A Participation Game in a Mechanism Implementing a Proportional Cost-sharing Rule *

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Abstract In this study, a participation game in a mechanism to produce a pure public good is examined; in this game, agents decide simultaneously whether they will participate in the mechanism or not, and only the agents that selected participation decide the level of public good provision and distribute the cost of the public good in proportion to the benefits that participants receive from the public good. We focus on the economy in which the public good can be provided in this economy, then the participation game has a Nash equilibrium that supports a Pareto efficient allocation and some of such Nash equilibria are strong equilibria (Aumann, 1959). However, the results are not obtained in the economy in which the public good is provided in multiple units. We second show that, in the economy where at most two units of the public good are provided, if agents are identical and some conditions are satisfied, then no Nashe quilibrium supports Pareto efficient allocations.

Keywords: Participation game, Proportional cost-sharing allocation rule, Public project, Multi-unit public good, Strong equilibrium

JEL Classification Numbers: C72, D62, D71, H41.

1 Introduction

This paper studies a participation problem in a mechanism to produce a pure public good. From the theory of implementation, the construction of a mechanism can solve the "free-rider" problem in economies with public goods. For example, Bagnoli and Lipman (1989), Jackson and Moulin (1992), and Bag (1997) constructed mechanisms to implement desirable allocation rules in the economy with a discrete public good.

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However, implementation theory supposes the participation of all agents, and each agent does not have the right to decide whether he participates in the mechanism or not. Palfrey and Rosenthal (1984), Saijo and Yamato (1999), and Dixit and Olson (2000) pointed out the importance of strategic behavior of agents as they can decide whether or not to participate in the mechanisms. In the real world, as for example the participation problems in international environmental treaties, agents often have the right to make such decisions, and they may have an incentive not to enter the mechanism, hoping that other agents will participate in the mechanism and provide a public good. This will generate another kind of a free-rider problem.

These authors formulated a participation game in a public good mechanism. In the game, each agent simultaneously chooses either participation or non-participation in the mechanism. If an agent chooses participation, he pays the expense requested by the mechanism and the public good is produced. If an agent selects non-participation, he can enjoy the public good at no costs. Palfrey and Rosenthal (1984) and Dixit and Olson (2000) analyzed the participation game in the case in which the public good is discrete and at most one unit of the public good is produced. They showed that there exist Nash equilibria that support efficient allocations in the game. Saijo and Yamato (1999) examined the participation game in the case of a perfectly divisible public good. They considered a mechanism that implements the Lindahl allocation rule. They showed that not every agent enters the mechanism at Nash equilibria and proved that efficient allocations of the economy are not achieved at the equilibrium of the game in many cases. Hence, it depends on the form of the provision of a public good whether the equilibrium of the participation game achieves the efficient allocations or not.

In this paper, we examine the participation game that is similar to that of the earlier literature. We consider an economy in which the public good is discrete and there is a mechanism that implements a proportional cost-sharing rule: The public good is produced in a way that maximizes the total surplus of participants, and the cost of producing the public good is distributed among participants in proportion to the benefits that participants receive from the public good. First, we examine the participation game where only one unit of the public good can be provided, which is similar to Palfrey and Rosenthal (1984) and Dixit and Olson (2000). (We, hereafter, call this game a *participation game with a public project*.) In this case, there is a Nash equilibrium at which a Pareto efficient allocation is achieved and some of such Nash equilibria are strong equilibria introduced by Aumann (1959). Second, we extend our analysis to the case of a multi-unit public good. In particular, we focus on the participation game in which the public good is discrete and at most two units of the public good. In this case, a Nash equilibrium of the game can not necessarily support the efficient allocation. We show that no Nash equilibrium supports a Pareto efficient allocation if agents are identical and some conditions hold. We can conclude from these results that the assumption that only one unit of the public good can be produced is essential to the general existence of a Nash equilibrium that supports an efficient allocation and that of a strong equilibrium. Moreover, we obtain the implication that is similar to those of Saijo and Yamato (1999) even in the participation game in which the public good is discrete and at most two units of the public good can be produced.

Before the model is introduced, let us discuss the relationship between our work

and other work. First, we allow the case in which agents' preferences are heterogeneous. Earlier literature on the participation game has focused on the case of identical agents. Second, we consider the possibility that agents form a coalition and coordinate the participation decisions. We analyze the effect of such coalitional behavior on the participation decision. Palfrey and Rosenthal (1984), Saijo and Yamato (1999), and Dixit and Olson (2000) have focused solely on Nash equilibria, disregarding the effects. In this paper, analyses are presented of strong equilibria and Nash equilibria. A strong equilibrium is a strategy profile that is stable against all possible coalitional deviations. This is a very demanding equilibrium concept, and many games that are of interest to economists do not have a strong equilibrium. In this paper, we identify a sufficient condition for the existence of a strong equilibrium in games of the provision of pure public goods.

2 A participation game with a public project

We consider the problem of undertaking a (pure) public project and distributing its cost. Let n be the number of agents. We denote the set of agents by $N = \{1, \ldots, n\}$. Let $y \in \{0, 1\}$ be the public project: y = 1 if the project is undertaken, and y = 0if not. Let $\theta_i > 0$ denote agent *i*'s *willingness to pay* for the project or *benefit* from the project. Let $x_i \ge 0$ denote a transfer from agent *i*. Each agent *i* has a preference relation which is represented by a quasi-linear utility function $V_i(y, x_i) = \theta_i y - x_i$. The cost of the project is c > 0.

In this paper, we assume that there exists a mechanism that implements the proportional cost-sharing rule. We consider a two-stage game. In the first stage, each agent simultaneously decides whether she participates in the mechanism or not. In the second stage, following the rule of the mechanism, only the agents that selected participation in the first stage decide the implementation of the project and the distribution of its cost. As a result, the *proportional cost-sharing allocation* only for participants' preferences is achieved.

First, we formally define the outcome of the second stage. Let P be a set of participants and let $(y^P, (x_j^P)_{j \in N})$ be the outcome of the second stage when P is the set of participants. We denote $\theta_P = \sum_{j \in P} \theta_j$ for all sets of participants P: θ_P is the sum that agents in P are willing to pay for the public project. For all subsets P of N, #P means the cardinality of the set P.

Assumption 1 Let P denote a set of participants. The allocation to participants $(y^P, (x_j^P)_{j \in P})$ that satisfies the following conditions is attained in the equilibrium of the mechanism:

if
$$\theta_P > c$$
, then $x_i^P = \frac{\theta_i}{\theta_P}c$ for all $i \in P$ and $y^P = 1$, and
if $\theta_P \leq c$, then $x_i^P = 0$ for all $i \in P$ and $y^P = 0$.

In this study, we are not concerned with an implementation problem of the proportional cost-sharing rule. However, there is such an mechanism. Jackson and Moulin(1992) constructed the mechanisms which implements a class of cost-sharing rules including the proportional cost-sharing rule in subgame perfect Nash equilibria and undominated Nash equilibria.

Assumption 2 Let $P \subseteq N$ be a set of participants. We assume $x_i^P = 0$ for all $i \notin P$, and every non-participant can also consume y^P .

Assumption 2 expresses the non-excludability of the project. In this assumption, participants bear the cost share for the project, but non-participants do not. In spite of this, non-participants can benefit from the project.

Given the outcome of the second stage, the participation decision stage can be reduced to the following simultaneous game. In the game, each agent *i* simultaneously chooses either $s_i = I$ (participation) or $s_i = O$ (non-participation), and then the set of participants is determined. Let P^s be the set of participants at an action profile $s = (s_1, \ldots, s_n)$. Then each agent *i* obtains the utility $V_i(y^{P^s}, x_i^{P^s})$ at the action profile *s*. That is, if the public project is undertaken, then participants share the cost of it in proportion to the benefits from the project. However, each non-participant can free-ride the public project. On the other hand, if it is not provided, then the payoffs of both participants and non-participants are zero. We call this reduced game *participation game* and formally define as follows.

Definition 1 (Participation game) A participation game is represented by

 $G = [N, S^n = \{I, O\}^n, (U_i)_{i \in N}]$, where U_i is the payoff function of i which associates a real number $U_i(s)$ with each strategy profile $s \in S^n$: if P^s designates the set of participants at s, then $U_i(s) = V_i(y^{P^s}, x_i^{P^s})$ for all i.

Our attention is limited to the pure strategy profiles.

We define equilibrium concepts of the participation game. The Nash equilibria of the participation game are defined as usual. First, a definition is given for a strict Nash equilibrium.

Definition 2 (Strict Nash equilibrium) A strategy profile $s^* \in S^n$ is a strict Nash equilibrium if for all $i \in N$ and for all $\hat{s}_i \in S \setminus \{s_i^*\}, U_i(s_i^*, s_{-i}^*) > U_i(\hat{s}_i, s_{-i}^*)$.

The strict Nash equilibrium is an equilibrium concept that is strongly robust to unilateral deviations. Every strict Nash equilibrium is a *trembling-perfect Nash equilibrium* in normal form games. The trembling-perfect Nash equilibrium is the notion of non-cooperative equilibria that is robust to the possibility that players make mistakes with small probability. From this viewpoint, the notion of strict Nash equilibrium can be considered as one of the plausible equilibria.

Before defining strong equilibrium, some notation is presented. For all $D \subseteq N$, denote the complement of D by -D. For all coalitions $D, s_D \in S^{\#D}$ represents a strategy profile for D. For all $s_N \in S^n$, denote s_N by s.

Definition 3 (Strong equilibrium) A strategy profile $s^* \in S^n$ is a strong equilibrium of G if there exit no coalition $T \subseteq N$ and its strategy profile $\tilde{s}_T \in S^{\#T}$ such that $\sum_{i \in T} U_i(\tilde{s}_T, s^*_{-T}) > \sum_{i \in T} U_i(s^*)$ for all $i \in T$.

A strong equilibrium is a strategy profile at which no coalition, taking the strategies of others as given, can jointly deviate in a way that increases the sum of the payoffs of its members. The strong equilibrium in Definition 3 is slightly different from that originally defined by Aumann (1959). The difference lies in the possibility of monetary transfers among agents in coalitions. Our definition allows that members of coalitions can freely send monetary transfers each other, but Aumann (1959)'s definition does not. Hence, in our model, members of a coalition can coordinate their participation decision through monetary transfers.¹ Since the definition implies that the strong equilibrium is based on the stability against all possible coalitional deviations, this equilibrium is also one of the preferable non-cooperative equilibrium concepts.

Note that all strict Nash equilibria and all strong equilibria are Nash equilibria. Note also that the set of strict Nash equilibria and that of strong equilibria are not always related by inclusion.

Example 1 Let $N = \{1, 2, 3\}$, $\theta_1 = \theta_2 = \theta_3 = 3/4$, and c = 1. The payoff matrix of this example is depicted in Table 1, where agent 1 chooses rows, agent 2 chooses columns, and agent 3 chooses matrices. The first entry in each box is agent 1's payoff, the second is agent 2's, and the third is agent 3's. There are two types of Nash equilibria. One is the Nash equilibrium with two participants and the other is the Nash equilibrium with no participants. Only the Nash equilibria with participation of two agents are strict Nash equilibria and strong equilibria.

	Ι	0			Ι	0
Ι	$\frac{5}{12}, \frac{5}{12}, \frac{5}{12}$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{4}$		Ι	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}$	0, 0, 0
O	$\frac{3}{4}, \frac{1}{4}, \frac{1}{4}$	0, 0, 0		0	0, 0, 0	0, 0, 0
Ι			0			

Table 1: Payoff matrix of Example 1

3 Nash equilibria of the participation game

In this section, we characterize the sets of participants attained at Nash equilibria. The set of feasible allocations is defined as A:

$$A = \left\{ (y, (x_j)_{j \in N}) \mid x_j \ge 0 \text{ for all } j \in N, \ y \in \{0, 1\} \text{ and } \sum_{j \in N} x_j \ge cy \right\}.$$

Assumption 3 $\theta_N > c$.

Definition 4 An allocation $(y, (x_j)_{j \in N})$ is called *Pareto efficient* if there exists no feasible allocation $(\hat{y}, (\hat{x}_j)_{j \in N})$ such that $V_i(\hat{y}, \hat{x}_i) \geq V_i(y, x_i)$ for all $i \in N$ with strict inequality for at least one $i \in N$.

¹It generally holds true that the set of strong equilibria in a game without monetary transfers contains the set of strong equilibria in the game with monetary transfers. But the converse is not always true.

We, hereafter, consider a case in which Assumption 3 holds. By Assumption 3, the public project is undertaken at every Pareto efficient allocation of the economy. In the next Lemma, we characterize the sets of participants supported as Nash equilibria.

- **Lemma 1** (1.1) Let $P \subseteq N$ be such that $\theta_P > c$. Then, P is a set of participants supported as a Nash equilibrium if and only if $\theta_P \theta_i \leq c$ for all $i \in P$.
- (1.2) Let $P \subseteq N$ be such that $\theta_P \leq c$. Then P is a set of participants supported as a Nash equilibrium if and only if $\theta_P + \theta_i \leq c$ for all $i \notin P$.

Proof. First, we show (1.1). Let P be a set of participants that satisfies $\theta_P > c$. Let us assume that P is supportable as a Nash equilibrium. Then,

$$V_i(y^P, x_i^P) \ge V_i(y^{P \setminus \{i\}}, x^{P \setminus \{i\}}) \text{ for all } i \in P, \text{ and}$$
$$V_i(y^P, x_i^P) \ge V_i(y^{P \cup \{i\}}, x_i^{P \cup \{i\}}) \text{ for all } i \notin P.$$

Since $\theta_P > c$, $V_i(y^P, x_i^P) = \theta_i - \frac{\theta_i}{\theta_P}c$ for all $i \in P$. If $\theta_P - \theta_j > c$ for some $j \in P$, then the agent j has an incentive to switch from I to O because $V_j(y^{P\setminus\{j\}}, x_j^{P\setminus\{j\}}) = \theta_j > \theta_j - \frac{\theta_j}{\theta_P}c = V_j(y^P, x_j^P)$, which is a contradiction. Therefore, we must have $\theta_P - \theta_i \leq c$ for all $i \in P$. Next, suppose that $\theta_P - \theta_i \leq c$ for all $i \in P$. Then, the conditions below are satisfied:

$$V_i(y^P, x_i^P) = \theta_i - \frac{\theta_i}{\theta_P}c > 0 = V_i(y^{P \setminus \{i\}}, x_i^{P \setminus \{i\}}) \text{ for all } i \in P, \text{ and}$$
$$V_i(y^P, x_i^P) = \theta_i > \theta_i - \frac{\theta_i}{\theta_P + \theta_i}c = V_i(y^{P \cup \{i\}}, x_i^{P \cup \{i\}}) \text{ for all } i \notin P.$$

Hence, P is supportable as a Nash equilibrium.

Secondly, we prove (1.2). Let P be such that $\theta_P \leq c$. If $\theta_P + \theta_i \leq c$ for all $i \notin P$, then $V_i(y^P, x_i^P) = V_i(y^{P \cup \{i\}}, x_i^{P \cup \{i\}}) = 0$ for all $i \in P$ and $V_i(y^P, x_i^P) = V_i(y^{P \cup \{i\}}, x_i^{P \cup \{i\}}) = 0$ for all $i \notin P$. Hence, P is attained at a Nash equilibrium. Conversely, assume that P is a set of participants that is supportable as a Nash equilibrium. Then, we must have that $V_i(y^P, x_i^P) \geq V_i(y^{P \cup \{i\}}, x_i^{P \cup \{i\}}) = 0$ for all $i \notin P$. We have $V_i(y^P, x_i^P) = 0$ for all $i \notin P$, because $\theta_P \leq c$. If there exists agent $j \notin P$ such that $\theta_P + \theta_j > c$, then we obtain $V_j(y^{P \cup \{j\}}, x_j^{P \cup \{j\}}) = \theta_j - \frac{\theta_j}{\theta_P + \theta_j}c = \frac{\theta_j}{\theta_P + \theta_j}(\theta_P + \theta_j - c) > 0$. This means that agent j has an incentive to deviate, which is a contradiction. Therefore, it follows that $\theta_P + \theta_i \leq c$ for all $i \notin P$. Q.E.D.

In the following Lemma, we show that there is a set of participants that satisfies (1.1) of Lemma 1.

Lemma 2 There exists a set of participants that satisfies (1.1) of Lemma 1 under Assumption 3 in the participation game. Therefore, there is a Nash equilibrium at which the project is carried out in the game.

Proof. Let P be a set of participants such that:

$$P \in \arg\min_{Q \subseteq N} \ \theta_Q \text{ such that } \theta_Q > c.$$
(1)

Note that there is at least one set of participants R satisfying $\theta_R > c$ by Assumption 3. Clearly, P satisfies $\theta_P > c$ and $\theta_P - \theta_i \leq c$ for all $i \in P$. Q.E.D.

Remark 1 The set of Nash equilibria in (1.1) of Lemma 1 coincides with that of strict Nash equilibria in the participation game. By Lemma 2, a strict Nash equilibrium exists in the participation game.

In the participation game, there may be a non-strict Nash equilibrium. For example, a Nash equilibrium in which no agents choose I is obviously not strict in Example 1. Note that, if non-strict Nash equilibria exist, then the project is not done in the equilibrium, and the allocation supported as the equilibrium is Pareto dominated by that attained at a strict Nash equilibrium. The following proposition shows that the set of strict Nash equilibria coincides with the set of Nash equilibria that support efficient allocations.

Proposition 1 A strategy profile is a strict Nash equilibrium if and only if it is a Nash equilibrium that supports an efficient allocation in the participation game.

Proof. First, we prove that every strict Nash equilibrium is a Nash equilibrium that supports an efficient allocation. Obviously, every strict Nash equilibrium is a Nash equilibrium. Hence, we need to show that every allocation achieved at a strict Nash equilibrium is Pareto efficient. Let $s \in S^n$ denote a strict Nash equilibrium and let P^s be the set of participants at s. Denote the allocation that is attained at s by $(y^{P^s}, (x_j^{P^s})_{j \in N})$. Note that $V_i(y^{P^s}, x_i^{P^s}) = \theta_i - \frac{\theta_i}{\theta_{P^s}}c$ for all $i \in P^s$ and $V_i(y^{P^s}, x_i^{P^s}) = \theta_i$ for all $i \notin P^s$. Suppose, on the contrary, a feasible allocation $(\hat{y}, (\hat{x}_j)_{j \in N})$ Pareto dominates $(y^{P^s}, (x_j^{P^s})_{j \in N})$. It must be satisfied that $V_i(\hat{y}, \hat{x}_i) = \theta_i$ for all $i \notin P^s$ because θ_i is the greatest payoff of agent i in A. Hence, there is at least one participants, and let $\varepsilon_j = V_j(\hat{y}, \hat{x}_j) - V_j(y^{P^s}, x_j^{P^s}) > 0$ for all $j \in J$. Since $V_j(y^{P^s}, x_j^{P^s}) = \theta_j - \frac{\theta_j}{\theta_{P^s}}c > 0$ for every $j \in J$, we must have $\hat{y} = 1$: otherwise, $V_j(\hat{y}, \hat{x}_i) = 0$. Then, we obtain that $V_j(\hat{y}, \hat{x}_i) = \theta_j - x_j^P + \varepsilon_j$ for all $j \in J$. By the argument above,

$$\widehat{x}_{j} = 0 \text{ for all } j \notin P^{s},$$

$$\widehat{x}_{j} = x_{j}^{P^{s}} - \varepsilon_{j} \text{ for all } j \in J, \text{ and}$$

$$\widehat{x}_{j} = x_{j}^{P^{s}} \text{ for all } j \in P^{s} \setminus J.$$

Summing up \widehat{x}_j for all $j \in N$ yields $\sum_{j \in N} \widehat{x}_j = \sum_{j \in P^s} x_j^{P^s} - \sum_{j \in J} \varepsilon_j = c - \sum_{j \in J} \varepsilon_j < c$, which contradicts feasibility of $(\widehat{y}, (\widehat{x}_j)_{j \in N})$. Hence, $(y^{P^s}, (x_i^{P^s})_{i \in N})$ is Pareto efficient.

Secondly, every Nash equilibrium that supports an efficient allocation is a strict Nash equilibrium. Let $s \in S^n$ be a Nash equilibrium that attains an efficient allocation. Denote the set of participants at s by P^s . Since the project is done at efficient allocations, we have $\theta_{P^s} > c$. Furthermore, it is satisfied that $\theta_{P^s} - \theta_i \leq c$ for all $i \in P^s$: if

there is an agent $j \in P^s$ such that $\theta_{P^s} - \theta_j > c$, then agent j has an incentive to deviate from s because $\theta_j > \theta_j - x_j^{P^s}$. This contradicts the idea that s is a Nash equilibrium. By Lemma 1 and Remark 1, s is a strict Nash equilibrium. Q.E.D.

4 Strong equilibria of the participation game

In this section, we characterize the set of strong equilibria and show that there is a strong equilibrium in the participation game. By Lemma 2 and Proposition 1, there exists a Nash equilibrium supporting an efficient allocation in the participation game. But not all Nash equilibria attaining efficient allocations are strong equilibria. Moreover, existence of the Nash equilibria does not necessarily imply that of strong equilibria. By Pareto efficiency, the grand coalition does not improve its members' payoffs. By the definition of Nash equilibrium, every singleton coalition does not have an incentive to deviate. Applying these arguments, all of the Nash equilibria supporting efficient allocations are strong equilibria, when the economy consists of at most two agents. However, in games with more than two agents, coalitions consisting of more than one and less than n agents may form and their members may be better off. In fact, this applies to the participation game. The following example shows that there exists a Nash equilibrium supporting an efficient allocation, which is not a strong equilibrium in the participation game.

Example 2 Let $N = \{1, 2, 3\}$ and let $\theta_1 = \theta_2 = 8$, $\theta_3 = 4$, and c = 10. Table 2 shows the payoff matrix of this example. This game has three strict Nash equilibria: $(s_1, s_2, s_3) = (O, I, I), (I, O, I)$ and (I, I, O). All the strict Nash equilibria support efficient allocations. We now focus on the strategy profile $s^* = (I, I, O)$. The payoffs at s^* are $U_1(s^*) = U_2(s^*) = 3$, and $U_3(s^*) = 4$. Suppose a coalition $C = \{2, 3\}$ is formed and deviate from s_C^* to $\tilde{s}_C = (O, I)$. Note that the public project is undertaken at (s_1^*, \tilde{s}_C) . The payoffs of agent 2 and 3 at (s_1^*, \tilde{s}_C) are $U_2(s_1^*, \tilde{s}_C) = 8$ and $U_3(s_1^*, \tilde{s}_C) = 2/3$. Hence, the aggregate payoff for C at (s_1^*, \tilde{s}_C) is 26/3, which is greater than the sum of payoffs of C at s^* . Therefore, the strategy profile s^* is not a strong equilibrium, while the other strict Nash equilibria are strong equilibria.

	Ι	0			Ι	0
Ι	4, 4, 2	$\frac{4}{3}, 8, \frac{2}{3}$		Ι	3, 3, 4	0, 0, 0
0	$8, \frac{4}{3}, \frac{2}{3}$	0, 0, 0		0	0, 0, 0	0, 0, 0
I			<i>O</i>			

Table 2: Payoff matrix of Example 2

Example 2 indicates that not all strict Nash equilibria are strong in the participation game. In this example, the sum of the benefits that participants receive from the project is 12 in all strong equilibria, which is the smallest sum of the benefits of participants that can be attained in the set of strict Nash equilibria. In the following subsection, we identify which strict Nash equilibrium is a strong equilibrium in the participation game.

4.1 A characterization of strong equilibria

Proposition 2 Let $s^* \in S^n$ denote a strict Nash equilibrium and let P^* be the set of participants at s^* . The strict Nash equilibrium s^* is a strong equilibrium of G if and only if there is not coalition T and its strategy profile $\hat{s}_T \in S^{\#T}$ such that

$$T_I^* \subsetneq P^*, \ \theta_{T_I^* \setminus \widehat{T}_I} > \theta_{\widehat{T}_I \setminus T_I^*} > 0, \text{and} \ \theta_{P^*} - \theta_{T_I^* \setminus \widehat{T}_I} + \theta_{\widehat{T}_I \setminus T_I^*} > c, \tag{2}$$

where $T_I^* = \{i \in T | s_i^* = I\}$ and $\hat{T}_I = \{i \in T | \hat{s}_i = I\}.$

Proof. Let s^* denote a strict Nash equilibrium. Denote the set of participants by P^* . Let T denote a coalition and let \hat{s}_T denote a profile of strategies for T. The set of participants at (\hat{s}_T, s^*_{-T}) is denoted by \hat{P} . If we define $T_I^* = P \cap T$ and $\hat{T}_I = \hat{P} \cap T$, then $\hat{P} = (P^* \setminus (T_I^* \setminus \hat{T}_I)) \cup (\hat{T}_I \setminus T_I^*)$. Note that $\theta_{\hat{P}} = \theta_{P^*} - \theta_{T_I^* \setminus \hat{T}_I} + \theta_{\hat{T}_I \setminus T_I^*}$.

We first show the following lemma.

Lemma 3 Only the deviations that satisfy (2) improve the sum of the payoffs that members of T obtain.

Proof of Lemma 3.

Claim 1 If $\theta_{\widehat{P}} \geq \theta_{P^*}$, then the sum of the payoffs that agents in T obtain before the deviation is greater than or equal to the sum of the payoffs that members of T receive before the deviation.

Proof of Claim 1. The sum of the payoffs of agents in T at s^* is

$$\theta_T - \frac{\theta_{T_I^*}}{\theta_{P^*}}c,\tag{3}$$

and that at (\hat{s}_T, s^*_{-T}) is

$$\theta_T - \frac{\theta_{\widehat{T}_I}}{\theta_{\widehat{P}}}c.$$
 (4)

Subtracting (4) from (3) yields

$$\begin{aligned} &-\frac{\theta_{T_{I}^{*}}}{\theta_{P^{*}}}c + \frac{\theta_{\widehat{T}_{I}}}{\theta_{\widehat{P}}}c \\ &= \frac{c}{\theta_{P^{*}}\theta_{\widehat{P}}}(\theta_{P^{*}}\theta_{\widehat{T}_{I}} - \theta_{\widehat{P}}\theta_{T_{I}^{*}}) \\ &= \frac{c}{\theta_{P^{*}}\theta_{\widehat{P}}}\left(\theta_{P^{*}}\theta_{\widehat{T}_{I}} - \theta_{T_{I}^{*}}\left(\theta_{P^{*}} - \theta_{T_{I}^{*}\setminus\widehat{T}_{I}} + \theta_{\widehat{T}_{I}\setminus T_{I}^{*}}\right)\right) \\ &= \frac{c}{\theta_{P^{*}}\theta_{\widehat{P}}}\left(\theta_{P^{*}}\left(\theta_{\widehat{T}_{I}} - \theta_{T_{I}^{*}}\right) - \theta_{T_{I}^{*}}\left(\theta_{\widehat{T}_{I}\setminus T_{I}^{*}} - \theta_{T_{I}^{*}\setminus\widehat{T}_{I}}\right)\right). \end{aligned}$$

Using the equation $\theta_{\widehat{T}_I} - \theta_{T_I^*} = \theta_{\widehat{T}_I \setminus T_I^*} - \theta_{T_I^* \setminus \widehat{T}_I}$, we obtain

$$\frac{c}{\theta_{P^*}\theta_{\widehat{P}}}\left(\theta_{P^*} - \theta_{T_I^*}\right) \left(\theta_{\widehat{T}_I \setminus T_I^*} - \theta_{T_I^* \setminus \widehat{T}_I}\right).$$
(5)

We have $\theta_{P^*} - \theta_{T_I^*} \ge 0$ because $T_I^* \subseteq P^*$. Since $\theta_{\widehat{P}} \ge \theta_{P^*}$, we obtain $\theta_{\widehat{T}_I \setminus T_I^*} \ge \theta_{T_I^* \setminus \widehat{T}_I}$. Therefore, (5) is greater than or equal to zero. (End of Proof of Claim 1)

By Claim 1, the deviations by T satisfies $\theta_{P^*} > \theta_{\widehat{P}}$ if they result in improvements. Since $\theta_{P^*} > \theta_{\widehat{P}}$, we obtain $\theta_{T_I^* \setminus \widehat{T}_I} > \theta_{\widehat{T}_I \setminus T_I^*}$.

Claim 2 If $\theta_{\hat{P}} \leq c$, the deviation does not increase the sum of payoffs of agents in T.

Proof of Claim 2. Note that the project is not undertaken at (\hat{s}_T, s^*_{-T}) ; thus, the sum of the payoffs that members of T receive after the deviation is zero. Since (3) is more than zero, the deviations after which $\theta_{\hat{P}} \leq c$ is satisfies are not profitable. (End of Proof of Claim 2)

Combining Claim 1 and Claim 2 gives $\theta_{P^*} > \theta_{\widehat{P}} > c$. By Lemma 1, $\theta_{P^*} - \theta_i \leq c$ for all $i \in P^*$. Therefore, $\theta_{P^*} - \theta_{T_I^* \setminus \widehat{T}_I} \leq c$. By Claim 2, $\theta_{\widehat{P}} = \theta_{P^*} - \theta_{T_I^* \setminus \widehat{T}_I} + \theta_{\widehat{T}_I \setminus T_I^*} > c$. Thus, we have $\theta_{\widehat{T}_I \setminus T_I^*} > 0$. Accordingly, it follows that $\theta_{P^*} > \theta_{\widetilde{P}} > c$ and $\theta_{T_I^* \setminus \widehat{T}_I} > \theta_{\widehat{T}_I \setminus T_I^*} > 0$.

Claim 3 If $T_I^* = P^*$, then the total payoff of T at s^* is equal to that at (\hat{s}_T, s_{-T}^*) .

Proof of Claim 3. Note that the difference between the total payoff of T at s^* and that at (\hat{s}_T, s^*_{-T}) is equal to (5). Therefore, if $T^*_I = P^*$, then (5) is equal to zero. (End of Proof of Claim 3)

By Claims 1, 2, and 3, the statement of Lemma 3 is proven. (End of Proof of Lemma 3)

It is clear from Lemma 3 that a strict Nash equilibrium is a strong equilibrium in the participation game if and only if there are no coalitional deviations that satisfies (2). Q.E.D.

Proposition 2 says that a deviation from a strict Nash equilibrium results in improvements if and only if there exists the following situation: at the strict Nash equilibrium, a proper subset of the set of participants and non-participants form a coalition and they can coordinate in a way in which the sum of the benefits from the project of participants decreases and the project is undertaken. In this situation, members of the coalition changing their strategies I to O get benefits, and those who alter O to I suffer losses. However, by transferring part of the benefits to the agents altering O to I, the members switching I to O can make up for the losses. As a result, all members of the coalition can improve their payoffs after this deviation.

From Proposition 2, we confirm that no deviations after which the total benefits from the project of participants increases are profitable. Therefore, it is not profitable that participants at a strict Nash equilibrium commit themselves to choose participation and induce non-participants at the equilibrium to select participation by transferring money to the non-participants. Since we can interpret that allocations are more equitable as the number of participants increases, we conclude that the coalitional deviations from a strict Nash equilibrium to attain more equitable allocations are not profitable.

Shinohara (2003b) also analyzed the similar participation game in a mechanism that implements a class of allocation rules including the proportional cost allocation rule. However, he assumed that monetary transfers among members in coalitions are impossible. He showed that the set of strict Nash equilibria and that of strong equilibria coincide in the participation game without monetary transfers. On the other hand, when monetary transfers are possible, the set of strict Nash equilibria contains that of strong equilibria, and the two sets do not necessarily coincide. Therefore, the set of strong equilibria in the game with monetary transfers is a subset of that in the game without monetary transfers. Proposition 2 shows that the set of strong equilibria may shrink in the presence of the monetary transfers.

The following corollary shows that every strict Nash equilibrium at which only one agent chooses I is a strong equilibrium.

Corollary 1 If there is an agent $i \in N$ such that $\theta_i > c$, then $\{i\}$ is a set of participants at a strong equilibrium.

Proof. Suppose that there is an agent $i \in N$ be such that $\theta_i > c$. Then, the set $\{i\}$ is supportable as a strict Nash equilibrium in the game. Let $s^* \in S^n$ be the strict Nash equilibrium at which $\{i\}$ is the set of participants. By Proposition 2, s^* is a strong equilibrium if and only if no coalitions deviate from s^* in a way that satisfies (2). Because the proper subset of $\{i\}$ is empty, it is clear that no deviations from s^* satisfy (2). Therefore, $\{i\}$ is attained at a strict Nash equilibrium. Q.E.D.

Finally, we mention multiplicity of strong equilibria. In Example 2, the sum of the benefits that participants receive at strong equilibria is unique. However, this does not hold true in some cases. By Corollary 1, every strict Nash equilibrium with only one participant is a strong equilibrium even if the benefit of the participant is more than θ_{Pmin} . For example, consider an example where n = 3, c = 10, $\theta_1 = 5$, $\theta_2 = 6$, and $\theta_3 = 12$. Then, $\{1, 2\}$ and $\{3\}$ are the sets of participants that are supportable as a strong equilibrium. Hence, strong equilibria may support multiple sum of the benefits of participants.

4.2 Existence of a strong equilibrium

Proposition 3 A strong equilibrium exists in the participation game.

Proof. Let P^{min} denote a set of participants such that $\theta_{P^{min}}$ is the smallest sum of the benefits that participants receive in the set of strict Nash equilibria. Let $s^{min} \in S^n$ be a strict Nash equilibrium at which P^{min} is the set of participants. We show that s^{min} is a strong equilibrium. By Proposition 2, it is sufficient to show that there is no deviation that satisfies (2). Suppose, on the contrary, that there is a coalition T and its strategy profile s_T such that $T_I^{min} \subsetneq P^{min}$, $\theta_{T_I^{min} \setminus T_I} > \theta_{T_I \setminus T_I^{min}} > 0$, and

 $\theta_{P^{min}} - \theta_{T_I^{min} \setminus T_I} + \theta_{T_I \setminus T_I^{min}} > c$, where $T_I^{min} = \{i \in T | s_i^{min} = I\}$ and $T_I = \{i \in T | s_i = I\}$. Note that the set of participants at (s_T, s_{-T}^{min}) is $(P^{min} \cup (T_I \setminus T_I^{min})) \setminus (T_I^{min} \setminus T_I)$. Let us describe this set of participants by \widetilde{P} .

First of all, note that $\theta_{P^{min}} > \theta_{\widetilde{P}}$. Since $\theta_{P^{min}}$ is the smallest sum of the benefits that participants receive from the project at a strict Nash equilibrium, \widetilde{P} can not be supported as a strict Nash equilibrium. Thus, by Lemma 1, there is at least one agent $i \in \widetilde{P}$ such that $\theta_{\widetilde{P}} - \theta_i > c$.

Claim 4 Let $i \in \widetilde{P}$ be such that $\theta_{\widetilde{P}} - \theta_i > c$. Then, $i \in \widetilde{P} \setminus P^{min}$.

Proof of Claim 4. Let $i \in \widetilde{P}$ be such that $\theta_{\widetilde{P}} - \theta_i > c$. Suppose, on the contrary, $i \in \widetilde{P} \cap P^{min}$. Then, $\theta_{P^{min}} - \theta_i \leq c$ holds. From this condition and $\theta_{T_I^{min}\setminus T_I} > \theta_{T_I\setminus T_I^{min}}$, we obtain $\theta_{P^{min}} - \theta_i - \theta_{T_I^{min}\setminus T_I} + \theta_{T_I\setminus T_I^{min}} < c$. As $\theta_{\widetilde{P}} = \theta_{P^{min}} - \theta_{T_I^{min}\setminus T_I} + \theta_{T_I\setminus T_I^{min}}$, we obtain that $\theta_{\widetilde{P}} - \theta_i < c$, which contradicts $\theta_{\widetilde{P}} - \theta_i > c$. Therefore, we have $i \in \widetilde{P} \setminus P^{min}$. (End of Proof of Claim 4)

Note that $\widetilde{P} \setminus P^{min} = T_I \setminus T_I^{min}$. From the conditions $\theta_{\widetilde{P}} = \theta_{P^{min}} - \theta_{T_I^{min} \setminus T_I} + \theta_{T_I \setminus T_I^{min}} > c$ and $\theta_{P^{min}} - \theta_{T_I^{min} \setminus T_I} \leq c$, we obtain $\theta_{T_I \setminus T_I^{min}} > 0$. Thus, $T_I \setminus T_I^{min}$ is not empty. Suppose, without loss of generality, that the set $T_I \setminus T_I^{min}$ consists of h agents. Denote this set by $\{j_1, \ldots, j_h\}$. In addition, let us assume that $\theta_{j_1} \leq \theta_{j_2} \leq \cdots \leq \theta_{j_h}$. First, consider the set $\widetilde{P} \setminus \{j_1\}$. Since \widetilde{P} is not supported as strict Nash equilibria, we have $\theta_{\widetilde{P} \setminus \{j_1\}} > c$: otherwise, we have

$$c \ge \theta_{\widetilde{P}} - \theta_{j_1} \ge \theta_{\widetilde{P}} - \theta_k \text{ for all } k \in T_I \setminus T_I^{min}, \text{ and} c \ge \theta_{P^{min}} - \theta_k > \theta_{\widetilde{P}} - \theta_k \text{ for all } k \in \widetilde{P} \cap P^{min},$$

which means that \widetilde{P} is supportable as a strict Nash equilibrium. This is a contradiction. If $\theta_{\widetilde{P}\setminus\{j_1\}} - \theta_{j_2} \leq c$, then $\widetilde{P}\setminus\{j_1\}$ is supportable as a strict Nash equilibrium since $\theta_{\widetilde{P}\setminus\{j_1\}} - \theta_j \leq c$ for all $j \in \widetilde{P}\setminus\{j_1\}$. This contradicts the idea that $\theta_{\widetilde{P}}$ is the smallest sum of the benefits of participants that is attained at strict Nash equilibria. If $\theta_{\widetilde{P}\setminus\{j_1\}} - \theta_{j_2} > c$, then consider the set of participants $\widetilde{P}\setminus\{j_1, j_2\}$. If $\theta_{\widetilde{P}\setminus\{j_1, j_2\}} - \theta_{j_3} \leq c$, then the set $\widetilde{P}\setminus\{j_1, j_2\}$ is supportable as a strict Nash equilibrium, which is a contradiction by the same reason above. If else, consider the set $\widetilde{P}\setminus\{j_1, j_2, j_3\}$. Applying the same argument and using the facts that $\theta_{Pmin} - \theta_{T_I^{min}\setminus T_I} \leq c$ and $\theta_{Pmin} - \theta_{T_I^{min}\setminus T_I} + \theta_{T_I\setminus T_I^{min}} > c$, we can find the set $\widetilde{P}\setminus K$ is supportable as a strict Nash equilibrium. This is a contradiction. Therefore, coalition T can not deviate in a way that satisfies (2). Q.E.D.

From Proposition 2, the set of strict Nash equilibria contains that of strong equilibria, but the converse is not always true. However, in the case of identical agents, every strict Nash equilibrium is a strong equilibrium.

Corollary 2 Suppose that agents are identical: $\theta_i = \theta_j$ for all pairs of agents $\{i, j\}$. Then, all strict Nash equilibria are strong equilibria in the participation game.

Proof. Let $\theta = \theta_i$ for all $i \in N$ and let P be a set of participants that is supported as a strict Nash equilibrium. By Lemma 1, P satisfies $\#P\theta > c$ and $(\#P-1)\theta \leq c$, or $c/\theta < \#P \leq (c/\theta) + 1$. Since #P is a natural number, we find from these inequalities that #P is unique. Therefore, $\#P\theta$ is the smallest sum of the benefits that participants receive from the project in the set of strict Nash equilibria. In the proof of Proposition 3, we show that a strict Nash equilibrium in which the sum of the benefits of the participants is the smallest in the set of strict Nash equilibria is strong. Thus, P is attained at a strong equilibrium of the game. Q.E.D.

In the participation game, strict Nash and strong equilibria are both non-empty and the set of strong equilibria is included in that of strict Nash equilibria. This is an interesting respect of our model, because strict Nash equilibria and strong equilibria are based on different stability and there is not always the inclusion relation between the two sets of equilibria.

It follows from Lemma 2 that there exists an efficient allocation which is supportable as a Nash equilibrium. Moreover, some of the efficient allocations are also supported as a strong equilibrium of the participation game. This feature can not be observed in models of the provision of perfectly divisible public goods. In a participation game in a mechanism producing a perfectly divisible public good, Nash equilibria frequently support inefficient allocations, and there do not necessarily exist strong equilibria (Saijo and Yamato, 1999; Shinohara, 2003a). In a game of the voluntary contribution of a perfectly divisible public good, the allocation supported as a Nash equilibrium is not efficient; hence, a strong equilibrium does not exist in the standard voluntary contribution game. However, in the participation game with a public project, strong equilibria exist and an efficient allocation can always be attained not only Nash equilibria but also strong equilibria. Therefore, the above phenomenon does not occur if there exists one public project. This is the second interesting characteristic of our model.

The following theorem summarizes the results that have been obtained so far.

Theorem In the participation game with a public project, (i) there is a Nash equilibrium at which the efficiency of an allocation is achieved, (ii) the set of Nash equilibria that supports efficient allocations coincides with the set of strict Nash equilibria, (iii) a strong equilibrium exists, and (iv) the set of strict Nash equilibria includes that of strong equilibria, but the converse inclusion relation does not necessarily hold.

Remark 2 Let us consider the participation game in which the project is undertaken if and only if the sum of the benefits that participants receive from the project is more than or equal to the cost c: for all sets of participants P, $\theta_P \ge c$ if and only if $y^P = 1$. In this participation game, there are not necessarily strict Nash equilibria. However, a Nash equilibrium at which an efficient allocation is attained exists in this game. If P designates the set of participants at such Nash equilibria, then it is characterized as $\theta_P \ge c$ and $\theta_P - \theta_i < c$ for all $i \in P$. We can similarly show that a strong equilibrium exists and the set of strong equilibria is a subset of the set of Nash equilibria that supports efficient allocations in this participation game.

5 Participation games with a multi-unit public good

5.1 A participation game in which at most two units of the public good can be produced

In this section, we consider a participation game in a mechanism that implements the proportional cost-sharing rule in which at most two units of the public good can be provided. Let Y be a *public good space* such that $Y = \{(y_1, y_2) \in \{0, 1\}^2 | y_1 \ge y_2\}$: if $y_1 = y_2 = 1$, then two units of the public good are produced; if $y_1 = 1$ and $y_2 = 0$, then one unit of the public good is produced; if $y_1 = y_2 = 0$, then zero units of the public good. Let c > 0 be the cost of producing one unit of the public good. Each agent *i* has a preference relation that is represented by the utility function $V_i : Y \times \mathbb{R}_+ \to \mathbb{R}_+$, which associates a real value $V_i(y, x_i) = \sum_{k \in \{1,2\}} \theta_i^k y_k - x_i$ with each element (y, x_i) in $Y \times \mathbb{R}_+$, where $\theta_i^k > 0$ denotes agent *i*'s marginal benefit from the *k*-th unit of the public good. We denote $\theta_P^k = \sum_{j \in P} \theta_j^k$ for all $k \in \{1,2\}$ and for all $P \subseteq N$. Let us assume that $\theta_i^1 > \theta_i^2$ for all $i \in N$ and $\theta_N^2 > c$. Thus, at every Pareto efficient allocation, two units of the public good is produced.

Assumption 4 There exists a mechanism that implements the following allocation rule. Let P denote a set of participants and $((x_j^P)_{j\in P}, y^P)$ be the allocation that is implemented by the mechanism. Then,

$$y^{P} = \max\{k \in \{0, 1, 2\} | \theta_{P}^{k} - c > 0\}, \text{ and}$$
for all $i \in P, x_{i}^{P} = \begin{cases} 0 & \text{if } y^{P} = 0, \\ \frac{\sum_{k=1}^{y^{P}} \theta_{i}^{k}}{\sum_{k=1}^{y^{P}} \theta_{P}^{k}} y^{P}c & \text{otherwise.} \end{cases}$

The following example indicates that a Nash equilibrium does not always support an efficient allocation and strong equilibria do not necessarily exist in the participation game with a multi-unit public good.

Example 3 Let $N = \{1, 2, 3, 4\}$. Suppose that $\theta_i^1 = 2$ and $\theta_i^2 = 0.8$ for all $i \in N$ and c = 1. Let P be a set of participants. Note that one unit of the public good is produced if #P = 1, and two units of the public good are provided if $\#P \ge 2$. Table 3 shows the payoffs to participants and non-participants in this example. From the table, we can easily find that one and only one agent enters the mechanism at every strict Nash equilibrium. However, these Nash equilibria are not strong equilibria, since three non-participants at the Nash equilibrium can gain higher payoffs if all of them jointly deviate from non-participation to participation; thus, a strong equilibrium does not exist in this example.

In the participation game with a public project, there is a Nash equilibrium that supports an efficient allocation, and a strong equilibrium exists. However, in the participation game with a multi-unit public good, efficient allocations are not necessarily

The number of participants	Payoffs to participants	Payoffs to non-participants
0	-	0
1	1	2
2	1.8	2.8
3	$\frac{32}{15}$	2.8
4	2.3	-

Table 3: Payoffs of Example 3

supportable as Nash equilibria and there are not always strong equilibria. In Example 3, one unit of the public good is produced at every Nash equilibrium and no strong equilibrium exists. This is a remarkable difference between the participation game with a public project and that with a multi-unit public good.

5.2 Non-existence of equilibria that support efficient allocations

In this subsection, we investigate whether or not a Nash equilibrium supports an Pareto efficient allocation in the participation game in which at most two units of the public good can be produced. For this, we first characterize the set of Nash equilibria at which two units of the public good are produced.

Proposition 4 Two units of the public good are produced at a Nash equilibrium in the participation game if and only if there is a set of participants $P \subseteq N$ that satisfies (i) $\theta_P^2 > c$, (ii) $\theta_P^2 - \theta_i^2 \leq c$ for all $i \in P$, and (iii) if there is an agent $i \in P$ such $\theta_P^1 - \theta_i^1 > c$, then $\theta_i^2 \geq \frac{\sum_{k=1}^2 \theta_i^k}{\sum_{k=1}^2 \theta_k^k} (2c)$.

Proof. (sufficiency) Suppose that there is a set P that satisfies the conditions (i), (ii), and (iii). By (i), two units of the public good are produced if P is a set of participant. By conditions (ii) and (iii), no agent $i \in P$ have an incentive to switch I to O. Cleary, no agents $i \notin P$ do not have an incentive to participate in the mechanism, given the participation of P. Hence, P is a set of participants that is supportable as a Nash equilibrium.

(necessity) Let us assume that there exists a Nash equilibrium at which two units of the public good are produced. Let P be the set of participant attained at the Nash equilibrium. Since two units of the public good are provided, condition (i) must be satisfied. If P do not satisfy (ii), then there exists agent i such that $\theta_P^2 - \theta_i^2 > c$. Hence, agent i has an incentive to deviate from I to O, which is a contradiction. Suppose that there is an agent $i \in P$ such that $\theta_P^1 - \theta_i^1 > c$ and $\theta_i^2 < \frac{\sum_{k=1}^2 \theta_i^k}{\sum_{k=1}^2 \theta_P^k}(2c)$. Then, he obtains the payoff θ_i^1 if he chooses I, and he receives the payoff $\sum_{k=1}^2 \theta_i^k - \frac{\sum_{k=1}^2 \theta_i^k}{\sum_{k=1}^2 \theta_P^k}(2c)$. Since $\theta_i^2 < \frac{\sum_{k=1}^2 \theta_i^k}{\sum_{k=1}^2 \theta_P^k}(2c)$, he has an incentive to switch from I to O. This is a contradiction. Therefore, P satisfies (i), (ii), and (iii). Q.E.D. We examine whether two units of the public good are produced at a Nash equilibrium or not. First consider the following case.

Case 1. For all $P \subseteq N$, P satisfies either $\theta_P^2 \leq c$ or $\theta_{P \setminus \{i\}}^1 > c$ for some $i \in P$.

Let P^* be a set of participants such that $\theta_{P^*}^2 > c$ and $\theta_{P^*\setminus\{i\}}^1 > c$ for some $i \in P^*$. If all agents in P^* participate in the mechanism, then two units of the public good are produced, and one unit of the public good is supplied if some agent in P^* does not choose participation. Since $\theta_{P^*\setminus\{i\}}^1 > c$ for some $i \in P^*$, we have $\#P \ge 2$. We focus on a case of identical agents: let $\theta^1 = \theta_i^1$ and $\theta^2 = \theta_i^2$ for all $i \in N$. By Proposition 4, the set P^* is supportable as a Nash equilibrium if and only if it satisfies (i), (ii), and (iii). By condition (i),

$$\#P^*\theta^2 > c. \tag{6}$$

By condition (ii), we have $(\#P^* - 1)\theta^2 \leq c$. Therefore,

$$\theta^2 \le \frac{c}{\#P^* - 1}.\tag{7}$$

By condition (iii),

$$\theta^2 \ge \frac{2c}{\#P^*}.\tag{8}$$

Note that it is sufficient to focus solely on equations (7) and (8). Subtracting $\frac{2c}{\#P^*}$ from $\frac{c}{\#P^*-1}$ yields

$$\frac{c}{\#P^*(\#P^*-1)}(2-\#P^*).$$
(9)

Since $\#P^* \ge 2$, we have $(9) \le 0$ with equality if $\#P^* = 2$. Therefore, it is impossible for a Nash equilibrium to support the provision of two units of the public good if $\#P^* > 2$. When $\#P^* = 2$, two units of the public good are produced only in the case of $\theta^2 = c$. Therefore, in this case, two units of the public good are hardly provided when agents are identical.

The following proposition summarizes the above results.

Proposition 5 Suppose that, for all $P \subseteq N$, P satisfies either $\theta_P^2 \leq c$ or $\theta_{P \setminus \{i\}}^1 > c$ for some $i \in P$. Suppose that agents are identical. Then, Nash equilibria do not support efficient allocations at almost all values θ^2 .

Proposition 5 confirms that the strategic behavior on the participation decisions often leads to the inefficiency of the allocations, even though a mechanism is constructed in a way that implements an efficient allocation rule. Hence, the implication that is similar to Saijo and Yamato (1999) can be derived even in the participation game in which at most two units of the public good can be produced.

Proposition 6 In the participation game in which at most two units of the public good, if a strategy profile is a strong equilibrium, then it is a Nash equilibrium that supports an efficient allocation.

Proof. Suppose, on the contrary, that there is a strong equilibrium $s \in S^n$ that supports an inefficient allocation. If zero units of the public good is produced at s, then every agent receives the payoff zero, and if one units of the public good is produced, then every participant i obtains $\theta_i^1 - \frac{\theta_i^1}{\theta_p^1}c$ and every non-participant j receives θ_j^1 . Note that the sum of the payoffs of all agents is zero, when no public good is provided; the sum of the payoffs to all agents is $\theta_N^1 - c > 0$, when one unit of the public good is supplied. On the other hand, when all agents choose I, the sum of the payoffs to all agents is $\sum_{k=1}^2 \theta_N^k - 2c$, which is greater than $\theta_N^1 - c$. Therefore, if the grand coalition forms and every member chooses I, then all members of N are better off, which is a contradiction. Q.E.D.

Obviously, every strong equilibrium is Pareto efficient within the set of strategy profiles. However, in this model, it is not clear that a strong equilibrium supports an efficient allocation, because the strategy sets of all agents consists of two alternatives. Proposition 6 shows that two units of the public good is provided at every strong equilibrium of this game. It follows from Propositions 5 and 6 that a strong equilibrium does not necessarily exist in the participation game with a multiunit public good.

Finally, we briefly mention the case in which Case 1 is not satisfied: there exists a set of participants P such that $\theta_P^1 - \theta_i^1 \leq c$ for all $i \in P$ and $\theta_P^2 > c$. Note that, for all $D \subseteq N$, if D is not empty, then $\theta_D^1 > \theta_D^2$. Thus, the above P satisfies $\theta_{P \setminus \{i\}}^2 < \theta_{P \setminus \{i\}}^1 \leq c$ for all $i \in P$ and $\theta_P^2 > c$. If all agents in P choose participation, then two units of the public good is provided. By Proposition 4, P can be supportable as a Nash equilibrium. Thus, two units of the public good are produced at a Nash equilibrium in this case.

6 Conclusions

We have investigated the participation game in the mechanism implementing the proportional cost-sharing rule. First, we considered the case of a public project. We show that, in this case, strict Nash equilibria exist, the set of strict Nash equilibria and the set of Nash equilibria that supports an efficient allocation coincide, there are strong equilibria, and the set of strict Nash equilibria contains that of strong equilibria. Secondly, we considered the case in which at most two units of the public good can be provided. In this case, there is not always a Nash equilibrium that supports an efficient allocation and there is not necessarily a strong equilibrium. We found from these results that the assumption that only one unit of the public good can be produced is essential to the existence of a Nash equilibrium that supports the efficient allocation and that of a strong equilibrium. We also found that the strategic behavior on the participation decisions leads to the inefficiency of the allocations even in the participation game in which at most two units of the public good can be produced.

Although efficient allocations are attained at the equilibria in the participation game with a public project, the allocations are less desirable from the viewpoint of equity. In the participation game with a multi-unit public good, there are not necessarily equilibria to support a Pareto efficient allocation. To attain more desirable allocations at equilibria of the participation game, it is desirable that all agents participate in the mechanism. It is left for future researches to study the possibility of constructing the mechanism, in which all agents participate at equilibria.

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