_n| < \epsilon$ for $n \gg 0$ implies $|z - \gamma_f(\theta)| < 3\epsilon$ and conclude the claim.

Since $h(z_n) \in \Pi_g(\theta_n)$, the same argument works for $h(z_n)$ and w. Hence we also claim that $w = \gamma_g(\theta) \in J_g$. It follows that the original $z_n \to z$ implies $h(z_n)$ accumulates only

on $\gamma_g(\theta)$ with $\theta \in \Theta(z)$.

By Theorem A.1, there exists a semiconjugacy $h_J : J_f \to J_g$ with $h_J \circ \gamma_f = \gamma_g$. Since $\gamma_f(\theta) = \gamma_f(\theta')$ for any θ and θ' in $\Theta(z)$, we have $\gamma_g(\theta) = \gamma_g(\theta')$. This implies that $h(z_n)$ accumulates on a unique point $\gamma_g(\theta)$. Thus h continuously extends to the Julia set by $h(\gamma_f(\theta)) := \gamma_g(\theta)$ for each $\theta \in \mathbb{T}$.

5. Global extension. Finally we define $h : \bar{\mathbb{C}} - K_f \to \bar{\mathbb{C}} - K_g$ by the conformal conjugacy between $f|_{\bar{\mathbb{C}} - K_f}$ and $g|_{\bar{\mathbb{C}} - K_g}$ given via Böttcher coordinates. This conjugacy and the semiconjugacy above continuously glued along the Julia set thus we have a semiconjugacy on the sphere.

Properties (2) and (3) are clear by construction. To check property (1), we need to show that $h^{-1} : \bar{\mathbb{C}} - I_f \to \bar{\mathbb{C}} - I_g$ is continuous. Continuity in $\bar{\mathbb{C}} - K_g$ and K_g° is obvious by construction. Take any point $w \in J_g - I_g$. A similar argument to step 4 shows that any sequence $w_n \to w$ within $\bar{\mathbb{C}} - I_g$ is mapped to a convergent sequence $z_n \to z$ within $\bar{\mathbb{C}} - I_f$ satisfying $\Theta(z) = \Theta(w) \subset \mathbb{T} - \Theta_g$. ∎

5 Continuity of pinching semiconjugacies

In this section we deal with continuity of the dynamics of the degeneration pair $(f \to g)$ as f tends to g. We will establish:

Theorem 5.1 *Let $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be the semiconjugacy associated with a degeneration pair $(f \to g)$ that is given in Theorem 4.1. Then h tends to identity as f tends to g.*

Here are two immediate corollaries:

Corollary 5.2 *The closures of $T_f(\theta, m, *)$ and $\Pi_f(\theta, *)$ in $\mathrm{Tess}(f)$ uniformly converge to those of $T_g(\theta, m, *)$ and $\Pi_g(\theta, *)$ in $\mathrm{Tess}(g)$ in the Hausdorff topology.*

Corollary 5.3 *As $f \to g$, the diameters of connected components of I_f uniformly tends to 0.*

Let us start with some terminologies for the proof. Two degeneration pair $(f_1 \to g_1)$ and $(f_2 \to g_2)$ are *equivalent* if $g_1 = g_2$ and both f_1 and f_2 are in the same hyperbolic component. For a degeneration pair $(f \to g)$ by $f \approx g$ we mean f is sufficiently close to g. In other words, the multiplier $r\omega$ of O_f is sufficiently close to ω, i.e., $r \approx 1$.

Formally we consider a family of equivalent degeneration pairs $\{(f \to g)\}$ parameterized by $0 < r < 1$ and its behavior when r tends to 1. To show the theorem, it suffices to show the following:

(i) For any compact set K in $\bar{\mathbb{C}} - K_g$, we have $K \subset \bar{\mathbb{C}} - K_f$ for all $f \approx g$ and $h \to \mathrm{id}$ on K.

(ii) For any compact set K in K_g°, we have $K \subset K_f^\circ$ for all $f \approx g$ and $h \to \mathrm{id}$ on K.

(iii) h is equicontinuous as $f \to g$ on the sphere.

In fact, any sequence h_k associated with $f_k \to g$ has a subsequential limit h_∞ which is identity on $\bar{\mathbb{C}} - J_g$ and continuous on $\bar{\mathbb{C}}$. Since $\bar{\mathbb{C}} - J_g$ is open and dense, the map h_∞ must be identity on the whole sphere.

5.1 Proof of (i)

Let $B_f : \bar{\mathbb{C}} - \mathbb{D} \to \bar{\mathbb{C}} - K_f^\circ$ be the extended Böttcher coordinate of K_f, i.e., $B_f : \bar{\mathbb{C}} - \bar{\mathbb{D}} \to \bar{\mathbb{C}} - K_f$ is a conformal map with $B_f(w^2) = f(B_f(w))$; $B_f(w)/w \to 1$ as $w \to \infty$; and $B_f(e^{2\pi i\theta}) := \gamma_f(\theta) \in J_f$. Now (i) follows immediately from this stronger claim:

Theorem 5.4 (Böttcher convergence) *As $f \to g$, we have a uniform convergence $B_f \to B_g$ on $\bar{\mathbb{C}} - \mathbb{D}$.*

Note that the uniform convergence on compact sets in $\bar{\mathbb{C}} - \bar{\mathbb{D}}$ is not difficult. Our proof is a mild generalization of the proof of Theorem 2.11 in [Po].

Proof. By Corollary A.2 one can easily check that $\bar{\mathbb{C}} - K_f$ converges to $\bar{\mathbb{C}} - K_g$ in the sense of Carathéodory kernel convergence with respect to ∞. Thus pointwise convergence $B_f \to B_g$ on each $z \in \bar{\mathbb{C}} - \bar{\mathbb{D}}$ is given by [Po, Theorem 1.8] and $B_f'(\infty) = B_g'(\infty) = 1$. To show the theorem, it is enough to show that K_f is uniformly locally connected as $f \to g$ by [Po, Corollary 2.4]. That is, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $f \approx g$ and any $a, b \in K_f$ with $|a - b| < \delta$, there exists a continuum E such that diam $E < \epsilon$.

Suppose we have a sequence of equivalent degenerating pairs ($f_n \to g$) such that: $f_n \to g$ uniformly; for f_n there exist a_n and a_n' in J_{f_n} with $|a_n - a_n'| \to 0$ which can not be contained in the same continuum in K_{f_n} of diameter less than $\epsilon_0 > 0$. We may set $a_n = \gamma_{f_n}(\theta_n)$ and $a_n' = \gamma_{f_n}(\theta_n')$ for some $\theta_n, \theta_n' \in \mathbb{T}$ since γ_{f_n} maps $\mathbb{T}$ onto J_{f_n}. By passing to a subsequence, we may also assume that $\theta_n \to \theta$ and $\theta_n' \to \theta'$. Since $\gamma_{f_n} \to \gamma_g$ uniformly by Corollary A.3, the assumption $|a_n - a_n'| \to 0$ implies that we have $\gamma_g(\theta) = \gamma_g(\theta') =: w \in J_g$.

Case 1: $\theta = \theta'$. We may assume that $\theta_n \leq \theta_n'$ and both tend to θ. Set $E_n := \{\gamma_{f_n}(t) : t \in [\theta_n, \theta_n']\}$, which is a continuum containing a_n and a_n'. Then for any $t \in [\theta_n, \theta_n']$, we have $|\gamma_{f_n}(t) - w| \leq |\gamma_{f_n}(t) - \gamma_g(t)| + |\gamma_g(t) - \gamma_g(\theta)| \to 0$ since $\gamma_{f_n} \to \gamma_g$ uniformly and γ_g is continuous. This implies diam $E_n \to 0$ and is a contradiction.

Case 2-1: $\theta \neq \theta'$ and $w \notin I_g$. First we show that $\gamma_{f_n}(\theta) = \gamma_{f_n}(\theta')$. Let $h_n : J_{f_n} \to J_g$ be the semiconjugacy given by Theorem A.1. Since $h_n \circ \gamma_{f_n} = \gamma_g$, we have

$$w = h_n \circ \gamma_{f_n}(\theta) = h_n \circ \gamma_{f_n}(\theta') \notin I_g.$$

By property 1 of Theorem A.1, this implies $\gamma_{f_n}(\theta) = \gamma_{f_n}(\theta')$. Now set

$$E_n := \left\{ \gamma_{f_n}(t) : |t - \theta| \leq |\theta_n - \theta| \text{ or } |t - \theta'| \leq |\theta_n' - \theta'| \right\},$$

which is a continuum containing a_n and a_n'. Again one can easily check that $|\gamma_{f_n}(t) - w| \to 0$ uniformly for any $\gamma_{f_n}(t) \in E_n$ and diam $E_n \to 0$.

Case 2-2: $\theta \neq \theta'$ and $w \in I_g$. There exists an $m \geq 0$ such that $g^m(w) = \beta_0$. Since $h_n \circ \gamma_{f_n} = \gamma_g$ we have $\gamma_{f_n}(\theta), \gamma_{f_n}(\theta') \in h_n^{-1}(w) \subset J_{f_n} \cap I_{f_n}$. If $q = 1$, then h_n is homeomorphism by Theorem A.1. Thus $\gamma_{f_n}(\theta) = \gamma_{f_n}(\theta')$ and a contradiction follows from the same argument as above.

Suppose $q > 1$. Then Case (a) ($q = q'$ and $l = l'$) by Proposition 2.1. In particular, we have $w_n \in \alpha_{f_n}$ such that $w_n \to w$ and $f_n^m(w_n)$ is an attracting periodic point $\alpha_{0,n} \in O_{f_n}$ which tends to β_0. Let $\lambda_n = r_n e^{2\pi i p/q}$ be the multiplier of O_{f_n} with $r_n \nearrow 1$. On a fixed small neighborhood of w, we have

$$f^{-m} \circ f^{lq} \circ f^m(z) = r_n^q z (1 + z^q + O(z^{2q}))$$
$$\longrightarrow g^{-m} \circ g^{lq} \circ g^m(z) = z (1 + z^q + O(z^{2q}))$$

by looking though suitable local coordinates as in Appendix A.2. (For simplicity, we abbreviate conjugations by the local coordinates.)

By Lemma A.7, we can find a small continuum $E'_n \subset K_{f_n}$ which joins w_n and preperiodic points $\gamma_{f_n}(\theta)$, $\gamma_{f_n}(\theta')$. Set E_n as in Case 2-1. Now $E'_n \cup E_n$ is a continuum containing a_n and a'_n. Since $\mathrm{diam}\,(E'_n \cup E_n) \to 0$, we have a contradiction again. ∎

5.2 Proof of (ii)

Let us start with the following theorem:

Theorem 5.5 (Linearization convergence) *Let K be any compact set in K°_g. Then $K \subset K^{\circ}_f$ for $f \approx g$ and $\Phi_f \to \Phi_g$ uniformly on K.*

Proof. One can easily check that $K \subset K^{\circ}_f$ if $f \approx g$ by Corollary A.2. Let $\beta_0 \in O_g \cap \partial B'_0$. We may assume that $K' = g^N(K)$ is sufficiently close to β_0 and contained in B'_0 by taking a suitable $N \gg 0$. Then K' is attracted to β_0 along the attracting direction associated with B'_0 by iteration of $g^{l'q'}$. For simplicity, set $\bar{l} := lq = l'q'$.

Recall that Φ_f and Φ_g semiconjugate $f^{\bar{l}}$ and $g^{\bar{l}}$ to F^q and G^q in the fundamental model respectively. We will construct other semiconjugacies $\tilde{\Phi}_f$ and $\tilde{\Phi}_g$ with the same property plus $\tilde{\Phi}_f \to \tilde{\Phi}_g$ on compact subsets of a small attracting petal in B'_0. Then we will show that they coincide.

By Appendix A.2, there exist local coordinates $\zeta = \psi_f(z)$ and $\zeta = \psi_g(z)$ with $\psi_f \to \psi_g$ near β_0 such that we can view $f^{\bar{l}} \to g^{\bar{l}}$ as

$$
\begin{aligned}
f^{\bar{l}}(\zeta) &= \Lambda \zeta \left(1 + \zeta^{q'} + O(\zeta^{2q'})\right) \\
\longrightarrow g^{\bar{l}}(\zeta) &= \zeta \left(1 + \zeta^{q'} + O(\zeta^{2q'})\right)
\end{aligned}
$$

where $\Lambda \to 1$. (To simplify notation, we abbreviate conjugations by these local coordinates. For example, by $f^{\bar{l}}(\zeta)$ we mean $\psi_f \circ f^{\bar{l}} \circ \psi_f^{-1}(\zeta)$.) Now there are two cases for Λ:

- In Case (a) ($q = q'$ and $l = l'$), the fixed point $\zeta = 0$ is attracting and $\Lambda = \lambda^q = r^q = R < 1$.

- In Case (b) ($q = 1 < q' = l/l'$), the fixed point $\zeta = 0$ is repelling and $|\Lambda| > 1$.

Next by taking branched coordinate changes $w = \Psi_f(\zeta) = -\Lambda^{q'}/(q'\zeta^{q'})$ and $w = \Psi_g(\zeta) = -1/(q'\zeta^{q'})$ respectively, we can view $f^{\bar{l}} \to g^{\bar{l}}$ as

$$
\begin{aligned}
f^{\bar{l}}(w) &= \Lambda^{-q'} w + 1 + O(1/w) \\
\longrightarrow g^{\bar{l}}(w) &= w + 1 + O(1/w).
\end{aligned}
$$

Case (a). Set $\tau = \Lambda^{-q'} = R^{-q} > 1$. By simultaneous linearization in Appendix A.3, we have convergent coordinate changes $W = u_f(w) \to u_g(w)$ on compact sets of $P_\rho := \{\mathrm{Re}\, w > \rho \gg 0\}$ such that $f^{\bar{l}} \to g^{\bar{l}}$ is viewed as

$$
\begin{aligned}
\tilde{F}(W) &:= f^{\bar{l}}(W) = \tau W + 1 \\
\longrightarrow \tilde{G}(W) &:= g^{\bar{l}}(W) = W + 1.
\end{aligned}
$$

Let us adjust $\tilde{F} \to \tilde{G}$ to $F^q \to G^q$ in the fundamental model. Recall that the map $F(W) = RW + 1$ has the attracting fixed point at $a = 1/(1 - R)$. On the other hand, the

map $\tilde{F}$ has the repelling fixed point $\tilde{a} = 1/(1 - R^{-q})$ instead. Set $T_f(W) := aW/(W - \tilde{a})$. Then $T_f(W) = qW(1 + O(W/\tilde{a})) \to T_g(W) = qW$ on any compact sets on the W-plane as $R \to 1$. By taking conjugations with T_f and T_g, we can view $\tilde{F} \to \tilde{G}$ as $F^q \to G^q$ on any compact sets of the domain of $\tilde{G}$.

Case (b). By Rouché's theorem, there exists a fixed point b of $f^{\bar{l}}(w) = \Lambda^{-q'}w + 1 + O(1/w)$ of the form $b = 1/(1 - \Lambda^{-q'}) + O(1)$. Indeed, this b is one of the image of the attracting cycle O_f hence its multiplier is $r < 1$. Set $S_f(w) := bw/(b - w)$. Then $S_f(w) = w(1 + O(w/b)) \to S_g(w) = w$ on any compact sets of w-plane as $r \to 1$. By taking conjugations by S_f and S_g, we can view $f^{\bar{l}}(w) \to g^{\bar{l}}(w)$ as

$$
\begin{aligned}
f^{\bar{l}}(w) &= \tau w + 1 + O(1/w) \\
\longrightarrow \quad g^{\bar{l}}(w) &= w + 1 + O(1/w)
\end{aligned}
$$

where $\tau = 1/r > 1$. By simultaneous linearization, we have convergent coordinate changes $W = u_f(w) \to u_g(w)$ on compact sets of P_ρ such that $f^{\bar{l}} \to g^{\bar{l}}$ is again viewed as

$$
\begin{aligned}
\tilde{F}(W) := f^{\bar{l}}(W) &= \tau W + 1 \\
\longrightarrow \quad \tilde{G}(W) := g^{\bar{l}}(W) &= W + 1.
\end{aligned}
$$

Since $q = 1$, we adjust $\tilde{F} \to \tilde{G}$ to $F \to G$ in the fundamental model. Set $\tilde{b} := 1/(1 - \tau)$ and $T_f(W) := \tilde{b}W/(\tilde{b} - W)$. Then $T_f(W) = W(1 + O(W/\tilde{b})) \to T_g(W) = W$ on any compact sets on the W-plane as $r \to 1$. By taking conjugations by T_f and T_g, we can view $\tilde{F} \to \tilde{G}$ as $F \to G$ on any compact sets of the domain of $\tilde{G}$.

Adjusting critical orbits. Now we denote these final local coordinates conjugating $f^{\bar{l}} \to g^{\bar{l}}$ to $F^q \to G^q$ by $\hat{\Phi}_f \to \hat{\Phi}_g$, where the convergence holds on compact subsets of a small attracting petal P' in B_0' corresponding to P_ρ in the w-plane.

We need to adjust the positions of the images of the critical orbits on the W-plane by $\hat{\Phi}_f \to \hat{\Phi}_g$ to those by Φ_f and Φ_g. We may assume that $g^{n\bar{l}}(0) \in P'$ for fixed $n \gg 0$. Then $f^{n\bar{l}}(0) \in P'$ for all $f \approx g$. Set $s := \hat{\Phi}_f(f^{n\bar{l}}(0))$ and $s' := \hat{\Phi}_g(g^{n\bar{l}}(0))$. Then $s \to s'$ as $f \to g$. On the other hand, we have

$$
\Phi_f(f^{n\bar{l}}(0)) = F^{nq}(\Phi_f(0)) = F^{nq}(0) = R^{nq-1} + \cdots + 1 =: R_n
$$

and $\Phi_g(g^{n\bar{l}}(0)) = nq$. Set $U_f(W) := k(W - a) + a$ and $U_g(W) := W + nq - s'$ where $k = (R_n - a)/(s - a)$. Then one can check that $U_f \to U_g$ on any compact sets in the W-plane as $f \to g$ and U_f and U_g commute with F and G respectively. By defining $\tilde{\Phi}_f$ and $\tilde{\Phi}_g$ by $U_f \circ \hat{\Phi}_f$ and $U_g \circ \hat{\Phi}_g$ respectively, we have $\tilde{\Phi}_f \to \tilde{\Phi}_g$ on compact sets of P' with $\tilde{\Phi}_f(f^{n\bar{l}}(0)) = R_n$ and $\tilde{\Phi}_g(g^{n\bar{l}}(0)) = nq$.

Finally we need to check that $\tilde{\Phi}_f = \Phi_f$ and $\tilde{\Phi}_g = \Phi_g$. The latter equality is clear by uniqueness of the Fatou coordinate ([Mi1, §8]). For the former, recall that $W = \Phi_f(z)$ is given by

$$
z \mapsto \phi_f(z) = w \mapsto w^q = W \mapsto a(1 - W) =: \Phi_f(z)
$$

and ϕ_f is uniquely determined under the condition of $\phi_f(0) = 1$ ([Mi1, §10]). Let us consider the local coordinate $\tilde{\phi}_f$ on a compact set of P' given by

$$
z \mapsto \tilde{\Phi}_f(z) = W \mapsto \left(1 - \frac{W}{a}\right)^{1/q} =: \tilde{\phi}_f(z) = w,
$$

23

where we take a suitable branch of qth root such that $\tilde{\phi}_f(f^{n\bar{l}}(0)) = \lambda^{nq}$ on the w-plane. Then $\tilde{\phi}_f(f(z)) = \lambda\tilde{\phi}_f(z)$. Since $\phi_f(0) = 1$ is equivalent to $\phi_f(f^{n\bar{l}}(0)) = \lambda^{nq}$, the map $\tilde{\phi}_f$ coincide with ϕ_f. This implies the former equality.

Now we may assume that $K' = g^N(K) \subset D \Subset P'$ for some open set D. If $f \approx g$, then $f^N(K) \subset D$ and we have the uniform convergence $\Phi_g \to \Phi_f$ on D. We finally obtain the uniform convergence on K by $\Phi_f(z) = F^{-N}(\Phi_f(f^N(z))) \to G^{-N}(\Phi_g(g^N(z)) = \Phi_g(z)$ for $z \in K$. ∎

Proof of (ii). We first work with the fundamental model. Suppose $\epsilon \searrow 0$, and set $R = 1 - \epsilon$. Then $F(W) = RW + 1$ fixes $a_\epsilon = 1/(1 - R) = \epsilon^{-1}$. For a fixed γ with $1/2 < \gamma < 1$, we define a compact set $Q_\epsilon \subset \mathbb{C}$ by:

$$Q_\epsilon := \left\{ W = a_\epsilon + \rho e^{(\pi - t)i} \in \mathbb{C} \ : \ |t| \le \epsilon^\gamma, \ |\rho - a_\epsilon| \le a_\epsilon \sin \epsilon^\gamma \right\}.$$

Let D be any bounded set in $\mathbb{C}$. For all $\epsilon \ll 1$, the compact set Q_ϵ contains D. Let $H : \mathbb{C} - [a_\epsilon, \infty) \to \mathbb{C}$ be the conjugacy between F and $G(W) = W + 1$ as in Section 4. Then one can easily check that $|\operatorname{Re} W - \operatorname{Re} H(W)| = O(\epsilon^{2\gamma - 1})$ and $|\operatorname{Im} W - \operatorname{Im} H(W)| = O(\epsilon^{2\gamma - 1})$ on Q_ϵ. Thus $H \to \operatorname{id}$ uniformly on D.

Let K be any compact set in K_g°, and let D be the $1/10$-neighborhood of $\Phi_g(K)$. For all $f \approx g$, we have $K \subset K_f^\circ$ and $\Phi_f(K) \subset D$ by Theorem 5.5. By the argument above on the fundamental model, the restriction $h|_K$ is a branch of $\Phi_g^{-1} \circ H \circ \Phi_f$ that converges to the identity. (The branch is determined by the tile-to-tile correspondence given by h.) ∎

5.3 Proof of (iii)

To show (iii) we need two propositions on properties of panels as $f \to g$. The first one is a refinement of Proposition 3.3, and the second one is on the convergence of panels with a fixed angle:

Proposition 5.6 (Uniformly small panels) *For any $\epsilon > 0$, there exists $N = N(\epsilon)$ such that for all $f \approx g$, $* = \pm$ and $\theta \in \Theta_g$ with $\operatorname{depth}(\theta) \ge N$,*

$$\operatorname{diam} \Pi_f(\theta, *) < \epsilon \quad and \quad \operatorname{diam} \Pi_g(\theta, *) < \epsilon.$$

Proposition 5.7 (Hausdorff convergence to a panel) *For fixed angle $\theta \in \Theta_g$ and signature $* = +$ or $-$, we have $\overline{\Pi_f(\theta, *)} \to \overline{\Pi_g(\theta, *)}$ as $f \to g$ in the Hausdorff topology.*

Let us show (iii) first by assuming them:

Proof of (iii). By (i) we have equicontinuity near ∞. Assume that there exist degeneration pairs $(f_k \to g)$ with semiconjugacies h_k as in Theorem 4.1, points a_k, $a_k' \in \mathbb{C}$ with $|a_k - a_k'| \to 0$, and $b_k = h_k(a_k)$, $b_k' = h_k(a_k')$ with $|b_k - b_k'| \ge \epsilon_0 > 0$.

Suppose that a_k, $a_k' \in \mathbb{C} - K_f^\circ$ thus b_k, $b_k' \in \mathbb{C} - K_g^\circ$. Then there exists w_k, $w_k' \in \mathbb{C} - \mathbb{D}$ such that $a_k = B_{f_k}(w_k)$, $a_k' = B_{f_k}(w_k')$ and $b_k = B_g(w_k)$, $b_k' = B_g(w_k')$. By Theorem 5.4, we have $B_{f_k} \to B_g$. Thus $|a_k - a_k'| \to 0$ implies $|b_k - b_k'| \to 0$, a contradiction.

Now it is enough to show the case where a_k, $a_k' \in K_{f_k}$ thus b_k, $b_k' \in K_g$. By taking subsequences, we may assume that $a_k \to a$, $a_k' \to a$, $b_k \to b$, and $b_k' \to b'$ with $|b - b'| \ge \epsilon_0/2 > 0$. Since $K_{f_k} \to K_g$ in the Hausdorff topology, a, b and b' are all in K_g.

First let us consider the case where a is bounded distance away from J_g. Then we have a compact neighborhood E of a such that $h_k|_E \to \operatorname{id}|_E$ and $a_k, a_k' \in E$ for all $k \gg 0$. This implies that $|b_k - b_k'| \to 0$, a contradiction.

Next we consider the case where $a \in J_g$. For $a_k \to a$ and $b_k \to b$, we will claim that $a = b$. Then by the same argument we have $a = b'$ and this is a contradiction.

For $a_k \in K_{f_k}$, we take any $\theta_k \in \mathbb{T}$ such that: $a_k = \gamma_{f_k}(\theta_k)$ if $a_k \in J_{f_k}$; otherwise a_k is contained in the closure of $\Pi_{f_k}(\theta_k)$. (Then $b_k = \gamma_g(\theta_k)$ or b_k is in the closure of $\Pi_g(\theta_k)$.) By passing to a subsequence, we may assume that $\theta_k \to \theta$ for some $\theta \in \mathbb{T}$.

If $\theta_k \notin \Theta_g$, we define its depth by ∞. Then there are two more cases according to $\limsup \operatorname{depth}(\theta_k) = \infty$ or not.

If $\limsup \operatorname{depth}(\theta_k) = \infty$, we take a subsequence again and assume that $\operatorname{depth}(\theta_k)$ is strictly increasing. Then by Proposition 5.6 we have $|a_k - \gamma_{f_k}(\theta_k)| \to 0$. Since $\theta_k \to \theta$ and $\gamma_{f_k} \to \gamma_g$ uniformly (Corollary A.3), we have $|a_k - \gamma_g(\theta)| \to 0$, thus $a = \gamma_g(\theta)$. Similarly we conclude that $b = \gamma_g(\theta)$ and this implies a contradiction.

If $\limsup \operatorname{depth}(\theta_k) < \infty$, we take a subsequence again and assume that $\theta_k = \theta \in \Theta_g$ for all $k \gg 0$. By Proposition 5.7 $a_k \in \overline{\Pi_{f_k}(\theta)}$ are approximated by some $c_k \in \Pi_g(\theta)$ with $|a_k - c_k| \to 0$ thus $c_k \to a \in J_g$. Since $\overline{\Pi_g(\theta)} \cap J_g = \{\gamma_g(\theta)\}$, we have $a = \gamma_g(\theta)$. On the other hand, if $b_k \in \Pi_g(\theta)$ is bounded distance away from J_g, there exists a compact neighborhood $E' \subset K_g^\circ$ of b where $h_k|_{E'} \to \operatorname{id}|_{E'}$ and it leads to a contradiction. Thus $b \in J_g$ and it must be $\gamma_g(\theta)$. Now we obtain $a = b$. ∎

To complete the proof of Theorem 5.1, let us finish the proofs of the propositions.

Proof of Proposition 5.6. We modify the argument of Proposition 3.3. Suppose that there exist $f_k \to g$ which determine equivalent degeneration pairs $(f_k \to g)$ and θ_k with $n_k = \operatorname{depth}(\theta_k) \nearrow \infty$ such that $\operatorname{diam} \Pi_{f_k}(\theta_k, +) \geq \epsilon_0 > 0$ for all k. Then we can take a branch F_k of $f_k^{-n_k}$ such that F_k maps $\Pi_{f_k}(\theta_0^+, +)^\circ$ onto $\Pi_{f_k}(\theta_k, +)^\circ$ univalently.

Take a small ball $B \Subset T_g(\theta_0^+, 0, +)$ and fix a point $z \in B$. By (ii), we may assume that $B \Subset T_{f_k}(\theta_0^+, 0, +)$ for all $k \gg 0$. Since $F_k|_B$ avoid values near ∞, they form a normal family. By passing to a subsequence, we may also assume that there exists $\phi = \lim F_k|_B$ that is non-constant by assumption. Now we have a small open set $V \Subset \phi(B)$ with $V \subset F_k(B)$ for all $k \gg 0$, thus $f_k^{n_k}(V) \subset B \subset K_{f_k}^\circ$. This implies that $V \subset K_{f_k}^\circ$ for all $k \gg 0$ hence by Corollary A.2 we have $V \subset K_g^\circ$ too. Since V is open, there exist a tile $T = T_g(\theta, m, +)$ and a small ball B' such that $B' \Subset (T \cap V)^\circ$. By $B' \subset T$ and (ii) again, we have $B' \subset T_k := T_{f_k}(\theta, m, +)$ for all $k \gg 0$. Moreover, since $B' \subset V$ we have $f_k^{n_k}(B') \subset B \subset T_{f_k}(\theta_0^+, 0, +)$. Thus $f_k^{n_k}(T_k)$ must be $T_{f_k}(\theta_0^+, 0, +)$ but $f_k^{n_k}(T_k)$ has level $m + n_k \to \infty$. It is a contradiction.

Finally one can finish the proof by the same argument as Proposition 3.3. ∎

Proof of Proposition 5.7. It is enough to consider the case of $\theta = \theta_0^+$ and $* = +$. Recall that the attracting cycle O_f has the multiplier $re^{2\pi i p/q}$. We introduce a parameter $\epsilon \in [0, 1)$ of $f \to g$ such that $r^q = R = 1 - \epsilon$. Set $\Pi_\epsilon := \Pi_f(\theta_0^+, +)$ and $\Pi_0 := \Pi_g(\theta_0^+, +)$. Then the semiconjugacy $h = h_\epsilon$ sends Π_ϵ to Π_0. To show the statement it is enough to show the following: For any $\delta > 0$, we have $\Pi_0 \subset N_\delta(\Pi_\epsilon)$ and $\Pi_\epsilon \subset N_\delta(\Pi_0)$ for all $\epsilon \ll 1$, where $N_\delta(\cdot)$ denotes the δ-neighborhood.

It is easy to check $\Pi_0 \subset N_\delta(\Pi_\epsilon)$: We can take a compact set K such that $K \subset \Pi_0^\circ \Subset N_\delta(K)$. Since $h_\epsilon \to \operatorname{id}$ on K, we have $K \subset \Pi_\epsilon^\circ$ for all $\epsilon \ll 1$. Thus we have $\Pi_0 \subset N_\delta(K) \subset N_\delta(\Pi_\epsilon)$.

The proof of $\Pi_\epsilon \subset N_\delta(\Pi_0)$ is more technical. Here let us assume that $q = q'$, Case (a). Case (b) ($q = 1 < q'$) is merely analogous and left to the reader.

Local coordinates. Set $B := B(\beta_0, \delta)$. For fixed δ that is small enough, there exists a convergent family of local coordinates $\zeta = \psi_\epsilon(z) \to \psi_0(z)$ on B with the following

properties for all $0 \leq \epsilon \ll 1$:

- There exists $\delta' > 0$ such that $\Delta := B(0, \delta') \Subset \psi_\epsilon(B)$.

- Set $f_\epsilon := f^{lq}$, $f_0 := g^{lq}$, and $R_\epsilon = 1 - \epsilon$. Then $f_\epsilon(\zeta) = R_\epsilon \zeta \left(1 + \zeta^q + O(\zeta^{2q})\right)$ on Δ. (See Appendix A.2.)

- ψ_ϵ maps $\Pi_\epsilon \cap \psi_\epsilon^{-1}(\Delta)$ into $\Delta' := \{\zeta \in \Delta : -\pi/2q < \arg\zeta < 3\pi/2q\}$. (This is just a technical assumption.)

- Set $E_\epsilon := \{\zeta \in \Delta' : |\arg\zeta^q| \leq \pi/3, |\zeta^q| \geq \epsilon/2\}$. Then $f_0^{-1}(\overline{E_0}) \subset E_0 \cup \{0\}$ and $f_\epsilon^{-1}(\overline{E_\epsilon}) \subset E_\epsilon$ for all $0 < \epsilon \ll 1$. (See the argument of Lemma A.7).

Let us interpret the setting of Theorem 5.5 by using $\epsilon \in [0, 1)$. If $0 < \epsilon < 1$, we denote Φ_f, Ψ_f, u_f, T_f, and U_f by Φ_ϵ, Ψ_ϵ, u_ϵ, T_ϵ, and U_ϵ respectively. If $\epsilon = 0$ they denote Φ_g, Ψ_g, etc. In particular, we consider Ψ_ϵ only on Δ'. For later usage, we define $W = \chi_\epsilon(\zeta)$ for each $\zeta \in \psi_\epsilon(K_{f_\epsilon}^\circ \cap B)$ by $\chi_\epsilon := \Phi_\epsilon \circ \psi_\epsilon^{-1}$.

Now we can see f_ϵ on Δ' through $w = \Psi_\epsilon(\zeta)$ as $f_\epsilon(w) = \tau_\epsilon w + 1 + O(1/w)$ where $\tau_\epsilon := R_\epsilon^{-q}$. On this w-plane, take $P = P_\rho = \{\operatorname{Re} w \geq \rho \gg 0\}$ such that for all $0 \leq \epsilon \ll 1$, the set $\hat{P} := \Psi_\epsilon^{-1}(P)$ is contained in Δ' and that u_ϵ is defined on P. Note that for all $0 \leq \epsilon \ll 1$ we have $f_\epsilon(P) \subset P$ and $u_0(w) = w(1 + o(1))$ by Lemma A.6. One can also check that $\chi_\epsilon \circ \Psi_\epsilon^{-1}(w) = U_\epsilon \circ T_\epsilon \circ u_\epsilon(w)$ on P and

$$U_\epsilon \circ T_\epsilon \circ u_\epsilon(w) = U_0 \circ T_0 \circ u_0(w) + o(1) = qw(1 + o(1))$$

on compact sets of P.

Rectangles. For fixed positive integers M and N, we define the following compact sets in the W-plane:

$$C_0 := \{W \in \mathbb{C} : (N-1)q \leq \operatorname{Re} W \leq Nq, \ 0 \leq \operatorname{Im} W \leq Nq\}$$

$$Q_0 := \bigcup_{k=0}^{M} G^{-kq}(C_0) \quad \text{and} \quad C_0' := G^{-Mq}(C_0)$$

where $G(W) = W + 1$.

By taking sufficiently large N and M, we may assume the following:

(1) Set $\tilde{Q}_0 := \Pi_0 \cap \Phi_0^{-1}(Q_0)$ in the z-coordinate. Then $\Pi_0 - \tilde{Q}_0 \Subset \psi_0^{-1}(\Delta)$.

(2) In the w-coordinate, we have $\chi_0^{-1}(C_0) \subset \hat{P}$ and $\chi_0^{-1}(C_0') \subset E_0$.

See Figure 12. In fact, for any compact set K with $\Pi_0 - K \Subset \psi_0^{-1}(\Delta)$, the set $\Phi_0(K)$ is compact in $\overline{\overline{\mathbb{H}}}_W := \{\operatorname{Im} W \geq 0\}$ and covered by Q_0 if we take sufficiently large N and M. Thus we have (1). If $N \gg 0$, the set C_0 must be contained in $\chi_0(P)$. Since $\Pi_0 \cap \Phi_0^{-1}(C_0)$ is compact, it is uniformly attracted to the repelling direction by iteration of $(g|_{\Pi_0})^{-lq}$. Thus we have (2) by taking M much larger.

Perturbation. We fix such integers N and M. Now we consider perturbation of fixed rectangles C_0, C_0', and Q_0 with properties (1) and (2). By using the conjugacy $H = H_\epsilon : \mathbb{C} - [a, \infty) \to \mathbb{C}$ between $F = F_\epsilon$ and $G = F_0$, we define C_ϵ, C_ϵ' and Q_ϵ by their homeomorphic images by H_ϵ^{-1}. Since $H_\epsilon \to \mathrm{id}$ as $\epsilon \to 0$ on any compact sets (see the proof of (ii)), we have $C_\epsilon \to C_0$ etc. in the Hausdorff topology. Moreover, we have the following properties for all $\epsilon \ll 1$:
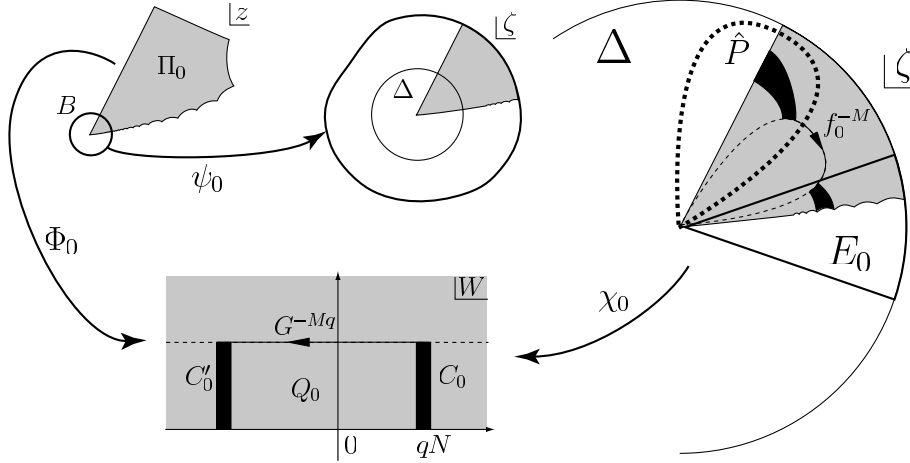
Figure 12: Taking M and N.

(1') Set $\tilde{Q}_\epsilon := \Pi_\epsilon \cap \Phi_\epsilon^{-1}(Q_\epsilon)$ in the z-coordinate. Then $\tilde{Q}_\epsilon \subset N_{\delta/2}(\tilde{Q}_0)$.

(2') In the ζ-coordinate, we have $\chi_\epsilon^{-1}(C_\epsilon) \subset \hat{P}$ and $\chi_\epsilon^{-1}(C_\epsilon') \subset E_\epsilon$.

In fact, since $\tilde{Q}_0 = h_\epsilon(\tilde{Q}_\epsilon)$ and is compact, property (1') follows by $\Phi_\epsilon \to \Phi_0$ as $\epsilon \to 0$. Property (2') holds because $\chi_\epsilon \to \chi_0$ on compact sets in $\hat{P}$ and $f^{lqM} \to g^{lqM}$.

Now it is enough to show $\Pi_\epsilon - \tilde{Q}_\epsilon \Subset \psi_\epsilon^{-1}(\Delta) \subset B$, which is equivalent to $\chi_\epsilon^{-1}(\overline{\mathbb{H}}_W - Q_\epsilon) \Subset \Delta$ in the ζ-coordinate. We consider the following three sets in $\overline{\mathbb{H}}_W$:

$$
\begin{aligned}
X_0 &:= \left\{ W \in \overline{\mathbb{H}}_W : \operatorname{Re} W \leq (N - M - 1)q, \ \operatorname{Im} W \leq Nq \right\} \\
Y_0 &:= \left\{ W \in \overline{\mathbb{H}}_W : \operatorname{Re} W \geq Nq, \ \operatorname{Im} W \leq Nq \right\} \\
Z_0 &:= \left\{ W \in \overline{\mathbb{H}}_W : \operatorname{Im} W \geq Nq \right\}
\end{aligned}
$$

Let X_ϵ, Y_ϵ, and Z_ϵ be their homeomorphic images by H_ϵ^{-1}. Then $X_\epsilon \cup Y_\epsilon \cup Z_\epsilon = \overline{\mathbb{H}}_W - Q_\epsilon^\circ$.

Note that $X_\epsilon = \bigcup_{k \geq 1} F_\epsilon^{-kq}(C_\epsilon')$ and $Y_\epsilon = \bigcup_{k \geq 1} F_\epsilon^{kq}(C_\epsilon)$. Since $f_\epsilon^{-1}(E_\epsilon) \subset E_\epsilon$ and $f_\epsilon(\hat{P}) \subset \hat{P}$ in the ζ-coordinate, (2') implies $\chi_\epsilon^{-1}(X_\epsilon) \subset E_\epsilon$ and $\chi_\epsilon^{-1}(Y_\epsilon) \subset \hat{P}$ thus we have $\chi_\epsilon^{-1}(X_\epsilon \cup Y_\epsilon) \subset \Delta$.

The proof is completed by showing $\chi_\epsilon^{-1}(Z_\epsilon) \subset \Delta$. It is enough to show that $\chi_\epsilon^{-1}(\partial Z_\epsilon) \subset \Delta$. Note that ∂Z_ϵ consists of two half lines, one is the interval $I_\epsilon := [a_\epsilon, \infty)$ where a_ϵ is the attracting fixed point of F_ϵ, and the other is $I_\epsilon' := H_\epsilon^{-1}(\partial Z_0)$, the one along the top edge of Q_ϵ.

First we show that $\chi_\epsilon^{-1}(I_\epsilon) \subset \Delta$. Recall that this is the image of a degenerating arc in the ζ-coordinate. Let $E_0' := \{\zeta \in \Delta' : |\arg(-\zeta^q)| \leq \pi/3\}$. Then one can check that $f_\epsilon(E_0') \subset E_0'$ and $f_\epsilon^{-1}(E_0) \subset E_0$ for all $\epsilon \ll 1$ as in the argument of Lemma A.7.

The real part of $g^{lqk}(0)$ in the w-coordinate increases as $k \to \infty$ thus the critical orbit of $f_0 = g^{lq}$ in Δ' is tangent to the attracting direction in the ζ-coordinate. Thus we may assume that $g^{lqn}(0)$ in the proof of Theorem 5.5 is contained in E_0'. Hence $f^{lqn}(0) = f_\epsilon^n(0)$ in the ζ-coordinate is contained in E_0' for all $\epsilon \ll 1$. Moreover, the property $f_\epsilon(E_0') \subset E_0'$ implies that the critical orbit of $f_\epsilon = f^{lq}$ in Δ' is eventually contained in E_0'. By construction of the degenerating arcs in Lemma 2.3 and by $f_\epsilon^{-1}(E_0) \subset E_0$, the arc $\chi_\epsilon^{-1}(I_\epsilon)$ must be contained in $E_0 \subset \Delta$.

Next we show that $\chi_\epsilon^{-1}(I_\epsilon') \subset \Delta$. Let s_ϵ and ℓ_ϵ be the top edges of quadrilaterals C_ϵ and Q_ϵ intersecting I_ϵ'. Then $\ell_\epsilon = \bigcup_{k \geq 0}^{M} F_\epsilon^{-kq}(s_\epsilon)$. Now it is enough to show that $\chi_\epsilon^{-1}(\ell_\epsilon)$ is contained in Δ since $\chi_\epsilon^{-1}(X_\epsilon \cup Y_\epsilon) \subset \Delta$.

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Take any point w_0 in $\Psi_\epsilon \circ \chi_\epsilon^{-1}(s_\epsilon) = (U_\epsilon \circ T_\epsilon \circ u_\epsilon)^{-1}(s_\epsilon)$ in the w-plane. We may assume that N is sufficiently large and $w_0 \in B(N + Ni, N/4)$ for all $\epsilon \ll 1$, since $U_\epsilon \circ T_\epsilon \circ u_\epsilon(w) = qw(1 + o(1))$ on compact sets of P. Moreover, we may assume that $\Psi_\epsilon(\partial \Delta) \subset B(0, N/4)$.

Recall that $f_\epsilon(w) = \tau_\epsilon w + 1 + O(1/w)$ and thus $f_\epsilon^{-1}(w) = \tau_\epsilon^{-1}(w - 1) + O(1/w)$. Take any w with $N/4 \leq |w| \leq 4N$. Then we have $|f_\epsilon^{-1}(w) - (w - 1)| = O(\epsilon N) + O(1/N)$. Thus for any fixed $\kappa \ll 1$, by taking $N \gg 0$ we have $|f_\epsilon^{-1}(w) - (w - 1)| \leq \kappa$ for all $\epsilon \ll 1$. This implies $|\arg(f_\epsilon^{-1}(w) - w)| \leq \arcsin \kappa$.
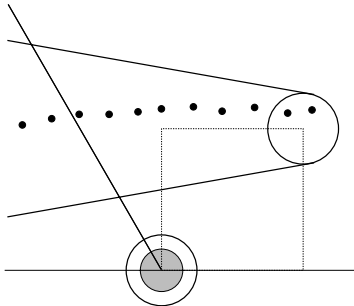

Figure 13: An orbit in the w-plane. The dotted square has height N.

By (2'), the orbit $w_k = f_\epsilon^{-k}(w_0)$ of w_0 lands on $\Psi_\epsilon(E_\epsilon)$ by at most M iteration of f_ϵ^{-1} (thus $\arg w_M > 2\pi/3$). For small enough κ, the point w_k satisfies $N/4 \leq |w_k| \leq 4N$ and $|\arg(w_k - w_0)| \leq \arcsin \kappa$ for all $k = 0, \ldots, M$. (See Figure 13.) This implies that $\Psi_\epsilon \circ \chi_\epsilon^{-1}(\ell_\epsilon)$ never crosses over $\Psi_\epsilon(\partial \Delta)$ thus we have $\chi_\epsilon^{-1}(\ell_\epsilon) \subset \Delta$. ∎

A Appendix

In this section we give some investigation on perturbation of parabolic cycle corresponding to the degeneration pair $(f \to g)$.

A.1 Pinching semiconjugacy on the Julia sets

Let $(f \to g)$ be a general degeneration pair. Recall that the attracting cycle $O_f = \{\alpha_1, \ldots, \alpha_l = a_0\}$ has multiplier $\lambda = r\omega = r \exp(2\pi i p/q)$ with $0 < r < 1$; and that the parabolic cycle $O_g = \{\beta_1, \ldots, \beta_{l'} = \beta_0\}$ has multiplier $\omega' = \exp(2\pi p'/q')$.

By applying Theorem 1.1 of [Ka1] to $(f \to g)$, we have:

Theorem A.1 *If $f \approx g$, there exists a unique semiconjugacy $h_J : J_f \to J_g$ with the following properties:*

1. *If $\operatorname{card} h_J^{-1}(w) \geq 2$ for some $w \in J_g$ then $w \in I_g$ and $\operatorname{card} h_J^{-1}(w) = q$ (thus $q = q' \geq 2$).*

2. *h_J is a homeomorphism iff $(f \to g)$ is of type $q = 1$.*

3. *$\sup_{z \in J_f} |z - h_J(z)| \to 0$ as $f \to g$.*

(See also Proposition 2.1.) The proof of Theorem 1.1 of [Ka1] is based on a pull-back argument and it does not use quasiconformal maps. Here is a useful corollary which easily follows from property 3:

Corollary A.2 *As $f \to g$, the Julia set J_f converges to J_g in the Hausdorff topology.*

Since $h_J \circ \gamma_f$ and γ_g determines the same ray equivalence, we have $h_J \circ \gamma_f = \gamma_g$. For $\theta \in \mathbb{T}$, put $\gamma_f(\theta)$ into z in property 3 of the theorem above. Then we have:

Corollary A.3 *As $f \to g$, the map $\gamma_f : \mathbb{T} \to J_f$ converges uniformly to $\gamma_g : \mathbb{T} \to J_g$.*

A.2 Normalized form of local perturbation

For a degeneration pair $(f \to g)$, the parabolic cycle O_g is approximated by an attracting or repelling cycle O'_f with the same period l' and multiplier $\lambda' \approx \omega' = e^{2\pi i p'/q'}$. (See Section 2.) Let $\alpha'_0 \in O'_f$ with $\alpha'_0 \to \beta_0$. Then by looking through the local coordinates $\psi_f(z) = z - \alpha'_0$ and $\psi_g(z) = z - \beta_0$ near β_0 one can view the convergence $f^{l'} \to g^{l'}$ as follows:

$$\psi_f \circ f^{l'} \circ \psi_f^{-1}(z) = \lambda' z + O(z^2)$$
$$\longrightarrow \ \psi_g \circ g^{l'} \circ \psi_g^{-1}(z) = \omega' z + O(z^2).$$

Here we claim that by replacing $\psi_f \to \psi_g$ with better local coordinates, we have a normalized form of convergence:

$$\psi_f \circ f^{l'} \circ \psi_f^{-1}(z) = \lambda' z + z^{q'+1} + O(z^{2q'+1})$$
$$\longrightarrow \ \psi_g \circ g^{l'} \circ \psi_g^{-1}(z) = \omega' z + z^{q'+1} + O(z^{2q'+1}).$$

More generally, we have:

Proposition A.4 *For $\epsilon \in [0,1]$, let $\{f_\epsilon\}$ be a family of holomorphic maps on a neighborhood of 0 such that as $\epsilon \to 0$,*

$$f_\epsilon(z) = \lambda_\epsilon z + O(z^2) \ \longrightarrow \ f_0(z) = \lambda_0 z + O(z^2)$$

where λ_0 is a primitive qth root of unity. Then we have a family of holomorphic maps $\{\phi_\epsilon\}$ such that

$$\phi_\epsilon \circ f_\epsilon \circ \phi_\epsilon^{-1}(z) = \lambda_\epsilon z + z^{q+1} + O(z^{2q+1}).$$

and $\phi_\epsilon \to \phi_0$ near $z = 0$.

Proof. First suppose that $f_\epsilon(z) = \lambda_\epsilon z + A_\epsilon z^n + O(z^{n+1})$ where $2 \le n \le q$. Let us consider a coordinate change by $z \mapsto z - B_\epsilon z^n$ with $B_\epsilon = A_\epsilon/(\lambda_\epsilon^{n+1} - \lambda_\epsilon)$. Note that $\lambda_\epsilon^{n+1} - \lambda_\epsilon$ is bounded distance away from 0 when $\epsilon \ll 1$, because λ_ϵ converges to a primitive qth root of unity. In particular, the coordinate change $z \mapsto z - B_\epsilon z^n$ also converges to $z \mapsto z - B_0 z^n$ near 0. By applying these coordinate changes, we can view the family $\{f_\epsilon\}$ as

$$f_\epsilon(z) = \lambda_\epsilon z + O(z^{n+1}).$$

By repeating this process until $n = q$, we have the family $\{f_\epsilon\}$ of the form

$$f_\epsilon(z) = \lambda_\epsilon z + C_\epsilon z^{q+1} + A'_\epsilon z^n + O(z^{n+1})$$

where $q + 2 \le n \le 2q$. Next for each ϵ take a linear coordinate change by $z \mapsto C_\epsilon^{1/q} z$ to normalize C_ϵ to be 1. By taking another coordinate change of the form $z \mapsto \zeta = z - B'_\epsilon z^n$ with $B'_\epsilon = A'_\epsilon/(\lambda_\epsilon^{n+1} - \lambda_\epsilon)$ again, we have

$$f_\epsilon(z) = \lambda_\epsilon z + z^{q+1} + O(z^{n+1}).$$

We can repeat this process until $n = 2q$ and we have the desired form of convergence. $\blacksquare$

For this new family $\{f_\epsilon(z) = \lambda_\epsilon z + z^{q+1} + O(z^{2q+1})\}$ and $n \geq 0$, one can easily check that

$$f_\epsilon^n(z) \;=\; \lambda_\epsilon^n z + C_{\epsilon,n} z^{q+1} + O(z^{2q+1})$$

where $C_{\epsilon,n}$ is given by the recursive formula $C_{\epsilon,n+1} = \lambda_\epsilon^{q+1} C_{\epsilon,n} + \lambda_\epsilon^n$. Let $n = q$ and set $\Lambda_\epsilon := \lambda_\epsilon^q$ ($\to 1$ as $\epsilon \to 0$). By taking linear coordinate changes with $z \mapsto (C_{\epsilon,q}/\Lambda_\epsilon)^{1/q} z$, we have the convergence of the form

$$
\begin{aligned}
f_\epsilon^q(z) &\;=\; \Lambda_\epsilon z \left(1 + z^q + O(z^{2q})\right) \\
\longrightarrow \quad f_0^q(z) &\;=\; z \left(1 + z^q + O(z^{2q})\right).
\end{aligned}
$$

By further coordinate changes with $w = \Psi_\epsilon(z) = -\Lambda_\epsilon^q/(q z^q)$, we have

$$
\begin{aligned}
\Psi_\epsilon \circ f_\epsilon^q \circ \Psi_\epsilon^{-1}(w) &\;=\; \Lambda_\epsilon^{-q} w + 1 + O(1/w) \\
\longrightarrow \quad \Psi_0 \circ f_0^q \circ \Psi_0^{-1}(w) &\;=\; w + 1 + O(1/w)
\end{aligned}
$$

on a neighborhood of infinity. Note that we have a similar representation for $f^{l'q'} \to g^{l'q'}$.

A.3   Simultaneous linearization

Recently T.Ueda [Ue] showed the simultaneous linearization theorem that explains hyperbolic-to-parabolic degenerations of linearizing coordinates. Here we give a simple version of the theorem which is enough for our investigation. For $R \geq 0$, let E_R denote the region $\{z \in \mathbb{C} : \operatorname{Re} z \geq R\}$.

Theorem A.5 (Ueda) *For $\epsilon \in [0,1]$, let $\{f_\epsilon\}$ be a family of holomorphic maps on $\{|z| \geq R > 0\}$ such that*

$$
\begin{aligned}
f_\epsilon(z) &\;=\; \tau_\epsilon z + 1 + O(1/z) \\
\longrightarrow f_0(z) &\;=\; z + 1 + O(1/z)
\end{aligned}
$$

uniformly as $\epsilon \to 0$ where $\tau_\epsilon = 1 + \epsilon$. If $R \gg 0$, then for any $\epsilon \in [0,1]$ there exists a holomorphic map $u_\epsilon : E_R \to \bar{\mathbb{C}}$ such that

$$u_\epsilon(f_\epsilon(z)) \;=\; \tau_\epsilon u_\epsilon(z) + 1$$

and $u_\epsilon \to u_0$ uniformly on compact sets of E_R.

Indeed, Ueda's original theorem in [Ue] claims that a similar holds for any radial convergence $\tau_\epsilon \to 1$ outside the unit disk. In [Ka4] an alternative proof is given and the error term $O(1/z)$ is refined to be $O(z^{-1/n})$ for any $n \geq 1$.

Lemma A.6 $u_0(z) = z(1 + o(1))$ *as* $\operatorname{Re} z \to \infty$.

Indeed, it is well-known that if $f_0(z) = z + 1 + a_0/z + \cdots$ then the Fatou coordinate is of the form $u_0(z) = z - a_0 \log z + O(1)$. See [Sh] for example.

A.4 Small invariant paths joining perturbed periodic points

For a degeneration pair $(f \to g)$ in Case (a) $(q = q')$, we may consider that the parabolic cycle O_g is perturbed into the attracting cycle O_f with the same period $l = l'$. (See Proposition 2.1). In this case, the convergence $f^{lq} \to g^{lq}$ is viewed as

$$f^{lq}(z) = r^q z + z^q + O(z^{2q}) \longrightarrow g^{lq}(z) = z + z^q + O(z^{2q})$$

with $r^q \nearrow 1$ through suitable local coordinates near $\beta_0 \in O_g$ as in Appendix A.2.

By taking an additional linear coordinate change by $z \mapsto z/r$, we consider a family of holomorphic maps $\{f_\epsilon\}$ of the form

$$f_\epsilon(z) = \lambda_\epsilon z (1 + z^q + O(z^{2q}))$$

instead, where we set $r^q = \lambda_\epsilon = 1 - \epsilon \nearrow 1$. Then the local solution of $f_\epsilon(z) = z$ is $z = 0$ or $z^q = \epsilon + O(\epsilon^2)$. The latter means q symmetrically arrayed repelling fixed points are generated by the perturbation of a parabolic point with multiplicity $q + 1$. Here we claim:

Lemma A.7 *For $\epsilon \ll 1$, there exist q f_ϵ-invariant paths of diameter $O(\epsilon^{1/q})$ joining the central attracting point $z = 0$ and each of symmetrically arrayed repelling fixed points.*

Proof. First we show that $D := \{z : |z|^q \le \epsilon/2\}$ satisfies $f_\epsilon(D) \subset D^\circ$. By checking the real part of $\log f_\epsilon(z)$, we have

$$|f_\epsilon(z)| \;=\; \lambda_\epsilon |z| (1 + \operatorname{Re} z^q + O(z^{2q})).$$

Since $\operatorname{Re} z^q \le \epsilon/2$ on D, we have $|f_\epsilon(z)| = |z|(1 - \epsilon/2 + O(\epsilon^2)) < |z|$.

Next we set

$$E \;:=\; \left\{ z : \frac{\epsilon}{2} \le |z^q| \le 4\epsilon \ \text{ and } \ |\arg z^q| \le \frac{\pi}{3} \right\}.$$

Note that E has q connected components around the repelling fixed points. Now we claim that E satisfies $f_\epsilon^{-1}(E) \subset E^\circ$. Since f_ϵ^{-1} is univalent near 0, it is enough to show that $f_\epsilon^{-1}(\partial E) \subset E^\circ$. Set

$$e_1 \;:=\; \left\{ z : |z^q| = \frac{\epsilon}{2} \ \text{ and } \ |\arg z^q| \le \frac{\pi}{3} \right\}$$

$$e_2 \;:=\; \left\{ z : |z^q| = 4\epsilon \ \text{ and } \ |\arg z^q| \le \frac{\pi}{3} \right\}$$

$$e_3^\pm \;:=\; \left\{ z : \frac{\epsilon}{2} \le |z^q| \le 4\epsilon \ \text{ and } \ \arg z^q = \pm\frac{\pi}{3} \right\}.$$

By checking $\log f_\epsilon^{-1}(z)$, we have

$$|f_\epsilon^{-1}(z)| \;=\; \lambda_\epsilon^{-1} |z| (1 - \lambda_\epsilon^{-q}\operatorname{Re} z^q + O(z^{2q}))$$
$$\arg f_\epsilon^{-1}(z) \;=\; \arg z - \lambda_\epsilon^{-q}\operatorname{Im} z^q + O(z^{2q}).$$

If $z \in e_1$, we have $\operatorname{Re} z^q \le \epsilon/2$ thus $|f_\epsilon(z)| \ge |z|(1 + \epsilon/2 + O(\epsilon^2)) > |z|$. If $z \in e_2$, we have $\operatorname{Re} z^q \ge 2\epsilon$ thus $|f_\epsilon(z)| \le |z|(1 - \epsilon + O(\epsilon^2)) < |z|$. For $z \in e_3^\pm$, set $|z^q| = \rho$ with $\epsilon/2 \le \rho \le 4\epsilon$. Then $\arg f_\epsilon^{-1}(z) = \arg z \mp (\sqrt{3}/2)\rho(1 + O(\rho))$. Thus we have $f_\epsilon^{-1}(\partial E) \subset E^\circ$ in total.

Take any q points $\{z_1, \cdots, z_q\}$ from each connected component of e_1. Let η_j be the segment joining z_j and $f_\epsilon(z_j)$. Then the path $\bigcup_{k \in \mathbb{Z}} f_\epsilon^k(\eta_j)$ has the desired property. ∎

Remark. In Case (b) ($q = 1 < q'$), the cycle O'_f in Appendix A.2 is repelling. By taking $f^{-l'q'} \to g^{-l'q'}$ near O_g, we have a similar form of the convergence

$$f_\epsilon(z) = \lambda_\epsilon z (1 + z^{q'} + O(z^{2q'}))$$

to the case of $q = q'$, where $\lambda_\epsilon = 1 - \epsilon + O(\epsilon^2) \in \mathbb{C}^*$. This λ_ϵ comes from the fact that the non-zero solutions of $f_\epsilon(z) = z$ has derivative $0 < r < 1$ (Since they are actually points in O_f in a different coordinate.) One can easily check that the argument of Lemma A.7 above also works for this f_ϵ and the statement is also true by replacing q with q'.

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